Compositio Mathematica **138:** 261–288, 2003.

Twisted Stability and Fourier–Mukai Transform I

KŌTA YOSHIOKA
Department of Mathematics, Faculty of Science, Kobe University, Kobe, 657, Japan.
e-mail: yoshioka@math.kobe-u.ac.jp

(Received: 4 September 2001; accepted in final form: 18 March 2002)

Abstract. In this paper, we consider the preservation of stability by using the notion of twisted stability. As applications, (1) we show that moduli spaces of stable sheaves on K3 and abelian surfaces are irreducible and (2) we compute Hodge polynomials of some moduli spaces of stable sheaves on Enriques surfaces.

Mathematics Subject Classification (1991). 14D20.

Key words. Fourier-Mukai functor, moduli of sheaves.

0. Introduction

Let X be an Abelian surface or a K3 surface over $\mathbb{C}$. Mukai introduced a lattice structure $\langle\ ,\ \rangle$ on $H^{ev}(X,\mathbb{Z}) := \bigoplus H^{2i}(X,\mathbb{Z})$ by

$$\begin{aligned}\langle x, y\rangle &:= -\int_X x^\vee \wedge y \\ &= \int_X (x_1 \wedge y_1 - x_0 \wedge y_2 - x_2 \wedge y_0),\end{aligned} \tag{0.1}$$

where $x_i \in H^{2i}(X,\mathbb{Z})$ (resp. $y_i \in H^{2i}(X,\mathbb{Z})$) is the $2i$th component of x (resp. y) and $x^\vee = x_0 - x_1 + x_2$. It is now called the Mukai lattice. For a coherent sheaf E on X, we can attach an element of $H^{ev}(X,\mathbb{Z})$ called the Mukai vector $v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}_X}$, where $\mathrm{ch}(E)$ is the Chern character of E and td_X is the Todd class of X. For a Mukai vector $v \in H^{ev}(X,\mathbb{Z})$ and an ample divisor H, let $M_H(v)$ be the moduli space of stable sheaves E with the Mukai vector $v(E) = v$ and $\overline{M}_H(v)$ the moduli space of semi-stable sheaves. An ample divisor H is general with respect to v, if the following condition holds:

($\natural$) for every μ-semi-stable sheaf E with $v(E) = v$, if $F \subset E$ satisfies

$(c_1(F), H)/\mathrm{rk}\,F = (c_1(E), H)/\mathrm{rk}\,E$,

then $c_1(F)/\mathrm{rk}\,F = c_1(E)/\mathrm{rk}\,E$.

The preservation of the stability by the Fourier-Mukai transform on X was investigated by many people (e.g. [BBH2, B-M, Mu5, Y4]). In [Y4], we introduced the twisted degree of coherent sheaf E by $\deg_G(E) = \deg(E \otimes G^\vee) = (c_1(E \otimes G^\vee), H)$, where G is a vector bundle on X. Then we showed that the Fourier–Mukai transform preserves Gieseker semi-stability, if the twisted degree is 0 and the polarization H is general. In this paper, we shall generalize our results to the case where H is not general. In this case, the Fourier-Mukai transform does not preserve Gieseker semi-stability. This fact is closely related to the following fact: If H is not general, then Gieseker semi-stability is not preserved by the twisting $E \mapsto E \otimes L$, where L is a line bundle. Thus Gieseker semi-stability depends on the choice of L. In order to understand this phenomenon, Matsuki and Wentworth [M-W] (also by Ellingsrud and Göttsche [E-G] and Friedman and Qin [F-Q]) introduced the notion of L-twisted semi-stability, where L is a $\mathbb{Q}$-line bundle. Hence we shall propose a formulation for our problem by using the twisted semi-stability. In Section 2, we shall show that the Fourier-Mukai transform preserves a suitable twisted semi-stability, if X is an Abelian surface (Theorem 2.3).

In [Y4], we showed that $M_H(v)$ is deformation equivalent to a moduli space of torsion free sheaves of rank 1, if v is primitive and the polarization H is general. In Section 3, we shall give another proof of this result by using results proved in Section 2. Moreover we shall show the following.

THEOREM 0.1. *Let X be an Abelian surface or a K3 surface. Let $v \in H^{ev}(X, \mathbb{Z})$ be a Mukai vector of* $\operatorname{rk} v > 0$. *Then $\overline{M}_H(v)$ is a normal variety, if $\langle v^2 \rangle > 0$ and H is general with respect to v.*

In Section 4, we shall consider the Fourier–Mukai transform on an Enriques surface associated to (-1)-reflection. In particular, we shall show a similar result to Theorem 2.3 (Proposition 4.3). As an application, we shall compute the Hodge polynomials of some moduli spaces (Theorem 4.6).

This paper is the first half part of [Y5].

1. Preliminaries

1.1. TWISTED STABILITY FOR TORSION FREE SHEAVES

Let X be a smooth projective surface and $K(X)$ the Grothendieck group of X. We fix an ample divisor H on X. For $G \in K(X) \otimes \mathbb{Q}$ with $\operatorname{rk} G > 0$, we define the G-twisted rank, degree, and Euler characteristic of $x \in K(X) \otimes \mathbb{Q}$ by

$$\begin{aligned} \operatorname{rk}_G(x) &:= \operatorname{rk}(G^\vee \otimes x), \\ \deg_G(x) &:= (c_1(G^\vee \otimes x), H), \\ \chi_G(x) &:= \chi(G^\vee \otimes x). \end{aligned} \tag{1.1}$$

For $t \in \mathbb{Q}_{>0}$, we get

$$\frac{\deg_G(x)}{\operatorname{rk}_G(x)} = \frac{\deg_{tG}(x)}{\operatorname{rk}_{tG}(x)}, \qquad \frac{\chi_G(x)}{\operatorname{rk}_G(x)} = \frac{\chi_{tG}(x)}{\operatorname{rk}_{tG}(x)}. \tag{1.2}$$

We shall define the G-twisted stability.

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DEFINITION 1.1. Let E be a torsion free sheaf on X. E is G-twisted semi-stable (resp. stable) with respect to H, if

$$\frac{\chi_G(F(nH))}{\operatorname{rk}_G(F)} \leqslant \frac{\chi_G(E(nH))}{\operatorname{rk}_G(E)}, \quad n \gg 0 \tag{1.3}$$

for $0 \subsetneq F \subsetneq E$ (resp. the inequality is strict).

For a $\mathbb{Q}$-divisor α, we define the α-twisted stability as the $\mathcal{O}_X(\alpha)$-twisted stability. This is nothing but the twisted stability introduced by Matsuki and Wentworth [M-W]. It is easy to see that the G-twisted stability is determined by $\alpha = \det(G)/\operatorname{rk} G$. Hence the G-twisted stability is the same as Matsuki-Wentworth stability.

DEFINITION 1.2. For $x \in K(X)$, we set

$$\gamma(x) := (\operatorname{rk} x, c_1(x), \chi(x)) \in \mathbb{Z} \oplus \operatorname{NS}(X) \oplus \mathbb{Z}. \tag{1.4}$$

Let $\mathcal{M}_H^G(\gamma)^{ss}$ be the moduli stack of G-twisted semi-stable sheaves E with $\gamma(E) = \gamma$ and $\mathcal{M}_H^G(\gamma)^s$ the open substack consisting of G-twisted stable sheaves. For usual stability, *i.e.*, $G = \mathcal{O}_X$, we denote $\mathcal{M}_H^{\mathcal{O}_X}(\gamma)^{ss}$ by $\mathcal{M}_H(\gamma)^{ss}$.

Remark 1.1. We take a sufficiently large integer m so that

(i) $H^0(X, E(mH)) \otimes \mathcal{O}_X \to E(m)$ is surjective for all $E \in \mathcal{M}_H^G(\gamma)^{ss}$,
(ii) $H^i(X, E(mH)) = 0$, $i > 0$ for all $E \in \mathcal{M}_H^G(\gamma)^{ss}$.

We set $N := \dim H^0(X, E(mH))$, $E \in \mathcal{M}_H^G(\gamma)^{ss}$. We shall consider the quot scheme $Q := \operatorname{Quot}^{\gamma}_{\mathcal{O}_X(-mH)^{\oplus N}/X/\mathbb{C}}$ parametrizing all quotients $\mathcal{O}_X(-mH)^{\oplus N} \to E$ with $\gamma(E) = \gamma$. Let Q^{ss} be an open subscheme of Q consisting of all quotients $\mathcal{O}_X(-mH)^{\oplus N} \to E$ such that

(i) E is a G-twisted semi-stable sheaf with respect to H with $\gamma(E) = \gamma$,
(ii) $H^0(X, \mathcal{O}_X^{\oplus N}) \to H^0(X, E(mH))$ is isomorphic.

Then $\mathcal{M}_H^G(\gamma)^{ss}$ is a quotient stack of Q^{ss} by the natural action of $\operatorname{GL}(N)$ on Q^{ss}:

$$\mathcal{M}_H^G(\gamma)^{ss} = [Q^{ss}/\operatorname{GL}(N)]. \tag{1.5}$$

Remark 1.2. Let $c_1(G)/\operatorname{rk} G = aH + \beta$, $a \in \mathbb{Q}$, $\beta \in H^{\perp}$ be the orthogonal decomposition. Then the twisted semi-stability condition only depends on β, *i.e.*, $\mathcal{M}_H^G(\gamma)^{ss} = \mathcal{M}_H^{\beta}(\gamma)^{ss}$.

THEOREM 1.1 [*M-W*] (*also see* [*E-G*]).

(i) *There is a coarse moduli scheme* $\overline{M}_H^G(\gamma)$ *of S-equivalence classes of G-twisted semi-stable sheaves.*
(ii) $\overline{M}_H^G(\gamma)$ *is projective.*

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(iii) *For different* G, G', *the relation between* $\overline{M}_H^G(\gamma)$ *and* $\overline{M}_H^{G'}(\gamma)$ *is described as Mumford–Thaddeus type flips*:

$$
\begin{array}{ccccccccc}
\overline{M}_H^{G_1}(\gamma) & & & & \overline{M}_H^{G_2}(\gamma) & & & & \overline{M}_H^{G_n}(\gamma) \\
& \searrow & & \swarrow & & \searrow & & \cdots \ \swarrow & \\
& & \overline{M}_H^{G_{1,2}}(\gamma) & & & & \overline{M}_H^{G_{2,3}}(\gamma) & &
\end{array}
\tag{1.6}
$$

where $G = G_1$, $G' = G_n$.

DEFINITION 1.3. $M_H^G(\gamma)$ is the open subscheme of $\overline{M}_H^G(\gamma)$ consisting of G-twisted stable sheaves and $M_H(\gamma)^{\mu-s}$ the open subscheme consisting of μ-stable sheaves. Usually we denote $\overline{M}_H^{\mathcal{O}_X}(\gamma)$ by $\overline{M}_H(\gamma)$ and $M_H^{\mathcal{O}_X}(\gamma)$ by $M_H(\gamma)$.

Since μ-stability does not depend on G, $M_H(\gamma)^{\mu-s}$ is a subscheme of $\overline{M}_H^G(\gamma)$ for all G.

DEFINITION 1.4. For a pair (H, G) of an ample divisor H and an element $G \in K(X) \otimes \mathbb{Q}$, (H, G) is general with respect to γ, if the following condition holds for every $E \in \mathcal{M}_H^G(\gamma)^{ss}$:

For $0 \subsetneq F \subsetneq E$,

$$
\frac{\chi_G(F(nH))}{\mathrm{rk}_G(F)} = \frac{\chi_G(E(nH))}{\mathrm{rk}_G(E)}, \quad n \gg 0 \tag{1.7}
$$

implies that $\gamma(F)/\mathrm{rk}\, F = \gamma(E)/\mathrm{rk}\, E$.

The following is easy (cf. [M-W]).

LEMMA 1.2. *For an ample divisor* H *and* $\gamma \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$, *there is a general* (H, G).

2. Fourier–Mukai Transform on Abelian and K3 Surfaces

2.1. FOURIER–MUKAI TRANSFORM

Let X be a K3 surface or an Abelian surface. Let E be a coherent sheaf on X. Let

$$
\begin{aligned}
v(E) &:= \mathrm{ch}(E)\sqrt{\mathrm{td}_X} \\
&= \mathrm{rk}(E) + c_1(E) + (\chi(E) - \epsilon\,\mathrm{rk}(E))\varrho_X \in H^{ev}(X, \mathbb{Z})
\end{aligned}
\tag{2.1}
$$

be the Mukai vector of E, where $\epsilon = 0, 1$ according as X is an Abelian surface or a K3 surface and ϱ_X is the fundamental class of X, i.e, $\int_X \rho_X = 1$. For these surfaces, it is common to use the Mukai vector of E instead of using $\gamma(E)$. Hence we use the Mukai vector in this Section. For a Mukai vector v, we define $\mathcal{M}_H^G(v)^{ss}$, $\overline{M}_H^G(v)$, … as in Section 1.

Let $v_1 := r_1 + c_1 + a_1\varrho_X$, $r_1 > 0$, $c_1 \in \mathrm{NS}(X)$ be a primitive isotropic Mukai vector on X. We take a general ample divisor H with respect to v_1. We set $Y := M_H(v_1)$. Then Y is an Abelian surface (resp. a K3 surface), if X is an Abelian surface (resp. a K3 surface).

If X is an Abelian surface, then Y consists of μ-stable vector bundles. By the proof of [Y3, Lem. 2.1], the following Lemma holds.

LEMMA 2.1. *Assume that H is general with respect to v_1.*

(i) *If Y contains a non-locally free sheaf, then there is an exceptional vector bundle E_0 and $v_1 = \mathrm{rk}(E_0)v(E_0) - \varrho_X$. Moreover $Y \cong X$ and a universal family is given by*

$$\mathcal{E} := \ker(E_0 \boxtimes E_0^\vee \to \mathcal{O}_\Delta). \tag{2.2}$$

(ii) *If Y consists of locally free sheaves, then they are μ-stable.*

Assume that there is a universal family $\mathcal{E}$ on $X \times Y$. Let $p_X\colon X \times Y \to X$ (resp. $p_Y\colon X \times Y \to Y$) be the projection. We define $\mathcal{F}_\mathcal{E}\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ by

$$\mathcal{F}_\mathcal{E}(x) := \mathbf{R}p_{Y*}(\mathcal{E} \otimes p_X^*(x)), \quad x \in \mathbf{D}(X), \tag{2.3}$$

and $\hat{\mathcal{F}}_\mathcal{E}\colon \mathbf{D}(Y) \to \mathbf{D}(X)$ by

$$\hat{\mathcal{F}}_\mathcal{E}(y) := \mathbf{R}\,\mathrm{Hom}_{p_X}(\mathcal{E}, p_Y^*(y)), \quad y \in \mathbf{D}(Y), \tag{2.4}$$

where $\mathrm{Hom}_{p_X}(-,-) = p_{X*}\mathrm{Hom}_{\mathcal{O}_{X\times Y}}(-,-)$ is the sheaf of relative homomorphisms. Bridgeland [Br] showed that $\mathcal{F}_\mathcal{E}$ is an equivalence of categories and the inverse is given by $\hat{\mathcal{F}}_\mathcal{E}[2]$. $\mathcal{F}_\mathcal{E}$ is now called the Fourier–Mukai functor. We denote the ith cohomology sheaf $H^i(\mathcal{F}_\mathcal{E}(x))$ by $\mathcal{F}_\mathcal{E}^i(x)$.

$\mathcal{F}_\mathcal{E}$ induces an isomorphism $K(X) \to K(Y)$ and an isometry of Mukai lattice $\mathcal{F}_\mathcal{E}\colon H^{ev}(X,\mathbb{Z}) \to H^{ev}(Y,\mathbb{Z})$. We also have a commutative diagram:

$$\begin{array}{ccc}
\mathbf{D}(X) & \xrightarrow{\mathcal{F}_\mathcal{E}} & \mathbf{D}(Y) \\
\downarrow & & \downarrow \\
K(X) & \xrightarrow{\mathcal{F}_\mathcal{E}} & K(Y) \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v} \\
H^{ev}(X,\mathbb{Z}) & \xrightarrow{\mathcal{F}_\mathcal{E}} & H^{ev}(Y,\mathbb{Z}).
\end{array} \tag{2.5}$$

For our purpose, the usual Fourier–Mukai functor $\mathcal{F}_\mathcal{E}$ is not sufficient. As in [Y4], we introduce a functor $\mathcal{H}_\mathcal{E}\colon \mathbf{D}(X) \to \mathbf{D}(Y)_{\mathrm{op}}$ which is a composite of $\mathcal{F}_\mathcal{E}$ with the taking dual functor $\mathcal{D}_X\colon \mathbf{D}(X) \to \mathbf{D}(X)_{\mathrm{op}}$:

$$\begin{aligned}\mathcal{H}_\mathcal{E}(x) &:= \mathcal{F}_\mathcal{E} \circ \mathcal{D}_X(x) \\ &= \mathbf{R}\,\mathrm{Hom}_{p_Y}(p_X^*(x), \mathcal{E}), \quad x \in \mathbf{D}(X).\end{aligned} \tag{2.6}$$

By the Grothendieck–Serre duality,

$$\mathcal{H}_\mathcal{E}(x) = \mathcal{F}_\mathcal{E} \circ \mathcal{D}_X(x) = \mathcal{D}_Y(\mathbf{R}\,\mathrm{Hom}_{p_Y}(\mathcal{E}, p_X^*(x)))[-2]. \tag{2.7}$$

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Hence $\mathcal{H}_{\mathcal{E}}$ gives an equivalence of categories and the inverse is given by

$$\widehat{\mathcal{H}}_{\mathcal{E}}(y) := \mathbf{R}\operatorname{Hom}_{p_X}(p_Y^*(y), \mathcal{E}), \quad y \in \mathbf{D}(Y)_{\mathrm{op}}. \tag{2.8}$$

$\mathcal{H}_{\mathcal{E}}$ induces an isomorphism $K(X) \to K(Y)$ and an isometry $H^{ev}(X, \mathbb{Z}) \to H^{ev}(Y, \mathbb{Z})$. We denote them by $\mathcal{H}_{\mathcal{E}}$. We denote the ith cohomology sheaf $H^i(\mathcal{H}_{\mathcal{E}}(x))$ by $\mathcal{H}^i_{\mathcal{E}}(x)$.

2.2. PRESERVATION OF STABILITY

In [Y4], we considered the preservation of stability under $\mathcal{H}_{\mathcal{E}}$. By using the twisted stability introduced by Matsuki and Wentworth [M-W], we shall generalize [Y4, Sect. 8.2]. In order to state our theorem (Theorem 2.3), we prepare some notation. We set $w_1 := v(\mathcal{E}_{|\{x\}\times Y}) = r_1 + \tilde{c}_1 + \tilde{a}_1\varrho_Y$, $x \in X$. We have an isomorphism $\mathrm{NS}(X) \otimes \mathbb{Q} \to v_1^{\perp} \cap \varrho_X^{\perp}$ by sending $D \in \mathrm{NS}(X) \otimes \mathbb{Q}$ to $D + \frac{1}{r_1}(D, c_1)\varrho_X \in v_1^{\perp} \cap \varrho_X^{\perp}$. Since $\mathcal{H}_{\mathcal{E}}$ is an isometry of Mukai lattice, we get an isomorphism $v_1^{\perp} \cap \varrho_X^{\perp} \to w_1^{\perp} \cap \varrho_Y^{\perp}$. Thus we have an isomorphism δ: $\mathrm{NS}(X) \otimes \mathbb{Q} \to \mathrm{NS}(Y) \otimes \mathbb{Q}$ given by

$$\delta\big(c_1(L)\big) = c_1\left(p_{Y*}\left[\operatorname{ch}\mathcal{E}\sqrt{\operatorname{td}_X}\,p_X^*\left(c_1(L) + \frac{1}{r_1}(c_1(L), c_1)\varrho_X\right)^{\vee}\right]\right). \tag{2.9}$$

For a $\mathbb{Q}$-line bundle $L \in \operatorname{Pic}(X) \otimes \mathbb{Q}$, we choose a $\mathbb{Q}$-line bundle $\hat{L}$ on Y such that $\delta\big(c_1(L)\big) = c_1(\hat{L})$. By a result of Li [Li] (or [BBH2]) and [Y4, Lem. 7.1], $\hat{H}$ is ample, if Y consists of μ-stable vector bundles. By [Y3, Lem. 2.1], Y consists of μ-stable vector bundles unless $\mathcal{E}$ is given by (2.2). In this case, a direct computation (or [Li]) shows that $\hat{H}$ is ample.

We consider the following two conditions.

(#1) $\hat{H}$ is general with respect to w_1.
(#2) $\mathcal{E}_{|\{x\}\times Y}$ is stable with respect to $\hat{H}$.

Remark 2.1. If X is Abelian or Y consists of nonlocally free sheaves, then the assumption (#1, 2) holds for all general H. For another example, see [BBH1].

PROBLEM [Y4]. Is $\mathcal{E}_{|\{x\}\times Y}$ always stable with respect to $\hat{H}$?

For a coherent sheaf E on X (resp. F on Y), we set $\deg(E) := \big(c_1(E), H\big)$ $\big($resp. $\deg(F) := \big(c_1(F), \hat{H}\big)\big)$. We consider the twisted degree $\deg_{G_1}(E)$ and $\deg_{G_2}(F)$, where $G_1 := \mathcal{E}_{|X\times\{y\}}$ and $G_2 := \mathcal{E}_{|\{x\}\times Y}$. We also define the twisted degree of a Mukai vector v by $\deg_{G_1}(v) := \deg_{G_1}(E)$, where $E \in K(X)$ satisfies $v(E) = v$.

LEMMA 2.2 [Y4, Lem. 8.3]. $\deg_{G_1}(v) = \deg_{G_2}\big(\mathcal{H}_{\mathcal{E}}(v)\big)$.

Every Mukai vector v can be uniquely written as

$$v = lv_1 - a\varrho_X + d\left[H + \frac{1}{r_1}(H, c_1)\varrho_X\right] + \left[D + \frac{1}{r_1}(D, c_1)\varrho_X\right], \tag{2.10}$$

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where $l, a, d \in \mathbb{Q}$, and $D \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap H^{\perp}$.

It is easy to see that $l = -\langle v, \varrho_X \rangle / r_1$, $a = \langle v, v_1 \rangle / r_1$ and $d = \deg_{G_1}(v)/\big(r_1(H^2)\big)$.

DEFINITION 2.1. For a Mukai vector v, we set $l(v) := -\langle v, \varrho_X \rangle / \operatorname{rk} v_1$, $a(v) := \langle v, v_1 \rangle / \operatorname{rk} v_1$.

Since $\mathcal{H}_{\mathcal{E}}(v_1) = \varrho_Y$ and $\widehat{\mathcal{H}}_{\mathcal{E}}(w_1) = \varrho_X$, we get

$$\begin{aligned} &\mathcal{H}_{\mathcal{E}}\left(l v_1 - a \varrho_X + \left[\mathrm{d}H + D + \frac{1}{r_1}(\mathrm{d}H + D, c_1)\varrho_X \right] \right) \\ &\quad = l \varrho_Y - a w_1 + \left[\mathrm{d}\hat{H} + \hat{D} + \frac{1}{r_1}(\mathrm{d}\hat{H} + \hat{D}, \tilde{c}_1)\varrho_Y \right] \end{aligned} \tag{2.11}$$

where $\hat{D} \in \mathrm{NS}(X) \otimes \mathbb{Q} \cap \hat{H}^{\perp}$. We can now state our theorem.

THEOREM 2.3. *We assume the condition* (#1, 2) *holds. Assume that* $\deg_{G_1}(v) = 0$ *and* $l(v), a(v) > 0$. *Let* ε *be an element of* $K(X) \otimes \mathbb{Q}$ *such that* $v(\varepsilon) \in v_1^{\perp} \cap \varrho_X^{\perp}$, $|\langle v(\varepsilon)^2 \rangle| \ll 1$ *and* $(H, c_1(\varepsilon)) = 0$. *Then we have an isomorphism*:

$$\mathcal{M}_H^{G_1+\varepsilon}(v)^{ss} \to \mathcal{M}_{\hat{H}}^{G_2+\hat{\varepsilon}}(-\mathcal{H}_{\mathcal{E}}(v))^{ss}, \tag{2.12}$$

where $\hat{\varepsilon} = \mathcal{H}_{\mathcal{E}}(\varepsilon)$. *In particular, if* $c_1(G_1) \in \mathbb{Q}H$, *then* $c_1(G_2) \in \mathbb{Q}\hat{H}$ *and we have an isomorphism* $\mathcal{M}_H(v)^{ss} \to \mathcal{M}_{\hat{H}}(-\mathcal{H}_{\mathcal{E}}(v))^{ss}$.

Remark 2.2. If $\langle v^2 \rangle > 0$, then we see that $a(v) > 0$.

The proof of Theorem 2.3 is almost the same as that in [Y4, Thm. 8.2]. Before proving Theorem 2.3, we prepare three Lemmas.

LEMMA 2.4. *Assume that* $\deg_{G_1}(v) < 0$, *or* $\deg_{G_1}(v) = 0$ *and* $a(v) > 0$. *Then* $\mathrm{Hom}(\mathcal{E}_{|X \times \{y\}}, E) = 0$ *for all* $y \in Y$ *and* $E \in \mathcal{M}_H^{G_1}(v)^{ss}$.

Proof. Obviously the claim holds, if $\deg_{G_1}(v) < 0$. Hence, we assume that $\deg_{G_1}(v) = 0$ and $a(v) > 0$. Since H is general with respect to v_1, $\mathcal{E}_{|X \times \{y\}}$ is G_1-twisted stable. Since E is G_1-twisted semi-stable, it is sufficient to show that $-a(\mathcal{E}_{|X \times \{y\}})/l(\mathcal{E}_{|X \times \{y\}}) > -a(v)/l(v)$. Since $v(\mathcal{E}_{|X \times \{y\}}) = v_1$, we get

$$\frac{-a(\mathcal{E}_{|X \times \{y\}})}{l(\mathcal{E}_{|X \times \{y\}})} - \frac{-a(v)}{l(v)} = \frac{a(v)}{l(v)} > 0. \tag{2.13}$$

□

LEMMA 2.5. *For a* μ*-semi-stable sheaf* E *with* $\deg_{G_1}(E) \geqslant 0$, *there is a finite subset* $S \subset Y$ *such that*

$$\mathrm{Hom}(E, \mathcal{E}_{|X \times \{y\}}) = 0 \tag{2.14}$$

for all $y \in Y \setminus S$.

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Proof. If $\deg_{G_1}(E) > 0$, then $\mathrm{Hom}(E, \mathcal{E}_{|X\times\{y\}}) = 0$ for all $y \in Y$. We assume that $\deg_{G_1}(E) = 0$. Considering the Jordan–Hölder filtration of E with respect to μ-stability, we may assume that E is μ-stable. If $\mathcal{E}_{|X\times\{y\}}$ is locally free, then by Lemma 2.1, $\mathcal{E}_{|X\times\{y\}}$ is μ-stable, and hence $E^{\vee\vee} \cong \mathcal{E}_{|X\times\{y\}}$. Therefore y is uniquely determined by E. Next we assume that $\mathcal{E}_{|X\times\{y\}}$ is not locally free. Under the notation of (2.2), if $E^{\vee\vee} \neq E_0$, then clearly $\mathrm{Hom}(E, E_0) = 0$. Hence $\mathrm{Hom}(E, \mathcal{E}_{|X\times\{y\}}) = 0$ for all $y \in Y$. If $E^{\vee\vee} = E_0$, then $\mathrm{Hom}(E, \mathcal{E}_{|X\times\{y\}}) = 0$ for $y \in Y \backslash \mathrm{Supp}(E^{\vee\vee}/E)$. □

LEMMA 2.6. *Let E be a coherent sheaf on X. Assume that*

(i) $\mathrm{Hom}(E, \mathcal{E}_{|X\times\{y\}}) = 0$ *except for a finite number of points* $y \in Y$,
(ii) $\mathrm{Ext}^2(E, \mathcal{E}_{|X\times\{y\}}) = 0$ *for all* $y \in Y$.

Then WIT_1 *holds for E, that is,* $\mathcal{H}^i_{\mathcal{E}}(E) = 0$ *for* $i \neq 1$, *and* $\mathcal{H}^1_{\mathcal{E}}(E)$ *is torsion free.*

Proof. Since $p_X^*(E)$ and $\mathcal{E}$ are flat over Y, there is a complex of locally free sheaves on Y

$$0 \to V^0 \xrightarrow{f} V^1 \xrightarrow{g} V^2 \to 0 \tag{2.15}$$

which is quasi-isomorphic to $\mathcal{H}_{\mathcal{E}}(E)$. By our assumptions, (a) f_y is injective except for a finite number of points $y \in Y$ and (b) g is surjective. Hence $\ker f = \mathrm{coker}\, g = 0$, which implies that WIT_1 holds. Since $\ker g$ is locally free, (a) implies that $\mathcal{H}^1_{\mathcal{E}}(E) = \mathrm{coker}(V^0 \to \ker g)$ is torsion free. □

Proof of Theorem 2.3. We shall first treat the case where $\varepsilon = 0$. By the symmetry of the condition, it is sufficient to show that WIT_1 holds for $E \in \mathcal{M}_H^{G_1}(v)^{ss}$ (i.e., $\mathcal{H}^i_{\mathcal{E}}(E) = 0$, $i \neq 1$) and $\mathcal{H}^1_{\mathcal{E}}(E)$ is G_2-twisted semi-stable with respect to $\hat{L}$. By Lemma 2.4, 2.5 and 2.6, WIT_1 holds for E and $\mathcal{H}^1_{\mathcal{E}}(E)$ is torsion free. We shall show that $\mathcal{H}^1_{\mathcal{E}}(E)$ is G_2-twisted semi-stable.

(I) $\mathcal{H}^1_{\mathcal{E}}(E)$ is μ-semi-stable: Assume that $\mathcal{H}^1_{\mathcal{E}}(E)$ is not μ-semi-stable. Let $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = \mathcal{H}^1_{\mathcal{E}}(E)$ be the Harder-Narasimhan filtration of $\mathcal{H}^1_{\mathcal{E}}(E)$ with respect to μ-semi-stability. We shall choose the integer k which satisfies $\deg_{G_2}(F_i/F_{i-1}) \geqslant 0, i \leqslant k$ and $\deg_{G_2}(F_i/F_{i-1}) < 0, i > k$. We claim that $\widehat{\mathcal{H}}^0_{\mathcal{E}}(F_k) = 0$ and $\widehat{\mathcal{H}}^2_{\mathcal{E}}(\mathcal{H}^1_{\mathcal{E}}(E)/F_k) = 0$. Indeed since F_i/F_{i-1}, $i \leqslant k$ are μ-semi-stable sheaves with $\deg_{G_2}(F_i/F_{i-1}) \geqslant 0$, Lemma 2.5 implies that $\widehat{\mathcal{H}}^0_{\mathcal{E}}(F_i/F_{i-1})$, $i \leqslant k$ are of dimension 0. Since $\widehat{\mathcal{H}}^0_{\mathcal{E}}(F_i/F_{i-1})$ are torsion free, $\widehat{\mathcal{H}}^0_{\mathcal{E}}(F_i/F_{i-1}) = 0, i \leqslant k$. Hence $\widehat{\mathcal{H}}^0_{\mathcal{E}}(F_k) = 0$. On the other hand, by Lemma 2.4, we also see that $\widehat{\mathcal{H}}^2_{\mathcal{E}}(F_i/F_{i-1}) = 0, i > k$. Hence we conclude that $\widehat{\mathcal{H}}^2_{\mathcal{E}}(\mathcal{H}^1_{\mathcal{E}}(E)/F_k) = 0$. So F_k and $\mathcal{H}^1_{\mathcal{E}}(E)/F_k$ satisfy WIT_1 and we get an exact sequence

$$0 \to \widehat{\mathcal{H}}^1_{\mathcal{E}}\big(\mathcal{H}^1_{\mathcal{E}}(E)/F_k\big) \to E \to \widehat{\mathcal{H}}^1_{\mathcal{E}}(F_k) \to 0. \tag{2.16}$$

By (2.11), $\deg_{G_1}\big(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_k)\big) = -\deg_{G_2}(F_k) < 0$. This means that E is not μ-semi-stable with respect to L. Therefore $\mathcal{H}^1_{\mathcal{E}}(E)$ is μ-semi-stable with respect to L.

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(II) $\mathcal{H}^1_{\mathcal{E}}(E)$ is G_2-twisted semi-stable: Assume that $\mathcal{H}^1_{\mathcal{E}}(E)$ is not G_2-twisted semi-stable. Then there is an exact sequence

$$0 \to F_1 \to \mathcal{H}^1_{\mathcal{E}}(E) \to F_2 \to 0 \tag{2.17}$$

such that (i) F_2 is G_2-twisted stable and (ii) $-a(F_2)/l(F_2) < -a\big(\mathcal{H}^1_{\mathcal{E}}(E)\big)/l\big(\mathcal{H}^1_{\mathcal{E}}(E)\big) = -l(v)/a(v)$, where

$$v(F_2) = l(F_2)w_1 - a(F_2)\varrho_Y + \left(D_2 + \frac{(D_2, \tilde{c}_1)}{r_1}\varrho_Y\right), \quad D_2 \in \mathrm{NS}(Y) \otimes \mathbb{Q} \cap \hat{H}^{\perp}.$$

Since $-a(F_2)/l(F_2) < -l(v)/a(v) < 0$, Lemmas 2.4, 2.5 and 2.6 imply that $\widehat{\mathcal{H}}^0_{\mathcal{E}}(F_2) = \widehat{\mathcal{H}}^2_{\mathcal{E}}(F_2) = 0$. We also obtain that $\widehat{\mathcal{H}}^0_{\mathcal{E}}(F_1) = 0$. Hence, we have an exact sequence

$$0 \to \widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2) \to E \to \widehat{\mathcal{H}}^1_{\mathcal{E}}(F_1) \to 0. \tag{2.18}$$

Since $\widehat{\mathcal{H}}^1_{\mathcal{E}}(\mathcal{H}^1_{\mathcal{E}}(E)) = E$, $\widehat{\mathcal{H}}^2_{\mathcal{E}}(F_1) = 0$. Thus WIT_1 also holds for F_1. By (ii), we see that

$$\begin{aligned}\frac{-a(v)}{l(v)} - \frac{-a(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2))}{l(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2))} &= \frac{-a(v)}{l(v)} + \frac{l(F_2)}{a(F_2)} \\ &= \frac{-a(v)a(F_2) + l(v)l(F_2)}{l(v)a(F_2)} < 0.\end{aligned} \tag{2.19}$$

This means that E is not G_1-twisted semi-stable. Therefore $\mathcal{H}^1_{\mathcal{E}}(E)$ is G_2-twisted semi-stable.

We next treat general cases. Since $|\langle v(\varepsilon)^2\rangle| \ll 1$, we have an inclusion $\mathcal{M}_H^{G_1+\varepsilon}(v)^{ss} \subset \mathcal{M}_H^{G_1}(v)^{ss}$ and the complement consists of E which fits in an exact sequence:

$$0 \to E_1 \to E \to E_2 \to 0 \tag{2.20}$$

where E_1 is a G_1-twisted semi-stable sheaf such that $v(E_1) = l_1 v_1 - a_1\varrho_X + \delta_1$, $\delta_1 \in v_1^{\perp} \cap \varrho_X^{\perp} \cap H^{\perp}$, $a_1/l_1 = a(v)/l(v)$ and $-\langle v(E_1), v_1 + \varepsilon\rangle/l_1 > -\langle v, v_1 + \varepsilon\rangle/l(v)$. Then we see that $-\langle \delta_1, \varepsilon\rangle/l_1 > -\langle \delta, \varepsilon\rangle/l(v)$, where $\delta := v - (l(v)v_1 - a(v)\varrho_X) \in v_1^{\perp} \cap \varrho_X^{\perp} \cap H^{\perp}$. Applying $\mathcal{H}^1_{\mathcal{E}}$ to the exact sequence (2.20), we get an exact sequence

$$0 \to \mathcal{H}^1_{\mathcal{E}}(E_2) \to \mathcal{H}^1_{\mathcal{E}}(E) \to \mathcal{H}^1_{\mathcal{E}}(E_1) \to 0. \tag{2.21}$$

Since $-\langle \mathcal{H}_{\mathcal{E}}(\delta_1), \mathcal{H}_{\mathcal{E}}(\varepsilon)\rangle/a_1 > -\langle \mathcal{H}_{\mathcal{E}}(\delta), \mathcal{H}_{\mathcal{E}}(\varepsilon)\rangle/a(v)$, we get that

$$-\langle v\big(\mathcal{H}^1_{\mathcal{E}}(E_1)\big), \mathcal{H}_{\mathcal{E}}(\varepsilon)\rangle/a_1 < -\langle v\big(\mathcal{H}^1_{\mathcal{E}}(E)\big), \mathcal{H}_{\mathcal{E}}(\varepsilon)\rangle/a(v). \tag{2.22}$$

Therefore $\mathcal{H}^1_{\mathcal{E}}(E)$ is not $\big(G_2 + \mathcal{H}_{\mathcal{E}}(\varepsilon)\big)$-twisted semi-stable. □

PROPOSITION 2.7. *Assume that $M_H(v)^{\mu-s}$ is an open dense subscheme of $\overline{M}_H(v)$. If $\deg_{G_1}(v) = 0$, then $\mathcal{H}_{\mathcal{E}}$ induces a birational map $\overline{M}_H(v) \cdots \to \overline{M}_H(\mathcal{H}_{\mathcal{E}}(v))$ which is described as Mumford–Thaddeus type flips*:

$$\begin{array}{ccccccccc} \overline{M}_H^{\alpha_1}(v) & & & & \overline{M}_H^{\alpha_2}(v) & & & & \overline{M}_{\hat{H}}^{\alpha_n}(\mathcal{H}_{\mathcal{E}}(v)) \\ & \searrow & & \swarrow & & \searrow & & \cdots \swarrow & \\ & & \overline{M}_H^{\alpha_{1,2}}(v) & & & & \overline{M}_H^{\alpha_{2,3}}(v) & & \end{array} \tag{2.23}$$

where $\alpha_i, \alpha_{i,i+1} \in \mathrm{NS}(X) \otimes \mathbb{Q}$ *and* $\alpha_1 = \alpha_n = 0$.

Proof. By Theorem 1.1, we have Mumford–Thaddeus type flips $\overline{M}_H(v) \cdots \to \overline{M}_H^{G_1}(v)$ and $\overline{M}_{\hat{H}}^{G_2}(\mathcal{H}_{\mathcal{E}}(v)) \cdots \to \overline{M}_{\hat{H}}(\mathcal{H}_{\mathcal{E}}(v))$. By the proof of Theorem 2.3, we see that $\mathcal{H}_{\mathcal{E}}$ preserves S-equivalence classes of twisted semi-stable sheaves, and hence we have an isomorphism $\overline{M}_H^{G_1}(v) \cong \overline{M}_{\hat{H}}^{G_2}(\mathcal{H}_{\mathcal{E}}(v))$. Therefore we get our claim. □

EXAMPLE 2.1. Let X be a K3 surface and H an ample divisor on X. Assume that $H^{\perp} = \mathbb{Z}D$ and $(D^2) = -2n$, $n > 2$. We set $v = 2 + (1-2n)\varrho_X$. Then there is a non-trivial extension

$$0 \to I_x(D) \to E \to \mathcal{O}_X(-D) \to 0, \tag{2.24}$$

where $x \in X$. We can easily show that E is a stable sheaf with $v(E) = v$. We consider the Fourier–Mukai transform defined by $\mathcal{E} = I_\Delta \otimes p_X^*\mathcal{O}_X(D)$. Since $\mathrm{Ext}^2(E, \mathcal{E}_x) = \mathrm{Hom}(\mathcal{E}_x, E)^\vee \neq 0$, E does not satisfy WIT_1 with respect to $\mathcal{H}_{\mathcal{E}}$. In this case, we get the following diagram

$$\begin{array}{ccccc} \overline{M}_H(v) & & \cdots \to & & \overline{M}_H^D(v) \cong \overline{M}_H(\mathcal{H}_{\mathcal{E}}(v)) \\ & \searrow & & \swarrow & \\ & & \overline{M}_H^{tD}(v) & & \end{array} \tag{2.25}$$

where $t = 1/4n$.

Remark 2.3. Let (X, H) be a polarized K3 surface which has a divisor ℓ such that

$$(H^2) = 2, \quad (\ell^2) = -12, \quad (H, \ell) = 0 \tag{2.26}$$

and $H^0\big(X, \mathcal{O}_X(\ell + 2H)\big) = 0$. Then $Y := M_H(2 + \ell - 3\varrho_X)$ is isomorphic to X and there is a universal family $\mathcal{E}$ on $X \times Y$. In [B-M], Bruzzo and Maciocia showed that the Fourier–Mukai transform $\mathcal{F}_{\mathcal{E}}$ gives an isomorphism

$$\mathcal{M}_H\big(1 + (1-n)\varrho_X\big)^{ss} \cong \mathcal{M}_{\hat{H}}\big((1+2n) - n\hat{\ell} + (1-3n)\varrho_Y\big)^{ss}. \tag{2.27}$$

Moreover every element E of $\mathcal{M}_{\hat{H}}\big((1+2n) - n\hat{\ell} + (1-3n)\varrho_Y\big)$ fits in a non-trivial extension

$$0 \to E' \to E \to \mathcal{O}_Y \to 0 \tag{2.28}$$

where $E' \in \mathcal{M}_{\hat{H}}\big(n(2 - \hat{\ell} - 3\varrho_Y)\big)^{ss}$. Then we can show that $E \mapsto E^\vee$ induces an isomorphism $\mathcal{M}_{\hat{H}}\big((1+2n) - n\hat{\ell} + (1-3n)\varrho_Y\big)^{ss} \to \mathcal{M}_{\hat{H}}^{\hat{\ell}/2}\big((1+2n) + n\hat{\ell} + (1-3n)\varrho_Y\big)^{ss}$. Thus we get an isomorphism

$$\mathcal{M}_H\big(1 + (1-n)\varrho_X\big)^{ss} \cong \mathcal{M}_{\hat{H}}^{\hat{\ell}/2}\big((1+2n) + n\hat{\ell} + (1-3n)\varrho_Y\big)^{ss}, \tag{2.29}$$

which is nothing but the isomorphism given by $\mathcal{H}_{\mathcal{E}^\vee}$.

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3. Irreducibility of $\overline{M}_H(v)$

3.1. A SPECIAL CASE OF THEOREM 2.3

We shall give an application of Theorem 2.3. Let X be an Abelian surface or a K3 surface such that $\mathrm{NS}(X) = \mathbb{Z}e \oplus \mathbb{Z}f$, $(e^2) = (f^2) = 0$ and $(e, f) = 1$.

COROLLARY 3.1. *Assume that* (1) *X is an Abelian surface, Y is the dual Abelian surface and $\mathcal{E}$ is the Poincaré line bundle on $X \times Y$, or* (2) *X is a* K3 *surface, $Y = X$ and $\mathcal{E}$ is the ideal sheaf of the diagonal $\Delta \subset X \times X$. Assume that $e + kf$ is an ample divisor. We set $D := e - kf$. Then $\mathcal{H}_{\mathcal{E}}$ induces an isomorphism of stacks*

$$\mathcal{M}_{e+kf}(r + cD - a\varrho_X)^{ss} \to \mathcal{M}_{\hat{e}+k\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss}, \tag{3.1}$$

where $r, a > 0$ and $c \geqslant 0$. Moreover, if k is a sufficiently large integer depending on r and $\langle (r + cD - a\varrho_X)^2 \rangle$, then

$$\mathcal{M}_{e+nf}(r + cD - a\varrho_X)^{ss} \cong \mathcal{M}_{\hat{e}+n\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss} \tag{3.2}$$

if $0 < n - k \ll 1$, where $n \in \mathbb{Q}$.

Remark 3.1. Semi-stability with respect to $e + nf$, $n \in \mathbb{Q}$ is defined by $H = m(e + nf) \in \mathrm{NS}(X)$, $m > 0$. We also remark that $\hat{e} + n\hat{f}$, $0 < n - k \ll 1$ is general with respect to $a - c\hat{D} - r\varrho_Y$.

Remark 3.2. The assumption $a > 0$ is very weak, because of the inequality $(D^2) < 0$.

Proof. By Theorem 2.3, we get the first claim. We next show the second claim. We note that $(D^2) = -2k \ll 0$. By Lemma 5.2 in Appendix, $e + kf$ is a general polarization with respect to $r + cD - a\varrho_X$. The same is true for $e + nf$, $n > k$. Hence, $\mathcal{M}_{e+kf}(r + cD - a\varrho_X)^{ss} \cong \mathcal{M}_{e+nf}(r + cD - a\varrho_X)^{ss}$. In order to prove our claim, it suffices to show that $\mathcal{M}_{\hat{e}+n\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss} = \mathcal{M}_{\hat{e}+k\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss}$. We first show that $\mathcal{M}_{\hat{e}+k\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss} \subset \mathcal{M}_{\hat{e}+n\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss}$. For $E \in \mathcal{M}_{e+kf}(r + cD - a\varrho_X)^{ss}$, assume that $\mathcal{H}^1_{\mathcal{E}}(E) \in \mathcal{M}_{\hat{e}+k\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss} \setminus \mathcal{M}_{\hat{e}+n\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss}$. Then there is an exact sequence

$$0 \to F_1 \to \mathcal{H}^1_{\mathcal{E}}(E) \to F_2 \to 0 \tag{3.3}$$

such that F_2 is semi-stable with respect to $\hat{e} + n\hat{f}$ and

$$\text{(i)} \quad \frac{\left(c_1(\mathcal{H}^1_{\mathcal{E}}(E)), \hat{e} + n\hat{f}\right)}{\operatorname{rk} \mathcal{H}^1_{\mathcal{E}}(E)} > \frac{(c_1(F_2), \hat{e} + n\hat{f})}{\operatorname{rk} F_2} \tag{3.4}$$

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or

$$\text{(ii)}\quad \frac{\left(c_1\left(\mathcal{H}^1_{\mathcal{E}}(E)\right), \hat{e}+n\hat{f}\right)}{\operatorname{rk}\mathcal{H}^1_{\mathcal{E}}(E)} = \frac{\left(c_1(F_2), \hat{e}+n\hat{f}\right)}{\operatorname{rk}F_2},\quad \frac{\chi\left(\mathcal{H}^1_{\mathcal{E}}(E)\right)}{\operatorname{rk}\mathcal{H}^1_{\mathcal{E}}(E)} > \frac{\chi(F_2)}{\operatorname{rk}F_2}. \tag{3.5}$$

Since $0 < n-k \ll 1$, (i) or (ii) implies that F_1 and F_2 are μ-semi-stable with respect to $\hat{e}+k\hat{f}$ with $(c_1(F_1), \hat{e}+k\hat{f}) = \left(c_1(F_2), \hat{e}+k\hat{f}\right) = 0$. Hence, $\operatorname{Hom}(F_1, \mathcal{E}_x) = \operatorname{Hom}(F_2, \mathcal{E}_x) = 0$ except for a finite number of points of X.

If (i) holds, then $\operatorname{Ext}^2(F_2, \mathcal{E}_{|X\times\{y\}}) = \operatorname{Hom}(\mathcal{E}_{|X\times\{y\}}, F_2)^\vee = 0$ for all $y \in Y$, because $\mathcal{E}_{|X\times\{y\}}$, $y \in Y$ is a stable sheaf with respect to $e+nf$ with $c_1(\mathcal{E}_{|X\times\{y\}}) = 0$ and $(c_1(F_2), \hat{e}+n\hat{f})/\operatorname{rk}F_2 < (-c\hat{D}, \hat{e}+n\hat{f})/\operatorname{rk}\mathcal{H}^1_{\mathcal{E}}(E) = -c(n-k)/\operatorname{rk}\mathcal{H}^1_{\mathcal{E}}(E) \leqslant 0$. Therefore F_1 and F_2 satisfies WIT_1 and we get an exact sequence

$$0 \to \widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2) \to E \to \widehat{\mathcal{H}}^1_{\mathcal{E}}(F_1) \to 0. \tag{3.6}$$

By Lemma 2.2,

$$(0 <)\frac{(c_1(E), e+nf)}{\operatorname{rk}\mathcal{H}^1_{\mathcal{E}}(E)} < \frac{\left(c_1\left(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2)\right), e+nf\right)}{\operatorname{rk}F_2}. \tag{3.7}$$

Since E is semi-stable with respect to $e+kf$ and $\left(c_1\left(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2)\right), e+kf\right) = 0$, $-\operatorname{rk}(F_2)/\operatorname{rk}\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2) \leqslant -\operatorname{rk}\mathcal{H}^1_{\mathcal{E}}(E)/\operatorname{rk}E$. Hence we see that

$$\frac{\left(c_1(E), e+nf\right)}{\operatorname{rk}E} < \frac{\left(c_1\left(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2)\right), e+nf\right)}{\operatorname{rk}\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2)}. \tag{3.8}$$

This implies that E is not semi-stable with respect to $e+nf$. Therefore (i) does not occur. If (ii) holds, then

$$\frac{c_1(\mathcal{H}^1_{\mathcal{E}}(E))}{\operatorname{rk}\mathcal{H}^1_{\mathcal{E}}(E)} = \frac{c_1(F_2)}{\operatorname{rk}F_2}.$$

Then by (3.5), $\mathcal{H}^1_{\mathcal{E}}(E)$ is not semi-stable with respect to $\hat{e}+k\hat{f}$, which is a contradiction. Thus $\mathcal{M}_{\hat{e}+k\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss} \subset \mathcal{M}_{\hat{e}+n\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss}$.

We next show that $\mathcal{M}_{\hat{e}+n\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss} \subset \mathcal{M}_{\hat{e}+k\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss}$. Assume that there is an element $F \in \mathcal{M}_{\hat{e}+n\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss} \setminus \mathcal{M}_{\hat{e}+k\hat{f}}(a - c\hat{D} - r\varrho_Y)^{ss}$. Then we see that there is an exact sequence

$$0 \to F_1 \to F \to F_2 \to 0 \tag{3.9}$$

such that (i) $\left(c_1(F), \hat{e}+n\hat{f}\right)/\operatorname{rk}F \leqslant \left(c_1(F_2), \hat{e}+n\hat{f}\right)/\operatorname{rk}F_2$, (ii) $\left(c_1(F), \hat{e}+k\hat{f}\right)/\operatorname{rk}F = \left(c_1(F_2), \hat{e}+k\hat{f}\right)/\operatorname{rk}F_2$, (iii) $\chi(F)/\operatorname{rk}F > \chi(F_2)/\operatorname{rk}F_2$ and (iv) F_2 is semi-stable with respect to $\hat{e}+k\hat{f}$. We set $\epsilon = 0, 1$ according as X is an Abelian surface or a K3 surface as in Section 2. We note that

(a) $\mathcal{E}_{|\{x\}\times Y}$, $x \in X$ is stable with respect to $\hat{e}+n\hat{f}$ with $\left(c_1(\mathcal{E}_{|\{x\}\times Y}), \chi(\mathcal{E}_{|\{x\}\times Y})\right) = (0, \epsilon)$,
(b) F is semi-stable with respect to $\hat{e}+n\hat{f}$ with $\left(c_1(F), \hat{e}+n\hat{f}\right) \leqslant 0$ and $\chi(F)/\operatorname{rk}F = \epsilon - r/a < \epsilon$.

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By (a) and (b), we get $\operatorname{Ext}^2(F, \mathcal{E}_{|\{x\}\times Y}) = \operatorname{Hom}(\mathcal{E}_{|\{x\}\times Y}, F)^\vee = 0$ for all $x \in X$. By (iii) and (iv), we see that $\operatorname{Ext}^2(F_2, \mathcal{E}_{|\{x\}\times Y}) = 0$ for all $x \in X$. Since F_1 and F_2 are μ-semi-stable sheaves of degree 0 with respect to $\hat{e} + k\hat{f}$, $\operatorname{Hom}(F_1, \mathcal{E}_{|\{x\}\times Y}) = \operatorname{Hom}(F_2, \mathcal{E}_{|\{x\}\times Y}) = 0$ except for a finite number of points $x \in X$. Lemma 2.6 implies that WIT_1 holds for F_1, F_2 and F with respect to $\widehat{\mathcal{H}}_{\mathcal{E}}$ and we have an exact sequence

$$0 \to \widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2) \to \widehat{\mathcal{H}}^1_{\mathcal{E}}(F) \to \widehat{\mathcal{H}}^1_{\mathcal{E}}(F_1) \to 0. \tag{3.10}$$

In the same way as in the proof of Theorem 2.3, we see that $E := \widehat{\mathcal{H}}^1_{\mathcal{E}}(F)$ is μ-semi-stable with respect to $e + kf$. Since $e + kf$ is general, we get $c_1\big(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2)\big)/\operatorname{rk}\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2) = c_1\big(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F)\big)/\operatorname{rk}\widehat{\mathcal{H}}^1_{\mathcal{E}}(F)$. On the other hand, (i) implies that

$$\frac{\big(c_1(E), e + nf\big)}{\operatorname{rk} F} \geqslant \frac{\big(c_1\big(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2)\big), e + nf\big)}{\operatorname{rk} F_2}. \tag{3.11}$$

By using (iii), we see that

$$\frac{\big(c_1(E), e + nf\big)}{\operatorname{rk} E} > \frac{\big(c_1\big(\widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2)\big), e + nf\big)}{\operatorname{rk} \widehat{\mathcal{H}}^1_{\mathcal{E}}(F_2)}, \tag{3.12}$$

which is a contradiction. Therefore our claim holds. □

Remark 3.3. If $e + kf$ is general with respect to $r + cD - a\varrho_X$ and $c > 0$, then $\operatorname{Hom}(E, \mathcal{E}_x) = 0$ for all $x \in X$ and $E \in \mathcal{M}_{e+kf}(r + cD - a\varrho_X)^{ss}$. Hence, $\mathcal{H}^1_{\mathcal{E}}(E)$ is locally free.

Remark 3.4. In general, $\hat{e} + k\hat{f}$ is not a general polarization with respect to $a - c\hat{D} - r\varrho_Y$. Indeed, let E be a nonlocally free μ-stable sheaf with $v(E) = r + cD - a\varrho_X$ on X. Assume that $E^{\vee\vee}/E = \mathbb{C}_x$, $x \in X$ and $a > 1$. Then we get an exact sequence

$$0 \to \mathcal{H}^1_{\mathcal{E}}(E^{\vee\vee}) \to \mathcal{H}^1_{\mathcal{E}}(E) \to \mathcal{H}^2_{\mathcal{E}}(\mathbb{C}_x) \to 0. \tag{3.13}$$

It is easy to see that $\mathcal{H}^2_{\mathcal{E}}(\mathbb{C}_x) \cong \mathcal{E}_x$. Hence, $\hat{e} + k\hat{f}$ is not general with respect to $a - c\hat{D} - r\varrho_Y$, if $c > 0$.

3.2. APPLICATION TO THE DEFORMATION TYPE OF $M_H(v)$.

Let X be an Abelian surface or a K3 surface.

DEFINITION 3.1. Let v be a Mukai vector of $\operatorname{rk} v > 0$. Then we can write it as $v = m(v)v_p$, where $m(v) \in \mathbb{Z}$ and v_p is a primitive Mukai vector of $\operatorname{rk} v_p > 0$.

In [Y4], we showed that $M_H(v)$ is deformation equivalent to a moduli space of rank 1 torsion free sheaves, if v is primitive. Here we assume that $\operatorname{rk} v > 0$ and H is general. We shall give a slightly different proof of this result, that is, we shall use O'Grady's arguments [O1, sect. 2]. One of the benefit of O'Grady's arguments is that we do not need to use algebraic space. This enables us to treat cases with non-primitive Mukai vector. For this purpose, we need the following proposition.

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PROPOSITION 3.2. *Let X_1 and X_2 be Abelian (or K3) surfaces, and let $v_1 := l(r+\xi_1)+a_1\varrho_{X_1}\in H^{ev}(X_1,\mathbb{Z})$ and $v_2 := l(r+\xi_2)+a_2\varrho_{X_2}\in H^{ev}(X_2,\mathbb{Z})$ be primitive Mukai vectors such that (1) $l, r>0$, (2) $\gcd(r,\xi_1)=\gcd(r,\xi_2)=1$, (3) $\langle v_1^2\rangle = \langle v_2^2\rangle = 2s$, and (4) $a_1\equiv a_2 \bmod l$. Then $M_{H_1}(v_1)$ and $M_{H_2}(v_2)$ are deformation equivalent, where H_i, $i=1,2$ are general ample divisors on X_i with respect to v_i.*

The proof is due to O'Grady. For the convenience of the reader, we give an outline of the proof.

Proof. We first assume that $\rho(X_i)\geqslant 2$, $i=1,2$. We note that the equivalence class $a_i \bmod l$ does not change under the operation $v_i\mapsto v_i\operatorname{ch}L_i$, $L_i\in\operatorname{NS}(X_i)$. Replacing v_i by $v_i\operatorname{ch}L_i$, we may assume that (i) ξ_i is primitive, (ii) $H_i' := \xi_i$ is ample (iii) $(\xi_i^2)=2k_i > (lr)^2\big(\langle v_i^2\rangle + 2(lr)^2\big)/4$ and (iv) ξ_i belongs to the same chamber as H_i belongs. Then $M_{H_i}(v_i)=M_{H_i'}(v_i)$. Replacing H_i by H_i', we assume that $v_i = l(r+H_i)+a_i\varrho_{X_i}$. Let X'' be an Abelian (or a K3) surface such that $\operatorname{NS}(X'')=\mathbb{Z}e\oplus\mathbb{Z}f$ with $(e^2)=(f^2)=0$ and $(e,f)=1$. We may assume that $H_i'' := e+k_if$ is an ample divisor on X''. We note that $(H_i''^2)=(\xi_i^2)=2k_i$. Since $H_i''^{\perp}=\mathbb{Z}(e-k_if)$ and $-\big((e-k_if)^2\big)=2k_i>(lr)^2\big(\langle v^2\rangle + 2(lr)^2\big)/4$, Lemma 5.2 implies that H_i'' is a general polarization with respect to $v_i'' := l(r+H_i'')+a_i\varrho_{X''}$. Then we see that $M_{H_i}(v_i)$ is deformation equivalent to $M_{H_i''}(v_i'')$ (cf. the proof of Proposition 3.6). We also note that there is no wall between H_i'' and f. Hence $M_{H_i''}(v_i'')=M_{e+kf}(v_i'')$, $k=\max\{k_1,k_2\}$. Since $2k_1l^2-2lra_1 = 2k_2l^2-2lra_2=2s$, we have $k_2-k_1 = r(a_2-a_1)/l\in\mathbb{Z}$. By our assumption $a_1\equiv a_2 \bmod l$, there is a line bundle L with $c_1(L)=(a_2-a_1)/l$. Then $v_1''\operatorname{ch}(L)=v_2''$, which implies that $M_{e+kf}(v_1'')=M_{e+kf}(v_2'')$. Therefore $M_{H_1}(v_1)$ is deformation equivalent to $M_{H_2}(v_2)$.

In the moduli space $\mathcal{M}_d$ of polarized K3 surfaces (or Abelian surfaces) (X,H) with $(H^2)=2d$, the locus $\{(X,H)\in\mathcal{M}_d | \rho(X)\geqslant 2\}$ consists of countably many hypersurfaces. By using Grothendieck's boundedness theorem, we see that

$$\mathcal{M}_d^n := \{(X,H)\in\mathcal{M}_d \mid H \text{ is not general with respect to } v_i\} \tag{3.14}$$

is an algebraic subset of $\mathcal{M}$. Hence if $\rho(X_i)=1$, we can deform (X_i,H_i) to (X_i',H_i') such that $\rho(X_i')\geqslant 2$ and H_i' is general with respect to v_i. Therefore we can reduce our problem to the first case. □

Now we consider deformation type of $M_H(v)$ for a primitive Mukai vector v. In particular, we shall give another proof of [Y4]. For a Mukai vector $v := l(r+c_1)+a\varrho_{X_1}\in H^*(X_1,\mathbb{Z})$ such that $r>0$, $\gcd(r,c_1)=1$ and $\gcd(l,a)=1$, we set $b=-a+l\lambda$, $k=-(c_1^2)/2+r\lambda$, $\lambda\gg 0$ so that $e+kf$ is ample. We consider X in the above notation. By Lemma 5.2, $e+nf$, $0<n-k\ll 1$ is a general polarization with respect to $l(r+(e-kf))-b\varrho_X$ (cf. Remark 3.1) Since $-b\equiv a \bmod l$, Proposition 3.2 implies that $M_H(l(r+c_1)+a\varrho_{X_1})$ is deformation equivalent to $M_{e+nf}(l(r+(e-kf))-b\varrho_X)$, where H is general with respect to v. By Corollary 3.1, we have an isomorphism

$$M_{e+nf}(l(r+(e-kf))-b\varrho_X)\to M_{\hat{e}+n\hat{f}}(b-l(\hat{e}-k\hat{f})-lr\varrho_Y). \tag{3.15}$$

Here we note that $\hat{e}+n\hat{f}$, $0<n-k\ll 1$ is also general with respect to $b-l(\hat{e}-k\hat{f})-l r\varrho_Y$. Since $(b,l)=1$, Proposition 3.2 implies that $M_{\hat{e}+n\hat{f}}(b-l(\hat{e}-k\hat{f})-lr\varrho_Y)$ is deformation equivalent to $M_{e+n'f}(b+(e-k'f)-b'\varrho_X)$, where $b'=lr+\lambda'$, $k'=l^2k+b\lambda'$, $\lambda'\gg 0$ and $0<n'-k'\ll 1$. Applying Corollary 3.1 again, we get an isomorphism

$$M_{e+n'f}(b+(e-k'f)-b'\varrho_X)\to M_{\hat{e}+n'\hat{f}}(b'-(\hat{e}-k'\hat{f})-b\varrho_Y). \tag{3.16}$$

If λ' is sufficiently large, then $M_{\hat{e}+n'\hat{f}}(b'-(\hat{e}-k'\hat{f})-b\varrho_Y)$ is deformation equivalent to $M_{\hat{e}+\hat{n}''f}(b'-(\hat{e}-k''\hat{f})-\varrho_Y)$ and $e+k''f$ is ample, where $k''=lr(1-b)+l^2k+\lambda'$ and $0<n''-k''\ll 1$. Since $k''\gg 0$, Corollary 3.1 implies that $M_{e+n''f}(1+(e-k''f)-b''\varrho_X)$ is isomorphic to $M_{\hat{e}+\hat{n}''f}(b'-(\hat{e}-k''\hat{f})-\varrho_Y)$. Therefore $M_H(v)$ is deformation equivalent to the moduli space of rank 1 torsion free sheaves.

We shall next treat cases with nonprimitive Mukai vector.

LEMMA 3.3. *Let v be a Mukai vector of $\operatorname{rk} v>0$ and $\langle v^2\rangle>0$. Let H be a general ample divisor with rerspect to v. We set*

$$\mathcal{M}_H(v)^{pss}:=\{E\in\mathcal{M}_H(v)^{ss}\mid E \text{ is properly semi-stable}\}. \tag{3.17}$$

Then $\dim\mathcal{M}_H(v)^{pss}\leqslant\langle v^2\rangle$. Moreover inequality is strict, unless $m(v)=2$ and $\langle v^2\rangle=8$.

For the proof, see [Y3, Lem. 1.7].

PROPOSITION 3.4. *Under the same assumptions, $\mathcal{M}_H(v)^{ss}$ is a locally complete intersection stack which contains $\mathcal{M}_H(v)^{s}$ as an open dense substack and the singular locus is at least of codimension 2. In particular $\mathcal{M}_H(v)^{ss}$ is normal.*

Proof. In the notation of Remark 1.1, we shall prove that Q^{ss} is a locally complete intersection scheme. The following argument is due to Li [Li]. We take a quotient $\mathcal{O}_X(-mH)^{\oplus N}\to E\in Q^{ss}$ and set $K:=\ker(\mathcal{O}_X(-mH)^{\oplus N}\to E)$. Then the Zariski tangent space of Q^{ss} at this quotient is $\operatorname{Hom}(K,E)$ and the obstruction class for an infinitesimal lifting belongs to the kernel of the surjective homomorphism $\operatorname{Ext}^1(K,E)\cong\operatorname{Ext}^2(E,E)\xrightarrow{tr}H^2(X,\mathcal{O}_X)$ ([Mu3]). In particular Q^s is smooth of dimension $\langle v^2\rangle+1+N^2$, where Q^s is the open subscheme of Q^{ss} parametrizing stable quotient sheaves. By Lemma 3.3, the dimension of all irreducible components of Q^{ss} are at most $\langle v^2\rangle+1+N^2$. On the other hand, if we set $s:=\dim\operatorname{Hom}(K,E)$ and $t:=\dim\operatorname{Ext}^1(K,E)-1$, then locally Q^{ss} is defined by t equations $f_1,\dots,f_t$ in a smooth scheme of dimension s. Since $s-t=\chi(K,E)+1=N^2-\chi(E,E)+1\geqslant\dim Q^{ss}$, $f_1,\dots,f_t$ is a regular sequence, which implies that Q^{ss} is a locally complete intersection scheme.

If $m(v)\neq 2$ or $\langle v^2\rangle>8$, then $\dim\mathcal{M}_H(v)^{pss}\leqslant\langle v^2\rangle-1$. Therefore the singular locus is at least of codimension 2. If $m(v)=2$ and $\langle v^2\rangle=8$, then a general member of $\mathcal{M}_H(v)^{pss}$ fits in a non-trivial extension.

$$0\to E_1\to E\to E_2\to 0 \tag{3.18}$$

where $E_1, E_2 \in \mathcal{M}_H(v/2)^{ss}$ and $E_1 \neq E_2$. Then E is simple, which implies that $\mathcal{M}_H(v)^{ss}$ is smooth at E. Therefore the singular locus is at least of codimension 2. For the last claim, we use Serre's criterion. □

DEFINITION 3.2. Let Y_1, Y_2 be normal schemes. Then $Y_1 \sim Y_2$, if there is a proper and flat morphism $\mathcal{Y} \to T$ over a smooth connected curve T such that every fiber is normal and $Y_i = \mathcal{Y}_{t_i}$ for some $t_1, t_2 \in T$. Deformation equivalence is an equivalence relation generated by $\sim$.

By the following lemma, the number of irreducible components is an invariant of this equivalence relation.

LEMMA 3.5. *Let T be a smooth curve and $Y \to T$ a flat and proper morphism. Assume that every fiber is normal. Then the number of irreducible components of Y_t, $t \in T$ is constant.*

Proof. Since Y_t is normal, every connected component is an integral scheme. Hence the number of irreducible components of Y_t is $h^0(Y_t, \mathcal{O}_{Y_t})$. By the upper-semicontinuity of $h^0(Y_t, \mathcal{O}_{Y_t})$, the number of irreducible components of Y_t is upper semi-continuous. On the other hand, by Zariski's connectivity theorem, the number of connected components of Y_t is lower semi-continuous. Therefore we get our lemma. □

By the same proof, we can show the following.

PROPOSITION 3.6. *Under the same assumption as in Lemma 3.3, $\overline{M}_H(v)$ is deformation equivalent to $\overline{M}_H(m(v)(1 - n\varrho_X))$, where $n = \langle v_p^2 \rangle/2$. In particular the number of irreducible components of $\overline{M}_H(v)$ is determined by $m(v)$.*

Proof. Let T be a smooth curve over $\mathbb{C}$ and $\varphi\colon (\mathcal{X}, \mathcal{L}) \to T$ be a family of polarized abelian or K3 surfaces. For a family of Mukai vectors $v \in R^*\varphi_*\mathbb{Z} = \bigcup_{t \in T} H^*(\mathcal{X}_t, \mathbb{Z})$, let $\psi : \overline{\mathfrak{M}}_{\mathcal{L}}(v) \to T$ be the relative moduli space of semi-stable sheaves on $\mathcal{X}_t$, $t \in T$ of Mukai vector v_t and $\mathfrak{M}_{\mathcal{L}}(v)$ the open subscheme consisting of stable sheaves. Since T is defined over a field of characteristic 0, $\overline{\mathfrak{M}}_{\mathcal{L}}(v)_t = \overline{M}_{\mathcal{L}_t}(v_t)$ for $t \in T$, where $\overline{M}_{\mathcal{L}_t}(v_t)$ is the moduli space of semi-stable sheaves on $\mathcal{X}_t$ (cf. [MFK, Thm. 1.1]). Since $\psi_{|\mathfrak{M}_{\mathcal{L}}(v)}\colon \mathfrak{M}_{\mathcal{L}}(v) \to T$ is smooth [Mu3], it is flat. Assume that $\mathcal{L}_t$ is general with respect to v_t for all $t \in T$. By Proposition 3.4, $\mathfrak{M}_{\mathcal{L}}(v)$ is a dense subscheme of $\overline{\mathfrak{M}}_{\mathcal{L}}(v)$. Since T is a smooth curve, ψ is also flat. Therefore $\psi\colon \overline{\mathfrak{M}}_{\mathcal{L}}(v) \to T$ is a proper and flat morphism. By Proposition 3.4, all $\overline{M}_{\mathcal{L}_t}(v_t)$, $t \in T$ are deformation equivalent. Then our claim follows from the same argument as in $m(v) = 1$ case. □

3.3. IRREDUCIBILITY OF $\overline{M}_H(v)$

We shall show that $\overline{M}_H(v)$ is irreducible. We may assume that X has an elliptic fibration $\pi\colon X \to C$. We also assume that there is a section σ of π and $\mathrm{NS}(X) =$

$\mathbb{Z}\sigma \oplus \mathbb{Z}f$, where f is a fiber of π. We note that $(\sigma^2) = 0$ or -2, according as X is an Abelian surface or a K3 surface. By [Y4, Thm. 3.15], we have an isomorphism $\overline{M}_{\sigma+kf}(r(1-n\varrho_X)) \cong \overline{M}_{\sigma+kf}(w)$, where $w = r((\sigma+(n+\epsilon)f)+\varrho_X)$. Hence, it is sufficient to show that $\overline{M}_{\sigma+kf}(w)$ is irreducible. From now on, we assume that $r \geqslant 2$. By Proposition 3.4, we shall show that $M_{\sigma+kf}(w)$ is irreducible.

DEFINITION 3.3. For a purely one-dimensional sheaf L on X, $\mathrm{Div}(L)$ is the divisor on X which is defined by the fitting ideal of L.

We set $\xi = r(\sigma+(n+\epsilon)f)$. Let Hilb_X^{ξ} be the Hilbert scheme of curves C on X such that $c_1(\mathcal{O}_X(C)) = \xi$. There is a natural map $j : \overline{M}_{\sigma+kf}(w) \to \mathrm{Hilb}_X^{\xi}$ sending $L \in \overline{M}_{\sigma+kf}(w)$ to $\mathrm{Div}(L)$. We want to estimate the dimension of locally closed subsets of $M_{\sigma+kf}(w)$:

$$\begin{aligned} N_1 &:= \{L \in M_{\sigma+kf}(w) | \mathrm{Div}(L) \text{ is not irreducible}\}, \\ N_2 &:= \{L \in M_{\sigma+kf}(w) | \mathrm{Div}(L) \text{ is not reduced}\}. \end{aligned} \tag{3.19}$$

Estimate of $\dim N_1$. We prepare some lemmas.

LEMMA 3.7. *Let* C_i, $i = 1, 2$ *be irreducible curves of genus* $g(C_i) \geqslant 2$. *Then* $(C_i, C_j) \geqslant 2$.

Proof. If $C_1 - C_2$ or $C_2 - C_1$ is effective, then $(C_1, C_2) \geqslant (C_2^2) \geqslant 2$ or $(C_1, C_2) \geqslant (C_1^2) \geqslant 2$. If $C_1 - C_2$ and $C_2 - C_1$ are not effective, then $0 \geqslant \chi(\mathcal{O}_X(C_1 - C_2)) \geqslant (C_1 - C_2)^2/2$. Hence we see that $(C_1, C_2) \geqslant 2$. □

DEFINITION 3.4. For a Mukai vector $v \in H^{ev}(X, \mathbb{Z})$, $\mathcal{M}(v)$ is the stack of coherent sheaves E of $v(E) = v$.

LEMMA 3.8. *Let E be a purely one-dimensional sheaf such that* $\mathrm{Supp}\, E$ *consists of genus* $g \geqslant 2$ *curves. Then* $\dim \mathcal{M}(v(E)) = \langle v(E)^2 \rangle + 1$ *at E, if H is general.*

Proof. We set $v := v(E)$. Let $\mathcal{M}$ be an irreducible component of $\mathcal{M}(v)$ containing E and let E' be a general point of $\mathcal{M}$. We consider the Harder–Narasimhan filtration of E':

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E'. \tag{3.20}$$

We set $v_i = v(F_i/F_{i-1})$. By our assumption, we may assume that $\mathrm{Supp}\, F_i/F_{i-1}$ consist of curves of genus greater than 1. Hence, $\langle v_i^2 \rangle > 0$. Moreover, by Lemma 3.7, $\langle v_i, v_j \rangle \geqslant 2$. Let $\mathcal{F}^0(v_1, v_2, \ldots, v_s)$ be the stack of filtrations (3.20) such that $\mathrm{Hom}(F_i/F_{i-1}, F_j/F_{j-1}) = 0$ for $i < j$. By [Y3, Lem. 5.2],

$$\dim \mathcal{F}^0(v_1, v_2, \ldots, v_s) = \sum_{i=1}^{s} \dim \mathcal{M}_{\sigma+kf}(v_i)^{ss} + \sum_{i<j} \langle v_i, v_j \rangle. \tag{3.21}$$

Since $\langle v_i^2\rangle > 0$, we get $\dim \mathcal{M}_{\sigma+kf}(v_i)^{ss} = \langle v_i^2\rangle + 1$. Hence we see that

$$\begin{aligned}\langle v^2\rangle + 1 - \dim \mathcal{F}^0(v_1, v_2, \ldots, v_s) &= \langle v^2\rangle + 1 - \left(\sum_{i=1}^{s}(\langle v_i^2\rangle + 1) + \sum_{i<j}\langle v_i, v_j\rangle\right)\\ &= \sum_{i<j}\langle v_i, v_j\rangle - (s-1) > 0. \qquad (3.22)\end{aligned}$$

Since $\dim \mathcal{M}(v) \geqslant \langle v^2\rangle + 1$, we get our claim. □

LEMMA 3.9. *Assume that X is a K3 surface. Let E be a purely one-dimensional sheaf of* $\mathrm{Div}(E) = r\sigma$. *Then* $\dim \mathcal{M}(v(E)) = \langle v(E)^2\rangle + r^2 = -r^2$ *at E.*

Proof. We set $v := v(E)$. Let $\mathcal{M}$ be an irreducible component of $\mathcal{M}(v)$ containing E and let E' be a general point of $\mathcal{M}$. We consider the Harder–Narasimhan filtration of E':

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E'. \qquad (3.23)$$

We set $v_i = v(F_i/F_{i-1})$. Then $v_i = r_i\sigma + a_i\varrho_X$. It is easy to see that a_i is divisible by r_i and $\mathcal{M}_{\sigma+kf}(v_i)^{ss} = \{\mathcal{O}_\sigma(a_i/r_{i-1})^{\oplus r_i}\}$. Then $\dim \mathcal{M}_{\sigma+kf}(v_i)^{ss} = -r_i^2$. As in Lemma 3.8, let $\mathcal{F}^0(v_1, v_2, \ldots, v_s)$ be the stack of filtrations (3.23) such that $\mathrm{Hom}(F_i/F_{i-1}, F_j/F_{j-1}) = 0$ for $i < j$. By [Y3, Lem. 5.2],

$$\begin{aligned}\dim \mathcal{F}^0(v_1, v_2, \ldots, v_s) &= \sum_{i=1}^{s}\dim \mathcal{M}_{\sigma+kf}(v_i)^{ss} + \sum_{i<j}\langle v_i, v_j\rangle\\ &= -\sum_{i=1}^{s} r_i^2 - \sum_{i<j} 2r_ir_j = -r^2.\end{aligned} \qquad (3.24)$$

Therefore we get our claim. □

LEMMA 3.10. *Let E be a purely one-dimensional sheaf on X such that $v(E) = rf + a\varrho_X$, or $v(E) = r\sigma + a\varrho_X$. Assume that $\langle v(E)^2\rangle = 0$. Then* $\dim \mathcal{M}(v(E)) = r$ *at E.*

Proof. We set $v := v(E)$. Let $\mathcal{M}$ be an irreducible component of $\mathcal{M}(v)$ containing E and let E' be a general point of $\mathcal{M}$. We consider the Harder–Narasimhan filtration of E':

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E'. \qquad (3.25)$$

We set $v_i = v(F_i/F_{i-1})$. Then we see that $\langle v_i^2\rangle = 0$. As in the proof of Lemma 1.8 in [Y3], we see that $\dim \mathcal{M}_{\sigma+kf}(v_i)^{ss} = r_i$. By using [Y3, Lem. 5.2] again, we see that

$$\dim \mathcal{F}^0(v_1, v_2, \ldots, v_s) = \sum_{i=1}^{s}\dim \mathcal{M}_{\sigma+kf}(v_i)^{ss} + \sum_{i<j}\langle v_i, v_j\rangle = \sum_{i=1}^{s} r_i = r. \qquad (3.26)$$

Therefore we get our claim. □

For N_1, we get the following.

PROPOSITION 3.11. $\dim N_1 < \dim M_{\sigma+kf}(w)$.

Proof. Assume that Supp(L) is not irreducible. Then there is a filtration

$$0 \subset F_1 \subset F_2 \subset F_3 = L \tag{3.27}$$

such that (i) $\mathrm{Div}(F_1) = r_1\sigma$ (ii) F_2/F_1 is a pure dimension 1 sheaf of $\mathrm{Div}(F_2/F_1) = r_2 f$ and (iii) F_3/F_2 is a pure dimension 1 sheaf of $\mathrm{Div}(F_3/F_2) = C_3$, where C_3 consists of curves of genus greater than 1. We set $v_i := v(F_i/F_{i-1})$, $i = 1, 2, 3$. We note that the set of $\chi(F_3/F_2)$ and $\chi(F_2/F_1)$ are bounded. Indeed, if $\mathrm{Div}(L) = C \cup D$ with $\#(C \cap D) < \infty$, then $\chi(L_{|C}(-D)/T)/(C, \sigma + kf) \leqslant \chi(L)/(C + D, \sigma + kf)$, where T is the 0-dimensional subsheaf of $L_{|C}(-D)$. Hence $\chi(L_{|C}/T(D))$ is bounded above. Since $\chi(L_{|C}/T(D))$ is bounded below, $\chi(L_{|C}/T(D))$ is bounded. Applying this fact to $\chi(F_3/F_2)$ and $\chi(F_3/F_1)$, we get our claim. By the proof of Lemma 3.8, we may assume that $F_2 \neq 0$. We first note that $\mathrm{Ext}^2(F_i/F_{i-1}, F_j/F_{j-1}) = \mathrm{Hom}(F_j/F_{j-1}, F_i/F_{i-1})^\vee = 0$ for $i \neq j$.

(I) We first treat the case where X is a K3 surface. Assume that $F_2 \neq F_3$. By Lemmas 3.8, 3.9, 3.10, we see that

$$\begin{aligned} \operatorname{codim} \mathcal{F}^0(v_1, v_2, v_3) &= \dim \mathcal{M}_{\sigma+kf}(w)^{ss} - \left(\sum_i \dim \mathcal{M}_{\sigma+kf}(v_i)^{ss} + \sum_{i<j} \langle v_i, v_j\rangle\right) \\ &= \langle w^2\rangle + 1 - \left((\langle v_1^2\rangle + r_1^2) + (\langle v_2^2\rangle + r_2) + (\langle v_3^2\rangle + 1) + \sum_{i<j}\langle v_i, v_j\rangle\right) \\ &= \sum_{i<j}\langle v_i, v_j\rangle - r_1^2 - r_2 \\ &= -r_1^2 - r_2 + (r_1\sigma, r_2 f + C_3) + (r_2 f, C_3). \end{aligned} \tag{3.28}$$

By our assumption, $(c_1(w), \sigma) \geqslant 0$. Hence $(r_1\sigma, r_2 f + C_3) \geqslant 2r_1^2$. By our assumption, $(f, C_3) > 0$. Therefore $-r_1^2 - r_2 + (r_1\sigma, r_2 f + C_3) + (r_2 f, C_3) \geqslant r_1^2 > 0$. We next assume that $F_2 = F_3$. Then we see that

$$\operatorname{codim} \mathcal{F}^0(v_1, v_2) = -r_1^2 - r_2 + 1 + r_1 r_2. \tag{3.29}$$

Since $(c_1(w), \sigma) \geqslant 0$, $r_2 \geqslant 2r_1$. Then $-r_1^2 - r_2 + 1 + r_1 r_2 \geqslant r_1(r_1 - 2) + 1 \geqslant 1$, because $c_1(w)$ is not primitive. Therefore we get our claim.

(II) We next treat the case where X is an Abelian surface. Assume that $F_2 \neq F_3$. Then

$$\operatorname{codim} \mathcal{F}^0(v_1, v_2, v_3) = -r_1 - r_2 + (r_1\sigma, r_2 f + C_3) + (r_2 f, C_3). \tag{3.30}$$

Since C_3 consists of curves of genus greater that 1, $(\sigma, C_3) > 1$ and $(f, C_3) > 1$. Then $(-r_1 - r_2 + (r_1\sigma, r_2 f + C_3) + (r_2 f, C_3)) > 0$. If $F_2 = F_3$, then

$$\operatorname{codim} \mathcal{F}^0(v_1, v_2) = -r_1 - r_2 + 1 + r_1 r_2. \tag{3.31}$$

Since $c_1(w)$ is not primitive, $(r_1 - 1)(r_2 - 1) > 0$. Therefore we get our claim. □

Estimate of $\dim N_2$. For an integer $\lambda \geqslant 2$, let $\mathrm{Hilb}_X^{\xi}(\lambda)$ be the locally closed subset of Hilb_X^{ξ} consisting of λC, where C is an integral curve. By Proposition 3.11, it is sufficient to estimate the dimension of $\mathrm{Hilb}_X^{\xi}(\lambda)$. Let C be an integral curve. For $D = \lambda C$ and $w = D + r\varrho_X$, we set

$$M_{\sigma+kf}(w, D) := \{L \in M_{\sigma+kf}(w) \mid \mathrm{Div}(L) = D\}. \tag{3.32}$$

We fix a point $x \in C$. We also set

$$P_{\sigma+kf}(w, w+\varrho_X, D) := \left\{ L \subset L' \left| \begin{array}{l} L' \in M_{\sigma+kf}(w+\varrho_X), \mathrm{Div}(L') = D \\ L \in M_{\sigma+kf}(w), L'/L \cong \mathbb{C}_x \end{array} \right. \right\}. \tag{3.33}$$

Let π_w and $\pi_{w+\varrho_X}$ be natural projections sending $L \subset L'$ to L and L', respectively:

$$\begin{array}{ccc} & P_{\sigma+kf}(w, w+\varrho_X, D) & \\ \pi_w \swarrow & & \searrow \pi_{w+\varrho_X} \\ M_{\sigma+kf}(w, D) & & M_{\sigma+kf}(w+\varrho_X, D) \end{array}$$

For $L \in M_{\sigma+kf}(w, D)$, $-\dim \mathrm{Ext}^1(\mathbb{C}_x, L) + \dim \mathrm{Ext}^2(\mathbb{C}_x, L) = \chi(\mathbb{C}_x, L) = 0$. Combining the Serre duality, we see that $\dim \mathrm{Ext}^1(\mathbb{C}_x, L) = \dim \mathrm{Ext}^2(\mathbb{C}_x, L) = \dim \mathrm{Hom}(L, \mathbb{C}_x)$. By the following lemma, $\dim \pi_w^{-1}(L) \leqslant \lambda - 1$ and $\dim \pi_{w+\varrho_X}^{-1}(L') \leqslant \lambda - 1$.

LEMMA 3.12. *Let x be a smooth point of C. Let L be a purely one-dimensional sheaf such that* $\mathrm{Div}(L) = \lambda C$. *Then* $\dim L \otimes \mathbb{C}_x \leqslant \lambda$.

Proof. Let C' be a germ of a curve intersecting C at x transversely. Let $\mathcal{O}_{X,x}$ be the stalk of $\mathcal{O}_X$ at x. We take a free resolution of $L \otimes \mathcal{O}_{X,x}$:

$$0 \to \mathcal{O}_{X,x}^{\oplus n} \xrightarrow{A} \mathcal{O}_{X,x}^{\oplus n} \to L \otimes \mathcal{O}_{X,x} \to 0. \tag{3.35}$$

Then the local equation of $\mathrm{Div}(L)$ at x is given by $\det(A)$. By restricting the sequence to C', we get a free resolution of $L \otimes \mathcal{O}_{C',x}$. Then $\dim(L \otimes \mathcal{O}_{C',x})$ is given by the local intersection number $(\mathrm{Div}(D), C')_x = \lambda$. Therefore we get our claim. □

LEMMA 3.13. $\dim M_{\sigma+kf}(w + \varrho_X, D) = (D^2)/2 + 1$.

Proof. By [Y4, Thm. 3.15], $M_{\sigma+kf}(w + \varrho_X)$ is isomorphic to $M_{\sigma+kf}(r + f - rn\varrho_X)$. Since $r + f - rn\varrho_X$ is primitive, [Y4, Thms. 0.1 and 8.1] implies that it is irreducible. For a smooth curve $C \in \mathrm{Hilb}_X^{\xi}$, the fiber of $M_{\sigma+kf}(w + \varrho_X) \to \mathrm{Hilb}_X^{\xi}$ is $\mathrm{Pic}^{r+1}(C)$. It is easy to see that $\mathrm{Pic}^{r+1}(C)$ is a Lagrangian subscheme of $M_{\sigma+kf}(w + \varrho_X)$. By Matsushita [Mt], every fiber is of dimension $(\xi^2)/2 + 1$. □

LEMMA 3.14. *Let L be a stable sheaf of $v(L) = w$ and* $\mathrm{Div}(L) = D$, *and let L' be a coherent sheaf which fits in a nontrivial extension*

$$0 \to L \to L' \to \mathbb{C}_x \to 0 \tag{3.36}$$

where $x \in D$. Then L' is stable.

Proof. Assume that L' is not of pure dimension 1 and let T be the zero-dimensional subsheaf of L'. Then $T \to L' \to \mathbb{C}_x$ must be injective. Hence, it is isomorphic, which implies that the exact sequence split. Therefore L' is of pure dimension 1. If L' is not stable, then there is a subsheaf L_1 of L' such that $\frac{\chi(L_1)}{\lambda_1} > \frac{\chi(L')}{\lambda}$, where $\mathrm{Div}(L_1) = \lambda_1 C$. Hence $0 < \chi(L_1)\lambda - \chi(L')\lambda_1 = (\chi(L_1) - 1)\lambda - \chi(L)\lambda_1 + \lambda - \lambda_1$. Since $\chi(L) = r$ is divisible by λ and $\lambda - \lambda_1 < \lambda$, we get $(\chi(L_1) - 1)\lambda - \chi(L)\lambda_1 \geqslant 0$. Thus $\frac{\chi(L_1 \cap L)}{\lambda_1} \geqslant \frac{\chi(L)}{\lambda}$, which implies that L is not stable. Therefore L' must be stable. □

COROLLARY 3.15. $\dim M_{\sigma+kf}(w, D) \leqslant \dim P_{\sigma+kf}(w, w + \varrho_X, D) \leqslant \dim M_{\sigma+kf}(w + \varrho_X, D) + (\lambda - 1)$.

Proof. By Lemma 3.14, π_w is surjective. Since $\dim \pi^{-1}_{w+\varrho_X}(x) \leqslant \lambda - 1$ for $x \in M_{\sigma+kf}(w + \varrho_X, D)$, we get our inequality. □

Since $\lambda \geqslant 2$, we get that $2\lambda^2 - (\lambda^2 + \lambda + 1) > 0$. Then

$$\begin{aligned}\dim j^{-1}(\mathrm{Hilb}^{\xi}_X(\lambda)) &\leqslant (C^2)/2 + 1 + \lambda^2(C^2)/2 + 1 + (\lambda - 1)\\ &< (\lambda^2 + \lambda + 1)(C^2)/2 + 2\\ &< \lambda^2(C^2) + 2 = \dim M_{\sigma+kf}(w).\end{aligned} \tag{3.37}$$

Combining Proposition 3.11, we get the following proposition:

PROPOSITION 3.16. *We set*

$$\begin{aligned}M_{\sigma+kf}(w)_0 &:= \{L \in M_{\sigma+kf}(w) \mid \mathrm{Div}(L) \textit{ is an integral curve}\}\\ &= M_{\sigma+kf}(w) \setminus (N_1 \cup N_2).\end{aligned} \tag{3.38}$$

Then $M_{\sigma+kf}(w)_0$ is an open dense subscheme of $M_{\sigma+kf}(w)$.

PROPOSITION 3.17. *$M_{\sigma+kf}(w)$ is irreducible.*

Proof. By Proposition 3.16, it is sufficient to show that $M_{\sigma+kf}(w)_0$ is irreducible. Let C be an integral curve. Then $j^{-1}(C)$ is the compactified Jacobian of C. By [AIK], the compactified Jacobian of C is irreducible. Therefore $M_{\sigma+kf}(w)_0$ is irreducible. □

Combining all our results, we get the following theorem.

THEOREM 3.18. *Let X be an Abelian surface or a $K3$ surface and let v be a Mukai vector of* $\mathrm{rk}\, v > 0$ *and* $\langle v^2 \rangle > 0$. *Let H be a general ample divisor with respect to v. Then $\mathcal{M}_H(v)^{ss}$ is a normal and irreducible stack. In particular, $\overline{M}_H(v)$ is a normal variety.*

Remark 3.5. In [O2, O3], O'Grady studied the case where $m(v) = 2$. In particular, he constructed a symplectic desingularization of $\overline{M}_H(v)$, if $\langle v^2 \rangle = 8$.

4. Fourier–Mukai Transform on Enriques Surfaces

In this section, we consider the Fourier–Mukai transform on Enriques surface X. By using the Fourier–Mukai transform, we shall compute the Hodge polynomials of some moduli spaces of sheaves.

In our case, the Mukai vector $v(x)$ of $x \in K(X)$ is defined as an element of $H^*(X, \mathbb{Q})$:

$$v(x) := \mathrm{ch}(x)\sqrt{\mathrm{td}_X} = \mathrm{rk}(x) + c_1(x) + \left(\frac{\mathrm{rk}(x)}{2}\varrho_X + \mathrm{ch}_2(x)\right) \in H^*(X, \mathbb{Q}). \tag{4.1}$$

We also introduce Mukai's pairing on $H^*(X, \mathbb{Q})$ by $\langle x, y\rangle := -\int_X x^\vee \wedge y$. Then we have an isomorphism of lattices:

$$(v(K(X)), \langle\ ,\ \rangle) \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus E_8(-1). \tag{4.2}$$

DEFINITION 4.1. We call an element of $v(K(X))$ by the Mukai vector. A Mukai vector v is primitive, if v is primitive as an element of $v(K(X))$.

For a variety Y over $\mathbb{C}$, the cohomology with compact support $H_c^*(Y, \mathbb{Q})$ has a natural mixed Hodge structure. Let $e^{p,q}(Y) := \sum_k (-1)^k h^{p,q}(H_c^k(Y))$ be the virtual Hodge number and $e(Y) := \sum_{p,q} e^{p,q}(Y) x^p y^q$ the virtual Hodge polynimial of Y. By Remark 1.1, $\mathcal{M}_H^\alpha(v)^{ss}$ is described as a quotient stack $[Q^{ss}/GL(N)]$, where Q^{ss} is a suitable open subscheme of $\mathrm{Quot}_{\mathcal{O}_X^{\oplus N}/X}$. We define the virtual Hodge 'polynimial' of $\mathcal{M}_H^\alpha(v)^{ss}$ by

$$e(\mathcal{M}_H^\alpha(v)^{ss}) = e(Q^{ss})/e(GL(N)) \in \mathbb{Q}(x, y). \tag{4.3}$$

It is easy to see that $e(Q^{ss})/e(GL(N))$ does not depend on the choice of Q^{ss}. The following was essentially proved in [Y1, Sect. 3.2].

PROPOSITION 4.1. *Let X be a surface such that K_X is numerically trivial. Let (H, α) be a pair of ample divisor H and a $\mathbb{Q}$-divisor α. Then $e(\mathcal{M}_H^\alpha(v)^{ss})$ does not depend on the choice of H and α, if $(H, \mathcal{O}(\alpha))$ is general with respect to v (cf. Defn.* 1.4).

PROPOSITION 4.2. *Let v be a Mukai vector such that* $\mathrm{rk}(v)$ *is odd. Then $\mathcal{M}_H(v)^s$ is smooth of* $\dim \mathcal{M}_H(v)^s = \langle v^2\rangle + 1$.

Proof. For $E \in \mathcal{M}_H(v)^s$, we get $\det(E(K_X)) \not\cong \det(E)$. If there is a nonzero homomorphism $E \to E(K_X)$, then the stability condition implies that it is an isomorphism. Hence, $\mathrm{Ext}^2(E,E) = \mathrm{Hom}(E, E(K_X))^\vee = 0$. Since $-\chi(E,E) = \langle v(E), v(E)\rangle$, $\mathcal{M}_H(v)^s$ is smooth of $\dim \mathcal{M}_H(v)^s = \langle v^2\rangle + 1$. □

For a Mukai vector v, let $L_1, L_2 = L_1(K_X) \in \mathrm{Pic}(X)$ be line bundles on X such that $c_1(L_1)(= c_1(L_2)) = c_1(v)$. Then we have a decomposition

$$\mathcal{M}_H(v)^{ss} = \mathcal{M}_H(v, L_1)^{ss} \coprod \mathcal{M}_H(v, L_2)^{ss} \tag{4.4}$$

where $\mathcal{M}_H(v, L_i)^{ss}$, $i = 1, 2$, is the substack of $\mathcal{M}_H(v)^{ss}$ consisting of E such that $\det(E) = L_i$. We also have a decomposition $M_H(v) = M_H(v, L_1) \coprod M_H(v, L_2)$, where $M_H(v, L_i)$ is the subscheme of $M_H(v)$ consisting of E such that $\det(E) = L_i$.

We consider the Fourier-Mukai transform associated to (-1)-reflection. Let $v_0 := r + c_1 - (s/2)\varrho_X$ be a Mukai vector such that $\mathrm{rk}(v_0) > 0$ and $\langle v_0^2 \rangle = (c_1^2) + rs = -1$. Since (c_1^2) is even, r and s are odd. Let H be a general ample divisor with respect to v_0. Assume that there is a stable vector bundle E_0 with respect to H such that $v(E_0) = v_0$ (cf. Corollary 4.7). Then we see that

$$\mathrm{Hom}(E_0, E_0) = \mathbb{C}, \quad \mathrm{Ext}^1(E_0, E_0) = 0, \quad \mathrm{Ext}^2(E_0, E_0) = 0. \tag{4.5}$$

Let

$$ev_1\colon E_0^\vee \boxtimes E_0 \to \mathcal{O}_\Delta, \quad ev_2\colon E_0(K_X)^\vee \boxtimes E_0(K_X) \to \mathcal{O}_\Delta \tag{4.6}$$

be evaluation maps. We define a sheaf $\mathcal{E}$ on $X \times X$ by an exact sequence

$$0 \to \mathcal{E} \to E_0^\vee \boxtimes E_0 \oplus E_0(K_X)^\vee \boxtimes E_0(K_X) \xrightarrow{(ev_1, ev_2)} \mathcal{O}_\Delta \to 0. \tag{4.7}$$

Then $\mathcal{E}_{|\{x\}\times X}$ (resp. $\mathcal{E}_{|X\times\{x\}}$) is a stable sheaf with $v(\mathcal{E}_{|\{x\}\times X}) = 2\mathrm{rk}(E_0)v(E_0) - \varrho_X$ (resp. $v(\mathcal{E}_{|X\times\{x\}}) = 2\mathrm{rk}(E_0)v(E_0)^\vee - \varrho_X$). Thus $\mathcal{E}$ is a flat family of stable sheaves with $v(\mathcal{E}_{|\{x\}\times X}) = 2\mathrm{rk}(E_0)v(E_0) - \varrho_X$. By the construction of $\mathcal{E}$, $\mathcal{E}_{|\{x\}\times X}(K_X) \cong \mathcal{E}_{|\{x\}\times X}$, which implies that

$$\mathrm{Ext}^2(\mathcal{E}_{|\{x\}\times X}, \mathcal{E}_{|\{x\}\times X}) = \mathrm{Hom}(\mathcal{E}_{|\{x\}\times X}, \mathcal{E}_{|\{x\}\times X}(K_X))^\vee \cong \mathbb{C}. \tag{4.8}$$

Since $\langle v(E_0)^2 \rangle = -1$, we see that $\langle v(\mathcal{E}_{|\{x\}\times X})^2 \rangle = 0$. Hence, the Zariski tangent space is two-dimensional:

$$\mathrm{Ext}^1(\mathcal{E}_{|\{x\}\times X}, \mathcal{E}_{|\{x\}\times X}) \cong \mathbb{C}^{\oplus 2}. \tag{4.9}$$

Therefore X is a connected component of $M_H(v_1)$, where $v_1 = 2\mathrm{rk}(E_0)v(E_0) - \varrho_X$.

Then $\mathcal{H}_{\mathcal{E}}\colon \mathbf{D}(X) \to \mathbf{D}(X)_{\mathrm{op}}$ is an equivalence of categories. As a corollary of this fact, we get that $M_H(v_1) = X$. By our construction of $\mathcal{E}$, we see that

$$v(\mathcal{H}_{\mathcal{E}}(x)) = -(x^\vee + 2v(E_0)^\vee \langle x, v(E_0)\rangle). \tag{4.10}$$

If $E_0 = \mathcal{O}_X$ and $v(E) = r + c_1 + (s/2)\varrho_X$, then $v(\mathcal{H}_{\mathcal{E}}(E)) = s + c_1 + (r/2)\varrho_X$.

From now on, we assume that X is unnodal, i.e. there is no (-2)-curve. Let σ and f be elliptic curves on X such that $(\sigma, f) = 1$. Then

$$H^2(X, \mathbb{Z})_f = (\mathbb{Z}\sigma \oplus \mathbb{Z}f) \perp E_8(-1) \tag{4.11}$$

where $H^2(X, \mathbb{Z})_f$ is the torsion free quotient of $H^2(X, \mathbb{Z})$.

PROPOSITION 4.3. *We set* $G_1 := \mathcal{E}_{|\{x\}\times X}$ *and* $G_2 := \mathcal{E}_{|X\times\{x\}}$. *Assume that* $\deg_{G_1}(v) = 0$ *and* $l(v) := -\langle v, \varrho_X\rangle/\operatorname{rk} v_1 > 0$, $a(v) := \langle v, v_1\rangle/\operatorname{rk} v_1 > 0$. *Let* ε *be an element of* $K(X)\otimes\mathbb{Q}$ *such that* $v(\varepsilon)\in v_1^{\perp}\cap\varrho_X^{\perp}$, $|\langle v(\varepsilon)^2\rangle| \ll 1$ *and* $(H, c_1(\varepsilon)) = 0$. *Then* $\mathcal{H}_{\mathcal{E}}$ *induces an isomorphism*

$$\mathcal{M}_H^{G_1+\varepsilon}(v)^{ss} \to \mathcal{M}_{\hat{H}}^{G_2+\hat{\varepsilon}}(-\mathcal{H}_{\mathcal{E}}(v))^{ss}, \tag{4.12}$$

where $\hat{\varepsilon} = \mathcal{H}_{\mathcal{E}}(\varepsilon)$.

Proof. Since H is general with respect to $v(E_0)$, we see that $\mathcal{E}_{|\{x\}\times X}$ is G_1-twisted stable. Then we see that Lemma 2.4 holds. We next show that Lemma 2.5 holds. We may assume that E is μ-stable. If $E^{\vee\vee} \neq E_0, E_0(K_X)$, then $\operatorname{Hom}(E, E_0) = \operatorname{Hom}(E, E_0(K_X)) = 0$. If $E^{\vee\vee} = E_0, E_0(K_X)$, then $\operatorname{Hom}(E, \mathcal{E}_{|\{x\}\times X}) = 0$ for $x\in X\setminus \operatorname{Supp}(E^{\vee}/E)$. Thus Lemma 2.5 holds. Then the same proof of Theorem 2.3 works and we get our claim. □

COROLLARY 4.4. $M_H(r - (1/2)\varrho_X, \mathcal{O}_X) \cong \operatorname{Hilb}_X^{(r+1)/2}$ *for a general* H *with respect to* $r - (1/2)\varrho_X$.

PROPOSITION 4.5. *Assume that* $r, s > 0$. *Then* $e(M_H^{\alpha}(r + c_1 - (s/2)\varrho_X)) = e(M_H^{\alpha}(s - c_1 - (r/2)\varrho_X))$ *for a general* (H,α), *if* $(c_1^2) < 0$, *i.e.,* $\langle v^2\rangle < rs$, *where* $v = r + c_1 - (s/2)\varrho_X$. *In particular, if* $r > \langle v^2\rangle$, *then we get our claim.*

Proof. If $(c_1^2) < 0$, then the Hodge index theorem implies that there is a divisor H such that $(H, c_1) = 0$ and $(H^2) > 0$. By the Riemann–Roch theorem, we may assume that H is effective. Since X is unnodal, H is ample. If $E_0 = \mathcal{O}_X$, then $v(\mathcal{E}_{|\{x\}\times X}) = 2$. Hence v satisfies assumptions of Proposition 4.3. Then we get an isomorphism

$$M_H^{\mathcal{O}_X+\varepsilon}(r + c_1 - (s/2)\varrho_X) \to M_H^{\mathcal{O}_X+\varepsilon}(s - c_1 - (r/2)\varrho_X), \tag{4.13}$$

where $(H, \mathcal{O}_X + \varepsilon)$ is general with respect to v. By Proposition 4.1, we get our claim. □

THEOREM 4.6. *Let* $v = r + c_1 - (s/2)\varrho_X \in H^*(X, \mathbb{Q})$ *be a primitive Mukai vector such that* r *is odd. Then*

$$e(M_H(v, L)) = e(\operatorname{Hilb}_X^{(\langle v^2\rangle+1)/2}) \tag{4.14}$$

for a general H, *where* $L\in\operatorname{Pic}(X)$ *satisfies* $c_1(L) = c_1$. *In particular,*

(i) $M_H(v) \neq \emptyset$ *for a general* H *if and only if* $\langle v^2\rangle \geqslant -1$.
(ii) $M_H(v, L)$ *is irreducible for a general* H.

Proof. We first assume that $c_1 \in E_8(-1)$. We set $l = \gcd(r, c_1)$. Replacing v by $v\exp(\xi_1)$, $\xi_1 \in E_8(-1)$, we may assume that c_1/l is primitive and $s > \langle v^2\rangle$. Since v is primitive, $\gcd(l, s) = 1$. By Proposition 4.5, we get

$$e(M_H(r + c_1 - (s/2)\varrho_X)) = e(M_H(s - c_1 - (r/2)\varrho_X)). \tag{4.15}$$

Replacing $v = r + c_1 - (s/2)\varrho_X$ by $v' = s - c_1 - (r/2)\varrho_X$, we may assume that $r > \langle v^2 \rangle$. By the same argument as above, we may assume that $l = 1$ and c_1 is primitive. We set $D = \sigma - \frac{(\eta^2)}{2} f + \eta$, where $\eta \in E_8(-1)$ satisfies that $2(\eta, c_1) = s - 1$. Then $(D^2) = 0$ and $-s + 2(c_1, D) = -1$. Since $v\exp(D) = r + (c_1 + rD) - 1/2\varrho_X$, $e(M_H(v)) = e(M_H(r + (c_1 + rD) - 1/2\varrho_X))$. Since $r > \langle v^2 \rangle$, Proposition 4.5 implies that our claim holds for this case.

We shall next treat the general case. We use induction on r. We set $c_1 := d_1\sigma + d_2 f + \xi$. Replacing v by $v\exp(k\sigma)$, we may assume that $0 \leqslant |d_1| < r/2$. We first assume that $d_1 \neq 0$. We note that $(c_1, f) = d_1$. Replacing v by $v\exp(\eta)$, $\eta \in E_8(-1)$, we may assume that $s > \langle v^2 \rangle$. Then by Proposition 4.5, $e(M_H(v)) = e(M_H(s - c_1 - (r/2)\varrho_X))$ for a general H. We take an integer k such that $0 < r + 2d_1 k < 2|d_1| < r$. Then $v\exp(kf) = s + (-c_1 + skf) - r'/2\varrho_X$, where $r' = r + 2d_1 k$. Since $s > \langle v^2 \rangle$, Proposition 4.5, implies that $e(M_H(s + (-c_1 + skf) - r'/2\varrho_X)) = e(M_H(r' + (c_1 - skf) - (s/2)\varrho_X))$ for a general H. By induction hypothesis, we get our claim.

If $d_1 = 0$, then we may assume that $0 \leqslant |d_2| < r/2$. If $|d_2| > 0$, then we can apply the same argument and get our claim. If $d_1 = d_2 = 0$, then $c_1 \in E_8(-1)$, so we get our claim. □

COROLLARY 4.7. *If $\langle v^2 \rangle = -1$, then there is a stable vector bundle E_0 with respect to H with $v(E_0) = v$.*

Remark 4.1. By the proof, we also get the following: Let v be a primitive Mukai vector such that $\operatorname{rk} v$ is odd. Then $e(\mathcal{M}_H(mv)^{ss}) = e(\mathcal{M}_H(m(1 - (n/2)\varrho_X))^{ss})$. where $n = \langle v^2 \rangle$. □

5. Appendix

Let X be an Abelian (or a K3) surface. In this appendix, we give a sufficient condition on H to be general with respect to $v = r + \xi + a\varrho_X$.

LEMMA 5.1. *Assume that there is an exact sequence*

$$0 \to E_1 \to E \to E_2 \to 0 \tag{5.1}$$

such that E_1 and E_2 are μ-semi-stable sheaves with $v(E_1) = r_1 + \xi_1 + a_1\varrho_X$ and $v(E_2) = r_2 + \xi_2 + a_2\varrho_X$ respectively. Then

$$\frac{\langle v^2 \rangle}{r_1 r_2} \geqslant -\left(\left(\frac{\xi_1}{r_1} - \frac{\xi_2}{r_2}\right)^2\right) - \frac{2r^2}{r_1 r_2}\epsilon, \tag{5.2}$$

where $\epsilon = 0$ or 1 according as X is an Abelian surface or a $K3$ surface.

Proof. We note that $\langle v_i^2 \rangle / r_i^2 = ((\xi_i/r_i)^2) - 2a_i/r_i$, $i = 1, 2$. Then we see that

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$$\begin{aligned}\frac{\langle v^2\rangle}{r_1r_2} &= \frac{\langle v_1^2\rangle + \langle v_2^2\rangle + 2\langle v_1, v_2\rangle}{r_1r_2}\\ &= \frac{1}{r_1r_2}\langle v_1^2\rangle + \frac{1}{r_1r_2}\langle v_2^2\rangle + 2\left(\frac{\xi_1}{r_1}, \frac{\xi_2}{r_2}\right) - 2\left(\frac{a_1}{r_1} + \frac{a_2}{r_2}\right)\\ &= -\left(\left(\frac{\xi_1}{r_1} - \frac{\xi_2}{r_2}\right)^2\right) + \left(\frac{r_1+r_2}{r_1^2r_2}\langle v_1^2\rangle + \frac{r_1+r_2}{r_1r_2^2}\langle v_2^2\rangle\right). \end{aligned} \tag{5.3}$$

By the Bogomolov inequality, we have $\langle v_i^2\rangle \geqslant -2r_i^2\epsilon$. Hence our claim holds. □

LEMMA 5.2. *If*

$$\min\{-(D^2) | D \in \mathrm{NS}(X), (D, H) = 0, D \neq 0\} > r^2(\langle v^2\rangle + 2r^2)/4, \tag{5.4}$$

then H is a general polarization with respect to v.

Proof. If H is not general with respect to v, then there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{5.5}$$

such that (i) E_1 and E_2 are μ-semi-stable sheaves with $v(E_1) = r_1 + \xi_1 + a_1\varrho_X$ and $v(E_2) = r_2 + \xi_2 + a_2\varrho_X$ respectively (ii) $r_2\xi_1 - r_1\xi_2 \neq 0$ and (iii) $(r_2\xi_1 - r_1\xi_2, H) = 0$. Since $r_1r_2 \leqslant r^2/4$, (5.2) implies that $-((r_2\xi_1 - r_1\xi_2)^2) \leqslant r^2(\langle v^2\rangle + 2r^2)/4$, which is a contradiction. □

Acknowledgement

I would like to thank the referee for reading carefully this article and giving many suggestions.

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