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Partial C*-dynamics and Rokhlin dimension

FERNANDO ABADIE†, EUSEBIO GARDELLA‡ and SHIRLY GEFFEN§

† *Centro de Matemática, Facultad de Ciencias,*
Universidad de la República, Iguá 4225,
11400 Montevideo, Uruguay
(e-mail: fabadie@cmat.edu.uy)
‡ *Mathematisches Institut, Fachbereich Mathematik und Informatik der Universität*
Münster, Einsteinstrasse 62, 48149 Münsster, Germany
(e-mail: gardella@uni-muenster.de)
§ *Department of Mathematics, Ben-Gurion University of the Negev,*
Be'er Sheva 8410501, Israel
(e-mail: shirlyg@post.bgu.ac.il)

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Abstract. We develop the notion of the Rokhlin dimension for partial actions of finite groups, extending the well-established theory for global systems. The partial setting exhibits phenomena that cannot be expected for global actions, usually stemming from the fact that virtually all averaging arguments for finite group actions completely break down for partial systems. For example, fixed point algebras and crossed products are not in general Morita equivalent, and there is in general no local approximation of the crossed product $A \rtimes G$ by matrices over A. Using decomposition arguments for partial actions of finite groups, we show that a number of structural properties are preserved by formation of crossed products, including finite stable rank, finite nuclear dimension, and absorption of a strongly self-absorbing C^*-algebra. Some of our results are new even in the global case.

We also study the Rokhlin dimension of globalizable actions: while in general it differs from the Rokhlin dimension of its globalization, we show that they agree if the coefficient algebra is unital. For topological partial actions on spaces of finite covering dimension, we show that finiteness of the Rokhlin dimension is equivalent to freeness, thus providing a large class of examples to which our theory applies.

1. *Introduction*

Partial dynamical systems have implicitly been used in mathematics long before the notion was formalized, at least since the study of differential equations. Indeed, the flow of a differentiable vector field can be naturally regarded as a partial action of the reals. More precisely, given a smooth vector field v on a manifold M, for $x \in M$ let ϕ_x be the unique solution to the differential equation $\phi'(t) = v(\phi(t))$ with initial condition $\phi_x(0) = x$, and let $I_x \subseteq \mathbb{R}$ be the largest (open) neighbourhood of 0 on which ϕ_x is defined. For $t \in \mathbb{R}$, set

$$M_t = \{x \in M : t \in I_x\},$$

and let $\sigma_t : M_{-t} \to M_t$ be the diffeomorphism given by $\sigma_t(x) = \phi_x(t)$ for all $x \in M_{-t}$. The collection $\sigma = \{\sigma_t, M_t : t \in \mathbb{R}\}$ satisfies the crucial property that σ_{s+t} extends $\sigma_s \circ \sigma_t$, in the sense that whenever $x \in M_{-t}$ and $\sigma_t(x)$ belongs to M_{-s}, then x belongs to M_{-s-t} and $\sigma_{s+t}(x) = \sigma_s(\sigma_t(x))$. In modern language, this condition asserts that σ is a *partial action* of $\mathbb{R}$ on M.

Partial actions were originally introduced by Exel and McClanahan in the 1990s, by isolating and abstracting the conditions observed in the context described above: a partial action of a discrete group G on a topological space X is a collection $\{X_g : g \in G\}$ of open sets of X and homeomorphisms $\sigma_g : X_{g^{-1}} \to X_g$ such that $\sigma_1 = \mathrm{id}_X$ and σ_{gh} extends $\sigma_g \circ \sigma_h$ wherever the decomposition is well defined. The notion of a *global* (or ordinary) action is obtained by taking $X_g = X$ for all $g \in G$. We refer the reader to a recent book [6] for a modern treatment of this topic and historical references. The study of partial actions (both on topological spaces and on C^*-algebras) has been very fruitful, and has shed new light on the study of several objects. For example, the fact that the solutions of a differential equation on a *compact* manifold are defined on all $\mathbb{R}$ can be easily proved in this more abstract setting; see Proposition 2.4 in [1].

A typical example of a partial action is obtained by starting with a global action $\beta : G \to \mathrm{Aut}(B)$, a not necessarily invariant ideal A in B, and setting $A_g = A \cap \beta_g(A)$ with $\alpha_g = \beta_g|_{A_{g^{-1}}}$ for all $g \in G$. Partial actions obtained in this way are called *globalizable*, and the *globalization problem* involves determining whether a given partial action is globalizable and, if it is, describing its globalization; see §3 of [1]. As it turns out, not every partial action is globalizable (a necessary and sufficient condition is given in [7]). Even when a globalization exists, identifying it is often a challenging task, and its dynamical properties may differ significantly from those of α.

Given a partial group action α of a group G on a C^*-algebra A, one can construct its *crossed product* $A \rtimes_\alpha G$; see §I.8 in [6]. Large families of C^*-algebras can be naturally described as partial crossed products, typically with commutative C^*-algebras, even in situations where similar descriptions do not exist for global crossed products. For example, every unital approximately finite-dimensional (AF)-algebra arises as the crossed product of a partial homeomorphism of a totally disconnected compact space, whereas no unital AF-algebra arises as the crossed product of a homeomorphism.

It is therefore particularly important to develop tools to study partial crossed products. There have been a number of advances in this direction, for example, in reference to K-theory [5] and Takai duality [1]. On the other hand, the study of partial actions of *finite* groups remains conspicuously underdeveloped. Indeed, and even in the globalizable case, virtually all averaging arguments (and their consequences) that are standard for global actions completely break down in this setting. The lack of approximate identities that are compatible with the partial action is also a source of difficulties in this setting. The goal of the present work is to make advances in the study of the structure of crossed products by partial actions of finite groups.

In the modern literature in C^*-dynamical systems, several Rokhlin-type properties have an increasingly central role in the study of crossed products; see [**9**, **12**, **15**, **18**, **19**, **21**, **23**]. Their wide and fruitful applicability in the global setting make the extension of this theory to the partial setting worthy of exploration.

Motivated by [**18**], we define and study the notion of *Rokhlin dimension* in the partial setting; see Definition 2.1. The theory that we develop here exhibits phenomena that cannot be expected for global actions. Among others, the Rokhlin dimension of a globalizable partial action does not agree with that of its globalization; see Example 3.2. A notable exception is the unital case.

THEOREM 1.1. (Theorem 3.4) *Let α be a globalizable partial action of a finite group on a unital C^*-algebra, and let β be its globalization. Then*

$$\dim_{\mathrm{Rok}}(\alpha) = \dim_{\mathrm{Rok}}(\beta) \quad \textit{and} \quad \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) = \dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta).$$

Our original motivation was the study of the structure of the crossed product, particularly from the point of view of the classification programme for simple nuclear C^*-algebras; see [**4**]. As it turns out, this is technically much more complicated than in the global setting, and tackling this problem required us to first develop a decomposition theory for partial actions of finite groups into iterated extensions of relatively simpler systems; see [**2**] and particularly §6 there. Here, we prove the following.

THEOREM 1.2. (Theorem 4.7) *The following properties are inherited by crossed products or fixed point algebras by partial actions of finite groups with* $\dim_{\mathrm{Rok}} < \infty$.

(1) *Having finite nuclear dimension or decomposition rank; for example,*

$$\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq (|G| - 1)(\dim_{\mathrm{nuc}}(A) + 1)(\dim_{\mathrm{Rok}}(\alpha) + 1) + \dim_{\mathrm{nuc}}(A).$$

(2) *When $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$, having finite stable/real rank; for example,*

$$\mathrm{sr}(A \rtimes_\alpha G) \leq \frac{|G|(\mathrm{sr}(A) + \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) + 3) - 2}{2}.$$

(3) *When $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$, absorbing a given strongly self-absorbing C^*-algebra.*

(See [16] for previously known results regarding nuclear dimension of $C^\star$-algebras attached to partial dynamical systems.)

For unital partial actions, even more can be said; see Theorem 4.10. Most remarkably, the Universal Coefficient Theorem (UCT) is preserved in the commuting towers version. In this case, we also show that A^α is Morita equivalent to $A \rtimes_\alpha G$ (see Theorem 4.9) a fact that rather surprisingly fails if unitality of A is dropped (see Example 4.8).

Our structural results for crossed products are complemented by the fact that partial actions with finite Rokhlin dimension are relatively common. For example, we show that this notion is equivalent to freeness in the commutative setting.

THEOREM 1.3. (Theorem 5.10) *Let σ be a partial action of a finite group G on a locally compact Hausdorff space X with* $\dim(X) < \infty$. *Then* $\dim_{\mathrm{Rok}}(\sigma) < \infty$ *if and only if σ is* *free, in which case we have*

$$\dim_{\mathrm{Rok}}(\sigma) \leq (|G|-1)\dim(X).$$

The result in the global case is implicit [10, 17] and is an easy consequence of the existence of local cross-sections for the quotient map $\pi\colon X \to X/G$. However, the proof in the partial setting is considerably more complicated, since for free partial actions there may not exist local cross-sections for π. The proof in our context is quite involved and occupies most of §5. The main technical ingredient is the fact (Proposition 5.6) that an extension of topological partial actions with finite Rokhlin dimension again has finite Rokhlin dimension. Roughly speaking, one needs to lift Rokhlin towers from the quotient to the algebra, while at the same time respecting the domains of the partial action. The fact that the coefficient algebra is commutative seems to be crucial for this lifting problem to have a solution.

2. *Rokhlin dimension for partial actions of finite groups*

In this section we define Rokhlin dimension for partial actions.

Definition 2.1. Let $\alpha = ((A_g), (\alpha_g))_{g\in G}$ be a partial action of a finite group G on a C^*-algebra A. For $d \in \mathbb{N}$, we say that α has *Rokhlin dimension at most* d, and write $\dim_{\mathrm{Rok}}(\alpha) \leq d$, if for every $\varepsilon > 0$ and every finite subset $F \subseteq A$, there exist positive contractions $f_g^{(j)} \in A_g$, for $g \in G$ and $j = 0, \ldots, d$, satisfying:

(1) $\|(\alpha_g(f_h^{(j)}x) - f_{gh}^{(j)}\alpha_g(x))a\| < \varepsilon$ for all $g, h \in G$, $j = 0, \ldots, d$, $a \in F$ and $x \in A_{g^{-1}} \cap F$;

(2) $\|f_g^{(j)} f_h^{(j)} a\| < \varepsilon$ for $j = 0, \ldots, d$, $g, h \in G$ with $g \neq h$ and $a \in F$;

(3) $\|(\sum_{j=0}^{d}\sum_{g\in G} f_g^{(j)})a - a\| < \varepsilon$ for all $a \in F$;

(4) $\|(f_g^{(j)}b - bf_g^{(j)})a\| < \varepsilon$ for all $j = 0, \ldots, d$, $g \in G$, and $a, b \in F$.

Moreover, we say that α has *Rokhlin dimension with commuting towers at most* d, and write $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq d$, if for every $\varepsilon > 0$ and every finite subset $F \subseteq A$, there exist positive contractions $f_g^{(j)} \in A_g$, for $g \in G$ and $j = 0, \ldots, d$, satisfying conditions (1)–(4) above, in addition to:

(5) $\|(f_g^{(j)} f_h^{(k)} - f_h^{(k)} f_g^{(j)})a\| < \varepsilon$ for all $j, k = 0, \ldots, d$, $g, h \in G$ and $a \in F$.

In either case, we call the elements $f_g^{(j)}$ above *Rokhlin towers* for (F, ε).

We define the *Rokhlin dimension* of α by

$$\dim_{\mathrm{Rok}}(\alpha) = \min\{d \in \mathbb{N} \colon \dim_{\mathrm{Rok}}(\alpha) \leq d\},$$

and define the *Rokhlin dimension with commuting towers* $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha)$ similarly.

The multiplicative witness $a \in F$ that appears in conditions (1)–(5) also appears in the definition of Rokhlin dimension for *global* actions setting [17], and it can be omitted when A is unital. On the other hand, the witness $x \in F \cap A_{g^{-1}}$ is a conceptually new condition that cannot be omitted even if A is unital.

In the early stages of this project, it was unclear whether one should not instead require the stronger condition $\|\alpha_g(f_h^{(j)}x) - f_{gh}^{(j)}\alpha_g(x)\| < \varepsilon\|x\|$ for all $g, h \in G$, $j = 0, \ldots, d$, and *all* $x \in A_{g^{-1}}$. As it turns out, the stronger condition implies that the given partial action is in fact globalizable, which suggests that it was not the right notion to consider.

Unlike the case of global actions, the different elements within one tower are not 'interchangeable', as they tend to have different 'sizes'. In fact, the positive contraction corresponding to the unit of the group is usually much larger than the others. The following is an extreme example of this situation.

Example 2.2. Let G be a finite group, and let A be a unital C^*-algebra. We define the *trivial partial action* of G on A by setting $A_g = \{0\}$ for $g \in G \setminus \{1\}$. This partial action has Rokhlin dimension zero, with Rokhlin towers given by $f_1 = 1_A$ and $f_g = 0$ for $g \in G \setminus \{1\}$. Note that $A \rtimes_\alpha G = A$. Moreover, this is the only partial action of G on A with the 1-decomposition property (Definition 4.2); see Example 2.6 in [**2**].

Next, we show that condition (1) in Definition 2.1 can be strengthened in the case of *unital* partial actions (that is, partial actions whose domains are unital).

Remark 2.3. Let α be a unital partial action of a finite group G on a C^*-algebra A, with units $1_g \in A_g$, for $g \in G$. Then $1_{gh}1_g = \alpha_g(1_h 1_{g^{-1}})$ for all $g, h \in G$.

PROPOSITION 2.4. *Adopt the notation from Definition 2.1, and suppose that α is a unital partial action. For $g \in G$, denote by 1_g the unit of A_g. Then condition (1) in Definition 2.1 can be replaced by:*

(1') $\alpha_g(f_h^{(j)} 1_{g^{-1}}) = f_{gh}^{(j)} 1_g$ *for all $g, h \in G$, and for all $j = 0, \ldots, d$.*

Proof. We prove the proposition for $\dim_{\mathrm{Rok}}$, since the proof for $\dim_{\mathrm{Rok}}^{\mathrm{c}}$ is analogous. Using the identity $x = x1_{g^{-1}}$ for all $x \in A_{g^{-1}}$ and $g \in G$, one easily shows that (1') implies the following identity for all $g, h \in G$, $j = 0, \ldots, d$, and all $x \in A_{g^{-1}}$:

$$\alpha_g(f_h^{(j)} x) = f_{gh}^{(j)} \alpha_g(x).$$

This identity clearly implies (1). Conversely, let $\varepsilon > 0$ and a finite subset $F \subseteq A$ be given. Without loss of generality, we assume that F consists of contractions and that $\{1_g : g \in G\} \subseteq F$. Set $\varepsilon_0 = \varepsilon/(|G|(d+1)+1)$ and find Rokhlin towers $f_g^{(j)} \in A_g$, for $g \in G$ and $j = 0, \ldots, d$, satisfying conditions (1), (2), (3), and (4) in Definition 2.1 for (F, ε_0). Define positive contractions $\widetilde{f}_g^{(j)} \in A_g$, for all $g \in G$ and $j = 0, \ldots, d$, by $\widetilde{f}_g^{(j)} = \alpha_g(f_1^{(j)} 1_{g^{-1}})$. Since $1_{g^{-1}}$ belongs to F, condition (1) for $f_1^{(j)}$ gives

$$\|\widetilde{f}_g^{(j)} - f_g^{(j)}\| = \|\alpha_g(f_1^{(j)} 1_{g^{-1}}) - f_g^{(j)} 1_g\| < \varepsilon_0 \tag{2.1}$$

for all $g \in G$ and all $j = 0, \ldots, d$. We claim that these elements satisfy the conditions in Definition 2.1 with (1) replaced by (1').

We begin with (1'). For $g, h \in G$ and $j = 0, \ldots, d$, we have

$$\begin{aligned}\alpha_g(\widetilde{f}_h^{(j)} 1_{g^{-1}}) &= \alpha_g(\alpha_h(f_1^{(j)} 1_{h^{-1}}) 1_{g^{-1}}) \\ &= \alpha_g(\alpha_h(f_1^{(j)} 1_{h^{-1}} 1_{(gh)^{-1}})) \\ &= \alpha_{gh}(f_1^{(j)} 1_{(gh)^{-1}}) 1_g = \widetilde{f}_{gh}^{(j)} 1_g.\end{aligned}$$

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Finally, conditions (2), (3) and (4) for the $\widetilde{f}_g^{(j)}$ follow by combining (2.1) with conditions (2), (3) and (4) for $f_g^{(j)}$. We omit the details. $\square$

Remark 2.5. In the context of Proposition 2.4, one can show that condition (2) can be replaced by:
(2') $f_g^{(j)} f_h^{(j)} = 0$ for all $g, h \in G$ with $g \neq h$ and for all $j = 0, \ldots, d$.
Since we do not need this, we omit its proof. We stress the fact that it is in general *not* possible to replace (1) and (2) *simultaneously* with (1') and (2'), since the argument used to get (1') from (1) does not preserve (2'), and *vice versa*.

We close this section by proving that the finite Rokhlin dimension behaves well with respect to restriction to invariant ideals and passage to equivariant quotients.

PROPOSITION 2.6. *Let A be a C^*-algebra, let G be a finite group, and let α be a partial action of G on A. Let I be a G-invariant ideal of A. We denote by $\alpha|_I$ and $\overline{\alpha}$ the induced partial actions of G on I and A/I, respectively. Then*

$$\dim_{\mathrm{Rok}}(\alpha|_I) \leq \dim_{\mathrm{Rok}}(\alpha) \quad \textit{and} \quad \dim_{\mathrm{Rok}}(\overline{\alpha}) \leq \dim_{\mathrm{Rok}}(\alpha).$$

Similar estimates hold for $\dim_{\mathrm{Rok}}^{\mathrm{c}}$.

Proof. We prove the results for $\dim_{\mathrm{Rok}}$, s ince the case of $\dim_{\mathrm{Rok}}^{\mathrm{c}}$ is similar. We assume from now on that $d = \dim_{\mathrm{Rok}}(\alpha) < \infty$, otherwise there is nothing to prove.

We prove $\dim_{\mathrm{Rok}}(\alpha|_I) \leq \dim_{\mathrm{Rok}}(\alpha)$ first. Let $\varepsilon > 0$ and let a finite subset $F \subseteq I$ be given. Without loss of generality, we assume that F contains only contractions. Set $\varepsilon_0 = \varepsilon/(5|G|(d+1)+2)$. Apply Definition 2.1 to find Rokhlin towers $f_g^{(j)} \in A_g$ with $g \in G$ and $j = 0, \ldots, d$, for (F, ε_0). By making a small renormalization, we may assume that $\sum_{g\in G} \sum_{j=0}^{d} f_g^{(j)}$ is a contraction. By considering an approximate identity of I which is quasi-central in A, find $e \in I$ satisfying

$$\|e^{1/2} f_g^{(j)} - f_g^{(j)} e^{1/2}\| < \varepsilon_0, \quad \|be - b\| < \varepsilon_0, \text{ and } \quad \|eb - b\| < \varepsilon_0$$

for all $g \in G$, all $j = 0, \ldots, d$, and all $b \in \bigcup_{g\in G} \alpha_g(F \cap A_{g^{-1}})$.

For $g \in G$ and $j = 0, \ldots, d$, set $\widetilde{f}_g^{(j)} = e^{1/2} f_g^{(j)} e^{1/2}$. We claim that $\{\widetilde{f}_g^{(j)} : g \in G, j = 0, \ldots, d\}$ are Rokhlin towers with respect to (F, ε) for $\alpha|_I$. Note that $\widetilde{f}_g^{(j)}$ belongs to $A_g \cap I = I_g$. In order to check (1), let $g \in G$, let $x \in F \cap I_{g^{-1}} = F \cap A_{g^{-1}}$, let $a \in F$, and let $j = 0, \ldots, d$. Then

$$\begin{aligned}
\alpha_g(\widetilde{f}_h^{(j)} x)a &= \alpha_g(e^{1/2} f_h^{(j)} e^{1/2} x)a \approx_{\varepsilon_0} \alpha_g(f_h^{(j)} ex)a \\
&\approx_{\varepsilon_0} \alpha_g(f_h^{(j)} x)a \approx_{\varepsilon_0} f_{gh}^{(j)} \alpha_g(x)a \\
&\approx_{\varepsilon_0} f_{gh}^{(j)} e\alpha_g(x)a \approx_{\varepsilon_0} e^{1/2} f_{gh}^{(j)} e^{1/2} \alpha_g(x)a = \widetilde{f}_{gh}^{(j)} \alpha_g(x)a.
\end{aligned}$$

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Thus $\|\alpha_g(\widetilde{f}_h^{(j)}x)a - \widetilde{f}_{gh}^{(j)}\alpha_g(x)a\| < 5\varepsilon_0 < \varepsilon$, as desired. Condition (2) is easily checked and is left to the reader. For (3), let $a \in F$ be given. Then

$$\begin{aligned}\sum_{j=0}^{d}\sum_{g\in G}\widetilde{f}_g^{(j)}a &= \sum_{j=0}^{d}\sum_{g\in G}e^{1/2}f_g^{(j)}e^{1/2}a \\ &\approx_{|G|(d+1)\varepsilon_0}\sum_{j=0}^{d}\sum_{g\in G}f_g^{(j)}ea \\ &\approx_{\varepsilon_0}\sum_{j=0}^{d}\sum_{g\in G}f_g^{(j)}a \approx_{\varepsilon_0} a,\end{aligned}$$

where in the second-to-last step we use the fact that $\sum_{g\in G}\sum_{j=0}^{d} f_g^{(j)}$ is a contraction. Finally, to check (4), let $g \in G$, $j = 0, \ldots, d$, and $a \in F$. Then

$$\widetilde{f}_g^{(j)}a = e^{1/2}f_g^{(j)}e^{1/2}a \approx_{2\varepsilon_0} f_g^{(j)}a \approx_{\varepsilon_0} xf_g^{(j)} \approx_{2\varepsilon_0} xe^{1/2}f_g^{(j)}e^{1/2} = a\widetilde{f}_g^{(j)}.$$

We turn to the inequality $\dim_{\mathrm{Rok}}(\overline{\alpha}) \leq d = \dim_{\mathrm{Rok}}(\alpha)$. Write $\pi\colon A \to A/I$ for the canonical equivariant map. Let $\overline{F} \subseteq A/I$ be a finite set and let $\varepsilon > 0$. Let $F \subseteq A$ be any finite set satisfying $\pi(F) = \overline{F}$, and apply Definition 2.1 for α to find Rokhlin towers $f_g^{(j)} \in A_g$, with $g \in G$ and $j = 0, \ldots, d$, for (F, ε). It is then immediate that the positive contractions $\overline{f}_g^{(j)} = \pi(f_g^{(j)})$ witness the fact that $\dim_{\mathrm{Rok}}(\overline{\alpha}) \leq d$, as desired. □

3. *Rokhlin dimension and globalization*

The basic example of a partial action is obtained by starting with a global action $\beta\colon G \to \mathrm{Aut}(B)$ and a (not necessarily invariant) ideal A in B, and then setting $A_g = A \cap \beta_g(A)$ and $\alpha_g = \beta_g|_{A_{g^{-1}}}$ for all $g \in G$. Actions of this form are called *globalizable*, since they are induced by a global action. Here is the precise definition.

Definition 3.1. Let G be a finite group, let A be a C^*-algebra, and let α be a partial action of G on A. A triple (B, β, ι) consisting of a C^*-algebra B, a global action $\beta\colon G \to \mathrm{Aut}(B)$, and an embedding $\iota\colon A \to B$ is said to be an *enveloping action* for α if the following conditions are satisfied:

(a) A (identified with $\iota(A)$) is an ideal in B;
(b) $A_g = A \cap \beta_g(A)$ for all $g \in G$;
(c) $\alpha_g(a) = \beta_g(a)$ for all $a \in A_{g^{-1}}$ and all $g \in G$;
(d) $B = \overline{\mathrm{span}}\{\beta_g(a)\colon a \in A, g \in G\}$.

(If such a dynamical system (B, β) exists, then it is unique up to an equivariant isomorphism extending the identity on A by Theorem 3.8 in [1].) We say that α is *globalizable* if it has an enveloping action.

Not every partial action is globalizable, and even when it is, identifying its enveloping action may turn out to be challenging. Since there is a vast amount of literature concerning global actions with finite Rokhlin dimension, it would be very useful if one could relate

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the Rokhlin dimension of a (globalizable) partial action to its globalization. Unfortunately, these values do not necessarily agree.

Example 3.2. Set $X = S^1 \setminus \{1\}$ and $U = X \setminus \{-1\}$. Let $\sigma \in \mathrm{Homeo}(U)$ be given by $\sigma(x) = -x$ for all $x \in U$. Denote by γ the partial action of $\mathbb{Z}_2 = \{-1, 1\}$ on $C_0(X)$ determined by σ. Then γ is globalizable, with globalization $\widetilde{\gamma}\colon \mathbb{Z}_2 \to \mathrm{Aut}(C(S^1))$ induced by multiplication by -1.

Let $\delta\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{C} \oplus \mathbb{C})$ be the flip action. Set $\alpha = \gamma \otimes \delta$, which is globalizable with globalization given by $(B, \beta) = (C(S^1) \oplus C(S^1), \widetilde{\gamma} \otimes \delta)$. It is clear that $\dim_{\mathrm{Rok}}(\beta) = 0$, since we may take $p_{-1} = (1, 0)$ and $p_1 = (0, 1)$ in B.

We claim that $\dim_{\mathrm{Rok}}(\alpha) \neq 0$. From now on, we identify X with $(0, 2)$ and U with $(0, 1) \cup (1, 2)$. Arguing by contradiction, set $\varepsilon = 3/16$ and take $I_1 = [\varepsilon, 2 - \varepsilon]$ and $I_{-1} = [\varepsilon, 1 - \varepsilon] \cup [1 + \varepsilon, 2 - \varepsilon]$, let $a \in C_0(X)$ be constant equal to 1 on I_1 and linear otherwise, and let $b = \sigma(b) \in C_0(U) = C_0(X)_{-1}$ be constant equal to 1 on I_{-1} and linear otherwise. Let $f_{-1} = (\xi_{-1}, \eta_{-1}) \in C_0(U) \oplus C_0(U)$ and $f_1 = (\xi_1, \eta_1) \in C_0(X) \oplus C_0(X)$ be a Rokhlin tower for α with respect to $F = \{(a, a), (b, b)\}$ and ε. Then we have:

(a) $|\xi_{-1}(\sigma(x))b(x) - \eta_1(x)b(x)| < \varepsilon$ and $|\eta_{-1}(\sigma(x))b(x) - \xi_1(x)b(x)| < \varepsilon$;

(b) $\xi_{-1}(x)\xi_1(x)a(x) < \varepsilon$ and $\eta_{-1}(x)\eta_1(x)a(x) < \varepsilon$;

(c) $(1 - \xi_{-1}(x) - \xi_1(x))a(x) < \varepsilon$ and $(1 - \eta_{-1}(x) - \eta_1(x))a(x) < \varepsilon$

for all $x \in X$. Upon making a small renormalization, we may assume that

$$\xi_{-1}(x) + \xi_1(x) = 1 = \eta_{-1}(x) + \eta_1(x) \tag{3.1}$$

for all $x \in I_1$. In particular, $\xi_1(1) = 1$. Fix $x \in I_1$. Substituting (3.1) into (b), we get

$$(1 - \xi_1(x))\xi_1(x) < \varepsilon,$$

which yields either $\xi_1(x) > 3/4$ or $\xi_1(x) < 1/4$. Since ξ_1 is continuous, we must have either $\xi_1(I_1) \subseteq (3/4, 1]$ or $\xi_1(I_1) \subseteq [0, 1/4)$, and since $\xi_1(1) = 1$, it must be $\xi_1(I_1) \subseteq (3/4, 1]$ and thus

$$\xi_{-1}(I_1) \subseteq [0, 1/4). \tag{3.2}$$

An identical argument shows that

$$\eta_1(I_1) \subseteq (3/4, 1]. \tag{3.3}$$

Taking now $x \in I_{-1} \subseteq I_1$, and noting that $b(x) = 1$ and $\sigma(x) \in I_{-1}$ as well, we get

$$\frac{3}{16} = \varepsilon \overset{\text{(a)}}{>} |\xi_{-1}(\sigma(x))b(x) - \eta_1(x)b(x)| \overset{3.2,3.3}{>} \frac{3}{4} - \frac{1}{4},$$

which is a contradiction. We conclude that $\dim_{\mathrm{Rok}}(\alpha) > 0$.

We point out that one can construct towers that witness the fact that $\dim_{\mathrm{Rok}}(\alpha) \leq 1$, thus allowing us to conclude that $\dim_{\mathrm{Rok}}(\alpha) = 1$. However, we do not need this, so we omit it.

In contrast with the previous example, we will show in Theorem 3.4 that for globalizable partial actions which act on *unital* C^*-algebras, their Rokhlin dimension (with or without commuting towers) agrees with that of their globalization. The result is by no means obvious and its proof is quite technical. The following lemma, which deals exclusively with global actions, represents the first step in proving the inequality $\dim_{\mathrm{Rok}}(\alpha) \leq \dim_{\mathrm{Rok}}(\beta)$.

LEMMA 3.3. *Let $\beta\colon G \to \mathrm{Aut}(B)$ be an action of a finite group G on a unital C^*-algebra B. Let A be a unital ideal in B satisfying $B = \overline{\mathrm{span}}\{\beta_g(a)\colon a \in A, g \in G\}$. If $d = \dim_{\mathrm{Rok}}(\beta)$ is finite, then there exist Rokhlin towers $f_g^{(j)} \in B$ for β, satisfying condition (1') in Proposition 2.4, such that $f_1^{(0)}, \ldots, f_1^{(d)}$ belong to A. A similar statement holds when $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta) < \infty$.*

Proof. We divide the proof into claims.

Claim 1. There exist projections $p_g \in A$, for $g \in G$, which are central in B and satisfy $1_B = \sum_{g\in G} \beta_g(p_g)$.

Set $n = |G|$ and fix an enumeration $G = \{1_G = g_1, g_2, \ldots, g_n\}$. Since $B = \sum_{g\in G}\beta_g(A)$, we have $1_B \leq \sum_{g\in G} \beta_g(1_A)$. Note that the projections $\beta_g(1_A)$ are central and therefore together with 1_B generate a commutative C^*-algebra. In the rest of this claim, we regard 1_B and $\beta_g(1_A)$, for $g \in G$, as $\{0, 1\}$-valued functions on some compact Hausdorff space and identify them with their supports. In particular, the support of 1_B is equal to the union of the supports of $\beta_g(1_A)$, for $g \in G$. By successively removing the double intersections, adding the triple intersections, and proceeding similarly for higher degrees, we can write 1_B as follows:

$$\begin{aligned} 1_B = &\sum_{i=1}^{n} \beta_{g_i}(1_A) - \sum_{1\leq i<j\leq n} \beta_{g_i}(1_A)\beta_{g_j}(1_A) \\ &+ \sum_{1\leq i<j<k\leq n} \beta_{g_i}(1_A)\beta_{g_j}(1_A)\beta_{g_k}(1_A) + \cdots \\ &+ (-1)^{n-1}\beta_{g_1}(1_A)\beta_{g_2}(1_A)\cdots\beta_{g_n}(1_A). \end{aligned} \tag{3.4}$$

Observe that every element appearing in (3.4) is central in B. We now proceed to write the unit of B as a sum of orthogonal projections in the following manner. The first summand consists of all the elements appearing in (3.4) that have $\beta_{g_1}(1_A)$ as a factor. The second summand consists of all the elements appearing in (3.4) that have $\beta_{g_2}(1_A)$ as a factor but not $\beta_{g_1}(1_A)$. Continue inductively, and note that the process finishes after n steps; indeed, the only element left at the nth step is $\beta_{g_n}(1_A)$.

For $1 \leq k \leq n$, the kth summand is

$$\begin{aligned} P_k = &\beta_{g_k}(1_A) - \sum_{k<j\leq n} \beta_{g_k}(1_A)\beta_{g_j}(1_A) + \sum_{k<i<j\leq n} \beta_{g_k}(1_A)\beta_{g_i}(1_A)\beta_{g_j}(1_A) \\ &+ \cdots + (-1)^{n-k}\beta_{g_k}(1_A)\beta_{g_{k+1}}(1_A)\cdots\beta_{g_n}(1_A), \end{aligned}$$

and we have $1_B = P_1 + \cdots + P_n$. We want to see that P_k is a projection. Set

$$Q_k = \sum_{k<j\leq n} \beta_{g_j}(1_A) - \sum_{k<i<j\leq n} \beta_{g_i}(1_A)\beta_{g_j}(1_A) + \cdots + (-1)^{n-k+1}\prod_{j>k} \beta_{g_j}(1_A).$$

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Then Q_k is the unit of the ideal $\sum_{j>k} \beta_{g_j}(A)$ and therefore Q_k is a projection. Moreover, an easy computation shows that $P_k = \beta_{g_k}(1_A)(1_B - Q_k)$ and thus P_k is also a projection. Note that $P_k P_\ell = 0$ if $k \neq \ell$.

For $k = 1, \ldots, n$, set $p_k = \beta_{g_k}^{-1}(P_k)$, which can be written as

$$p_k = 1_A - \sum_{k<j\leq n} 1_A \beta_{g_k^{-1} g_j}(1_A) + \cdots + (-1)^{(n-k)} 1_A \beta_{g_k^{-1} g_{k+1}}(1_A) \cdots \beta_{g_k^{-1} g_n}(1_A).$$

Then p_k is a central projection, and it belongs to A because 1_A is a factor in each of its summands. Since $1_B = \sum_{k=1}^n \beta_{g_k}(p_k)$, this concludes the proof of Claim 1.

Let $\varepsilon > 0$ and let $F \subseteq B$ be finite. Without loss of generality, F is β-invariant, contains 1_B, and consists of contractions. Set $\varepsilon_0 = \varepsilon/|G|^2(d+2)$, and let $f_g^{(j)} \in B$, for $g \in G$ and $j = 0, \ldots, d$, be Rokhlin towers for (F, ε_0). Using Proposition 2.4 for the equality, and by replacing $f_g^{(j)}$ with $(1/(1+\varepsilon)) f_g^{(j)}$ for the inequality, we may assume

$$\beta_g(f_h^{(j)}) = f_{gh}^{(j)} \quad \text{and} \quad \sum_{g\in G} \sum_{j=0}^{d} f_g^{(j)} \leq 1 \tag{3.5}$$

for all $g, h \in G$ and all $j = 0, \ldots, d$.

Claim 2. There are positive elements $x_g^{(j)} \in A$, for $g \in G$ and $j = 0, \ldots, d$, for which:

(2.a) $f_1^{(j)} = \sum_{g\in G} \beta_g(x_g^{(j)})$;

(2.b) $\sum_{g\in G} x_g^{(j)} \in A$ is a positive contraction;

(2.c) $\beta_h(x_g^{(j)}) b \approx_{\varepsilon_0} b\beta_h(x_g^{(j)})$ for all $b \in F$, all $g, h \in G$ and all $j = 0, \ldots, d$; and

(2.d) $\|x_h^{(j)} \beta_g(x_t^{(j)})\| < \varepsilon_0$ for all $j = 0, \ldots, d$ and all $g, h, t \in G$ with $g \neq 1$.

Using Claim 1, fix projections $p_g \in A$, for $g \in G$, which are central in B and satisfy $1_B = \sum_{g\in G} \beta_g(p_g)$. For $j = 0, \ldots, d$, multiply both sides of the identity by $f_1^{(j)}$ to get

$$f_1^{(j)} = \sum_{g\in G} \beta_g(\beta_{g^{-1}}(f_1^{(j)}) p_g) = \sum_{g\in G} \beta_g(f_{g^{-1}}^{(j)} p_g).$$

Set $x_g^{(j)} = f_{g^{-1}}^{(j)} p_g$ for all $g \in G$ and $j = 0, \ldots, d$. Since p_g is central in B and belongs to A, it is clear that $x_g^{(j)}$ is a positive element in A. Condition (2.a) is satisfied by construction. Using centrality of p_g at the second step, we get

$$\sum_{g\in G} x_g^{(j)} = \sum_{g\in G} f_{g^{-1}}^{(j)} p_g \leq \sum_{g\in G} f_g^{(j)} \leq \sum_{j=0}^{d} \sum_{g\in G} f_g^{(j)} \overset{(3.5)}{\leq} 1,$$

and thus $\sum_{g\in G} x_g^{(j)}$ is a positive contraction, verifying (2.b). In order to check (2.c), let $g, h \in G$ and $j = 0, \ldots, d$. Since $\beta_h(x_g^{(j)}) = f_{hg^{-1}}^{(j)} \beta_h(p_g)$ and p_g is central, it follows that

$$\|\beta_h(x_g^{(j)}) b - b\beta_h(x_g^{(j)})\| \leq \|f_{hg^{-1}}^{(j)} b - b f_{hg^{-1}}^{(j)}\| < \varepsilon_0$$

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for all $b \in F$. We turn to (2.d). Given $j = 0, \ldots, d$ and $g, h, t \in G$ with $g \neq 1$, we have

$$x_h^{(j)} \beta_g(x_t^{(j)}) = f_{h^{-1}}^{(j)} p_h f_{gt^{-1}}^{(j)} \beta_g(p_t) = f_{h^{-1}}^{(j)} f_{gt^{-1}}^{(j)} \beta_g(p_t) p_h.$$

If $gt^{-1} \neq h^{-1}$ then $\|f_{h^{-1}}^{(j)} f_{gt^{-1}}^{(j)}\| < \varepsilon_0$ and hence $\|x_h^{(j)} \beta_g(x_t^{(j)})\| < \varepsilon_0$. Otherwise, we have $gt^{-1} = h^{-1}$ and thus $g = h^{-1}t$. In particular, $h \neq t$. Then

$$\|\beta_g(p_t) p_h\| = \|\beta_{h^{-1}t}(p_k) p_h\| = \|\beta_t(p_t) \beta_h(p_h)\| = 0,$$

since $\beta_t(p_t)$ is orthogonal to $\beta_h(p_h)$ by Claim 1. It follows that $x_h^{(j)} \beta_g(x_t^{(j)}) = 0$, and thus (2.d) is satisfied. This proves Claim 2.

Let $x_g^{(j)} \in A$, for $g \in G$ and $j = 0, \ldots, d$, be positive elements satisfying the conclusion of Claim 2. For $j = 0, \ldots, d$, set $a_1^{(j)} = \sum_{g \in G} x_g^{(j)} \in A$. For $g \in G$, we set $a_g^{(j)} = \beta_g(a_1^{(j)})$. Then $\beta_g(a_h^{(j)}) = a_{gh}^{(j)}$ for all $g, h \in G$ and $j = 0, \ldots, d$. In particular, condition (1) in Definition 2.1 is satisfied. To check condition (2), let $j = 0, \ldots, d$ and $g, h \in G$ with $g \neq h$. Then

$$\|a_g^{(j)} a_h^{(j)}\| = \|a_1^{(j)} \beta_{g^{-1}h}(a_1^{(j)})\| = \left\| \sum_{s,t \in G} x_s^{(j)} \beta_{g^{-1}h}(x_t^{(j)}) \right\| \overset{(2.d)}{\leq} |G|^2 \varepsilon_0 < \varepsilon.$$

Moreover, condition (3) follows from the following identity:

$$\begin{aligned}
\sum_{g \in G} \sum_{j=0}^{d} a_g^{(j)} &= \sum_{g \in G} \sum_{j=0}^{d} \beta_g\Big(\sum_{h \in G} x_h^{(j)} \Big) = \sum_{g,h \in G} \sum_{j=0}^{d} \beta_g(x_h^{(j)}) \\
&= \sum_{g,h \in G} \sum_{j=0}^{d} \beta_{gh}(x_h^{(j)}) = \sum_{g \in G} \sum_{j=0}^{d} \beta_g\Big(\sum_{h \in G} \beta_h(x_h^{(j)}) \Big) \\
&= \sum_{g \in G} \sum_{j=0}^{d} \beta_g(f_1^{(j)}) = \sum_{g \in G} \sum_{j=0}^{d} f_g^{(j)}.
\end{aligned}$$

Let $b \in F$, $g \in G$ and $j = 0, \ldots, d$. In order to check condition (4) in Definition 2.1, and since $a_g^{(j)} = \beta_g(a_1^{(j)})$ and F is β-invariant, it suffices to take $g = 1$. In this case, we have

$$\|a_1^{(j)} b - b a_1^{(j)}\| \leq \sum_{g \in G} \|x_g^{(j)} b - b x_g^{(j)}\| \overset{(2.c)}{\leq} |G| \varepsilon_0 < \varepsilon.$$

This proves the first part of the lemma.

Assume now that $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta) = d < \infty$, and choose the Rokhlin towers as above to moreover satisfy condition (5) in Definition 2.1. For $g, h \in G$ and $j, k = 0, \ldots, d$, we get

$$\begin{aligned} a_g^{(j)} a_h^{(k)} &= \beta_g(a_1^{(j)}) \beta_h(a_1^{(k)}) = \sum_{t,s \in G} \beta_g(x_t^{(j)}) \beta_h(x_s^{(k)}) \\ &= \sum_{t,s \in G} \beta_g(f_{t^{-1}}^{(j)} p_t) \beta_h(f_{s^{-1}}^{(k)} p_s) = \sum_{t,s \in G} f_{gt^{-1}}^{(j)} \beta_g(p_t) f_{hs^{-1}}^{(k)} \beta_h(p_s) \\ &\approx_{|G|^2 \varepsilon_0} \sum_{t,s \in G} f_{hs^{-1}}^{(k)} \beta_h(p_s) f_{gt^{-1}}^{(j)} \beta_g(p_t) = a_h^{(k)} a_g^{(j)}. \end{aligned}$$

□

We are now ready to prove the main result of this section: the Rokhlin dimension of a globalizable partial action on a unital C^*-algebra equals the Rokhlin dimension of its globalization. In particular, we obtain a large family of examples of partial actions with finite Rokhlin dimension.

THEOREM 3.4. *Let α be a globalizable partial action of a finite group G on a unital C^*-algebra A, and let $\beta \colon G \to \mathrm{Aut}(B)$ denote its globalization. Then*

$$\dim_{\mathrm{Rok}}(\alpha) = \dim_{\mathrm{Rok}}(\beta) \quad \textit{and} \quad \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) = \dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta).$$

Proof. The proofs for $\dim_{\mathrm{Rok}}$ and $\dim_{\mathrm{Rok}}^{\mathrm{c}}$ are very similar, so we provide full details for $\dim_{\mathrm{Rok}}$ and indicate how one modifies the proof to obtain the result for $\dim_{\mathrm{Rok}}^{\mathrm{c}}$. We divide the proof into two parts, namely the inequalities $\dim_{\mathrm{Rok}}(\alpha) \leq \dim_{\mathrm{Rok}}(\beta)$ and $\dim_{\mathrm{Rok}}(\alpha) \geq \dim_{\mathrm{Rok}}(\beta)$. Since A is unital and α is globalizable, it follows that A_g is unital for all $g \in G$, with unit given by $1_g = 1_A \beta_g(1_A)$.

To show that $\dim_{\mathrm{Rok}}(\alpha) \leq \dim_{\mathrm{Rok}}(\beta)$, it suffices to assume that $d = \dim_{\mathrm{Rok}}(\beta)$ is finite. Let $\varepsilon > 0$ and $F \subseteq A$ be given. Using Lemma 3.3, let $\widetilde{f}_g^{(j)} \in B$, for $g \in G$ and $j = 0, \ldots, d$, such that:

(a) $\beta_g(\widetilde{f}_h^{(j)}) = \widetilde{f}_{gh}^{(j)}$ for all $g, h \in G$ and $j = 0, \ldots, d$;

(b) $\|\widetilde{f}_g^{(j)} \widetilde{f}_h^{(j)}\| < \varepsilon$ for all $j = 0, \ldots, d$ and all $g, h \in G$ with $g \neq h$;

(c) $\|1_B - \sum_{g \in G} \sum_{j=0}^{d} \widetilde{f}_g^{(j)}\| \leq \varepsilon$;

(d) $\|\widetilde{f}_g^{(j)} b - b \widetilde{f}_g^{(j)}\| < \varepsilon$ for all $g \in G$, all $j = 0, \ldots, d$ and all $b \in F$.

Set $f_g^{(j)} = \widetilde{f}_g^{(j)} 1_g \in A_g$ for all $g \in G$ and all $j = 0, \ldots, d$. We claim that these are Rokhlin towers for α with respect to (F, ε). For $g, h \in G$ and $j = 0, \ldots, d$, we use at the second step that $\beta_g|_{A_{g^{-1}}} = \alpha_g$ to get

$$\alpha_g(f_h^{(j)} 1_{g^{-1}}) = \alpha_g(\widetilde{f}_h^{(j)} 1_h 1_{g^{-1}}) = \beta_g(\widetilde{f}_h^{(j)}) 1_{gh} 1_g \overset{(a)}{=} \widetilde{f}_{gh}^{(j)} 1_{gh} 1_g = f_{gh}^{(j)} 1_g,$$

thus verifying condition (1) in Definition 2.1. Moreover, $\|f_g^{(j)} f_h^{(j)}\| \leq \|\widetilde{f}_g^{(j)} \widetilde{f}_h^{(j)}\|$ and hence condition (2) is also satisfied by (b) above. Since

$$\sum_{g \in G} \sum_{j=0}^{d} f_g^{(j)} = \sum_{g \in G} \sum_{j=0}^{d} \beta_g(\widetilde{f}_1^{(j)}) 1_g = \sum_{g \in G} \sum_{j=0}^{d} \beta_g(\widetilde{f}_1^{(j)} 1_A) \beta_g(1_A) 1_A = \sum_{g \in G} \sum_{j=0}^{d} \widetilde{f}_g^{(j)} 1_A,$$

it follows from (c) that condition (3) is also satisfied. Finally, given $b \in F$, $g \in G$ and $j = 0, \ldots, d$, we have

$$\|f_g^{(j)}b - bf_g^{(j)}\| = \|\widetilde{f}_g^{(j)}b1_g - b\widetilde{f}_g^{(j)}1_g\| \leq \|\widetilde{f}_g^{(j)}b - b\widetilde{f}_g^{(j)}\| \overset{(d)}{=} \varepsilon,$$

establishing condition (4). It follows that $\dim_{\mathrm{Rok}}(\alpha) \leq d$, as desired. Note that

$$\|f_g^{(j)}f_h^{(k)} - f_h^{(k)}f_g^{(j)}\| \leq \|\widetilde{f}_g^{(j)}\widetilde{f}_h^{(k)} - \widetilde{f}_h^{(k)}\widetilde{f}_g^{(j)}\|$$

for all $g, h \in G$ and $j, k = 0, \ldots, d$. Thus, if $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta) \leq d$ and the Rokhlin towers $\widetilde{f}_g^{(j)}$ for β also satisfy condition (5) in Definition 2.1, then the Rokhlin towers $f_g^{(j)}$ for α also satisfy (5) and hence $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq d$.

We turn to the inequality $\dim_{\mathrm{Rok}}(\beta) \leq \dim_{\mathrm{Rok}}(\alpha)$, so we set $d = \dim_{\mathrm{Rok}}(\alpha)$ and assume that $d < \infty$. Let $\varepsilon > 0$ and let $F \subseteq B$ be a finite subset. Without loss of generality, we assume that F contains 1_g for all $g \in G$ and that it is β-invariant. Since B is generated by the β-translations of A (condition (d) in Definition 3.1), there exist $a_g \in A$ for $g \in G$ such that $1_B = \sum_{g \in G} \beta_g(a_g)$. Set $\varepsilon_0 = \varepsilon/|G| \max_{g \in G} \|a_g\|$. Using Proposition 2.4, let $f_g^{(j)} \in A_g$, for $g \in G$ and $j = 0, \ldots, d$, be positive contractions satisfying the following conditions:

(i) $\alpha_g(f_h^{(j)}1_{g^{-1}}) = f_{gh}^{(j)}1_g$ for all $g, h \in G$ and $j = 0, \ldots, d$;

(ii) $\|f_g^{(j)}f_h^{(j)}\| < \varepsilon$ for all $j = 0, \ldots, d$ and all $g, h \in G$ with $g \neq h$;

(iii) $\|\sum_{g \in G}\sum_{j=0}^{d} f_g^{(j)} - 1_A\| < \varepsilon$.

(iv) $\|f_g^{(j)}b - bf_g^{(j)}\| < \varepsilon$ for all $g \in G$, all $j = 0, \ldots, d$ and all $b \in F$.

For $g \in G$ and $j = 0, \ldots, d$, set $\widetilde{f}_g^{(j)} = \beta_g(f_1^{(j)}) \in B$. We claim that the $\widetilde{f}_g^{(j)}$ are Rokhlin towers for β with respect to (F, ε). Condition (1) in Definition 2.1 is clearly satisfied. In order to check (2), let $j = 0, \ldots, d$ and $g, h \in G$ with $g \neq h$ be given. Using that $f_1^{(j)} = f_1^{(j)}1_A$ at the second step, and that $1_{g^{-1}h} = 1_A\beta_{g^{-1}h}(1_A)$ at the third, we get

$$\begin{aligned}\|\widetilde{f}_g^{(j)}\widetilde{f}_h^{(j)}\| &= \|\beta_g(f_1^{(j)})\beta_h(f_1^{(j)})\| = \|f_1^{(j)}1_A\beta_{g^{-1}h}(1_Af_1^{(j)})\| \\ &= \|f_1^{(j)}1_{g^{-1}h}\beta_{g^{-1}h}(f_1^{(j)})\| = \|f_1^{(j)}\alpha_{g^{-1}h}(1_{h^{-1}g}f_1^{(j)})\| \\ &\overset{\text{(i)}}{=} \|f_1^{(j)}f_{g^{-1}h}^{(j)}\| \overset{\text{(ii)}}{<} \varepsilon_0.\end{aligned}$$

To check (3), it suffices to show that for any $a \in A$ and $h \in G$, we have

$$\left\|\sum_{j=0}^{d}\sum_{g \in G}\widetilde{f}_g^{(j)}\beta_h(a) - \beta_h(a)\right\| < \|a\|\varepsilon_0.$$

Indeed, once this is established, and since $1_B = \sum_{h \in G}\beta_h(a_h)$, it will follow that

$$\left\|\sum_{j=0}^{d}\sum_{g \in G}\widetilde{f}_g^{(j)} - 1_B\right\| < |G|\max_{g \in G}\|a_g\|\varepsilon_0 = \varepsilon,$$

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thus establishing (3). Let $a \in A$ and $h \in G$ be given; without loss of generality we assume that $\|a\| \leq 1$. Then:

$$\begin{aligned}\sum_{j=0}^{d}\sum_{g\in G}\widetilde{f}_g^{(j)}\beta_h(a) &= \sum_{j=0}^{d}\sum_{g\in G}\beta_g(f_1^{(j)})\beta_h(a)\\ &= \sum_{j=0}^{d}\sum_{g\in G}\beta_h(\beta_{h^{-1}g}(f_1^{(j)}1_A)1_A a)\\ &= \sum_{j=0}^{d}\sum_{g\in G}\beta_h(\beta_{h^{-1}g}(f_1^{(j)}1_A\beta_{g^{-1}h}(1_A))a)\\ &= \sum_{j=0}^{d}\sum_{g\in G}\beta_h(\alpha_{h^{-1}g}(f_1^{(j)}1_{g^{-1}h})a)\\ &\overset{\text{(i)}}{=} \sum_{j=0}^{d}\sum_{g\in G}\beta_h(f_{h^{-1}g}^{(j)}a)\\ &\overset{\text{(iii)}}{\approx}_{\varepsilon_0}\beta_h(a),\end{aligned}$$

as desired. Finally, to check condition (4), let $a \in F$, $g \in G$ and $j = 0, \ldots, d$ be given. Using at the last step that $\beta_{g^{-1}}(a) \in F$, we get

$$\begin{aligned}\|\widetilde{f}_g^{(j)}a - a\widetilde{f}_g^{(j)}\| &= \|\beta_g(f_1^{(j)})a - a\beta_g(f_1^{(j)})\|\\ &= \|f_1^{(j)}\beta_{g^{-1}}(a) - \beta_{g^{-1}}(a)f_1^{(j)}\| \overset{\text{(iv)}}{<} \varepsilon_0.\end{aligned}$$

This shows that $\dim_{\mathrm{Rok}}(\beta) \leq \dim_{\mathrm{Rok}}(\alpha)$. Observe that the Rokhlin towers for β that we constructed satisfy the following identity for all $j, k = 0, \ldots, d$ and $g, h \in G$:

$$\begin{aligned}\widetilde{f}_g^{(j)}\widetilde{f}_h^{(k)} &= \beta_g(f_1^{(j)})\beta_h(f_1^{(k)}) = \beta_g(f_1^{(j)}1_A\beta_{g^{-1}h}(1_A f_1^{(k)}))\\ &= \beta_g(f_1^{(j)}\alpha_{g^{-1}h}(1_{h^{-1}g}f_1^{(k)})) = \beta_g(f_1^{(j)}f_{g^{-1}h}^{(k)}).\end{aligned}$$

By taking adjoints, we also get $\widetilde{f}_h^{(k)}\widetilde{f}_g^{(j)} = \beta_g(f_{g^{-1}h}^{(k)}f_1^{(j)})$. In particular,

$$\|\widetilde{f}_g^{(j)}\widetilde{f}_h^{(k)} - \widetilde{f}_h^{(k)}\widetilde{f}_g^{(j)}\| \leq \|f_1^{(j)}f_{g^{-1}h}^{(k)} - f_{g^{-1}h}^{(k)}f_1^{(j)}\|.$$

Thus, if $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq d$ and the Rokhlin towers $f_g^{(j)}$ for α also satisfy condition (5) in Definition 2.1, then the Rokhlin towers $\widetilde{f}_g^{(j)}$ for β also satisfy (5) and hence $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta) \leq d$. □

4. *Structure of the crossed product*

Since its introduction in [18], the Rokhlin dimension has predominantly been used to study structural properties of the associated crossed products. Of greatest relevance are those properties related to the classification programme for nuclear C^*-algebras, such as the UCT, finiteness of the nuclear dimension, absorption of a strongly self-absorbing C^*-algebra [24], or divisibility properties on K-theory.

In this section, we explore the structure of the crossed products and fixed point algebras of partial actions with finite Rokhlin dimension. Our approach makes use of the decomposition property introduced and studied in [2], which we recall in §4.1. For unital partial actions, a more direct argument can be given, which even yields better results (notably in the zero-dimensional case). In this setting, we show that the crossed product is Morita equivalent to the fixed point algebra, a fact that fails for general partial actions of finite Rokhlin dimension.

4.1. *The decomposition property.* An important ingredient in our study of partial actions with finite Rokhlin dimension is our previous work [2] on decomposable partial actions. For the convenience of the reader, we make here a small digression.

Definition 4.1. Let G be a finite group. Given $n = 1, \ldots, |G|$, we define the *space of n-tuples* of G to be

$$\mathcal{T}_n(G) = \{\tau \subseteq G \colon 1 \in \tau \text{ and } |\tau| = n\}.$$

For $g \in G$, we set $\mathcal{T}_n(G)_g = \{\tau \in \mathcal{T}_n(G) \colon g \in \tau\}$. There is a canonical partial action $\mathtt{Lt}$ of G on $\mathcal{T}_n(G)$, with $\mathtt{Lt}_g \colon \mathcal{T}_n(G)_{g^{-1}} \to \mathcal{T}_n(G)_g$ induced by left translation by g. For $\tau \in \mathcal{T}_n(G)$, we write $G \cdot \tau \subseteq \mathcal{T}_n(G)$ for the orbit of τ with respect to $\mathtt{Lt}$.

We will adopt the following convention. Let α be a partial action of a finite group G on a C^*-algebra A, and let $n = 1, \ldots, |G|$. For $\tau \in \mathcal{T}_n(G)$, we write A_τ for the ideal $A_\tau = \bigcap_{g\in\tau} A_g$. Then $\alpha_g(A_\tau) = A_{g\tau}$ for $g \in G$ and $\tau \in \mathcal{T}_n(G)_{g^{-1}}$. For $\tau \in \mathcal{T}_n(G)$, we set $A_{G\cdot\tau} = \sum_{g\in\tau^{-1}} A_{g\tau}$. When $A = C_0(X)$ for a locally compact Hausdorff space X, we write X_τ for the spectrum of $C_0(X)_\tau$, and identify it canonically with $\bigcap_{g\in\tau} X_g$. We use similar notation for $X_{G\cdot\tau}$.

Definition 4.2. Let G be a finite group, let A be a C^*-algebra, and let $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ be a partial action of G on A. Given $n = 1, \ldots, |G|$, we say that α has the *n-decomposition property* if:

(a) $A = \overline{\sum_{\tau\in\mathcal{T}_n(G)} A_\tau}$; and

(b) $A_\tau \cap A_g = \{0\}$ for all $\tau \in \mathcal{T}_n(G)$ and all $g \in G$ such that $g \notin \tau$.

We say that α is *decomposable* if it has the n-decomposition property for some $n \in \mathbb{N}$. A partial action on a locally compact space X is said to have the *n-decomposition property* if the induced partial action on $C_0(X)$ has it.

Notation 4.3. Adopt the notation from Definition 4.2. For $\tau \in \mathcal{T}_n(G)$, we set $H_\tau = \{h \in G \colon h\tau = \tau\}$ and $m_\tau = (n/(|H_\tau|)) - 1$. Using Lemma 2.8 in [2], we fix elements $x_0^\tau = 1, x_1^\tau, \ldots, x_{m_\tau}^\tau \in G$ satisfying

$$\tau = H_\tau \sqcup H_\tau x_1^\tau \sqcup \cdots \sqcup H_\tau x_{m_\tau}^\tau.$$

Whenever τ is understood from the context, we will omit it from the notation for H_τ, m_τ and x_j^τ, for $j = 1, \ldots, m_\tau$. Let $\mathcal{O}_n(G)$ be the orbit space for the partial system described in Definition 4.1. We denote by $\kappa \colon \mathcal{T}_n(G) \to \mathcal{O}_n(G)$ the canonical quotient map and fix,

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for the rest of this work, a global section $\eta \colon \mathcal{O}_n(G) \to \mathcal{T}_n(G)$ for it. For $z \in \mathcal{O}_n(G)$, we write τ_z for $\eta(z)$, H_z for H_{τ_z}, and m_z for m_{τ_z}.

The following is part of Proposition 2.11 in [2].

PROPOSITION 4.4. *Let G be a finite group, let A be a C^*-algebra, let $n = 1, \ldots, |G|$, let α be a partial action of G on A with the n-decomposition property, and let $\tau \in \mathcal{T}_n(G)$. Adopt the conventions from Notation 4.3. Then:*

(1) *the restriction of $\alpha|_{H_\tau}$ to A_τ is a global action;*

(2) *there is a natural G-equivariant isomorphism*

$$\varphi\colon \bigoplus_{z\in\mathcal{O}_n(G)} A_{G\cdot\tau_z} \to A$$

given by $\varphi(a) = \sum_{z\in\mathcal{O}_n(G)} a_z$ for all $a = (a_z)_{z\in\mathcal{O}_n(G)}$.

By part (2) above, many facts about decomposable partial actions can be reduced to the G-invariant direct summands $A_{G\cdot\tau}$. In particular, for many purposes it suffices to work with a single $\tau \in \mathcal{T}_n(G)$ and the induced partial action on $A_{G\cdot\tau}$.

Next, we recall Theorem 6.1 from [2], which asserts that every partial action of a finite group is canonically an iterated extension of decomposable partial actions. It follows that many aspects of partial actions of finite groups can be reduced to the case of decomposable partial actions, as long as one has control over the resulting equivariant extension problem (which is in general quite complicated).

THEOREM 4.5. *Let G be a finite group, let A be a C^*-algebra, and let α be a partial action of G on A. Then there are canonical equivariant extensions*

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0,$$

for $2 \leq k \leq |G|$, satisfying the following properties:

(a) *$A^{(|G|)} = A$ and $\alpha^{|G|} = \alpha$;*

(b) *$\delta^{(k)}$ has the k-decomposition property;*

(c) *$\alpha^{(1)}$ has the 1-decomposition property.*

We close this subsection by showing that the Rokhlin dimension of a decomposable partial action can be computed in terms of the global subsystems $H_\tau \curvearrowright A_\tau$.

THEOREM 4.6. *Let G be a finite group, let A be a C^*-algebra, let $n = 1, \ldots, |G|$, and let α be a partial action of a finite group G on A with the n-decomposition property. Fix $\tau \in \mathcal{T}_n(G)$. Then*

$$\dim_{\mathrm{Rok}}(\alpha|_{H_\tau}) = \dim_{\mathrm{Rok}}(\alpha|_{A_{G\cdot\tau}}) \quad \textit{and} \quad \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha|_{H_\tau}) = \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha|_{A_{G\cdot\tau}}).$$

Consequently,

$$\dim_{\mathrm{Rok}}(\alpha) = \max_{\tau\in\mathcal{T}_n(G)} \dim_{\mathrm{Rok}}(\alpha|_{H_\tau}) \quad \textit{and} \quad \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) = \max_{\tau\in\mathcal{T}_n(G)} \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha|_{H_\tau}).$$

Proof. We begin by observing that the last two identities are consequences of the first two, by part (2) of Proposition 4.4.

We give a proof for the first equality; the proof for $\dim_{\mathrm{Rok}}^{\mathrm{c}}$ is similar. We start by showing $\dim_{\mathrm{Rok}}(\alpha|_{A_{G\cdot\tau}}) \leq \dim_{\mathrm{Rok}}(\alpha|_{H_\tau})$. For this we set $d = \dim_{\mathrm{Rok}}(\alpha|_{H_\tau})$ and assume that $d < \infty$. Adopt Notation 4.3, and fix $x_0 = 1, x_1, \ldots, x_m \in G$ with

$$\tau = H_\tau \sqcup H_\tau x_1 \sqcup \cdots \sqcup H_\tau x_m.$$

Let $F \subseteq A_{G\cdot\tau}$ be a finite subset consisting of contractions, and let $\varepsilon > 0$. Since $A_{G\cdot\tau}$ can be canonically identified with $\bigoplus_{\ell=0}^{m} A_{x_\ell^{-1}\tau}$, we may assume that F can be written as a disjoint union $F = F_0 \sqcup \cdots \sqcup F_m$, where $F_\ell \subseteq A_{x_\ell^{-1}\tau}$ for every $\ell = 0, \ldots, m$. Set $K = \bigcup_{\ell=1}^{m} \alpha_{x_\ell}(F_\ell) \subseteq A_\tau$, and let $\xi_h^{(j)} \in A_\tau$, for $h \in H_\tau$ and $j = 0, \ldots, d$, be Rokhlin towers for $\alpha|_{H_\tau}$ with respect to (K, ε). (Note that $\xi_h^{(j)} \neq 0$ for all $h \in H_\tau$ and all $j = 0, \ldots, d$.) For $g \in G$ and $j = 0, \ldots, d$, we set

$$f_g^{(j)} = \begin{cases} \alpha_{x_\ell^{-1}}(\xi_h^{(j)}) & \text{if } g = x_\ell^{-1}h \text{ for some } \ell = 0, \ldots, m \text{ and } h \in H_\tau, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $f_g^{(j)}$ is well defined (because $x_\ell^{-1}H_\tau \cap x_r^{-1}H_\tau = \emptyset$ if $\ell \neq r$) and that it is a positive contraction in $A_{G\cdot\tau} \cap A_g$. We claim that the $f_g^{(j)}$ satisfy the conditions in Definition 2.1 and thus witness the fact that $\dim_{\mathrm{Rok}}(\alpha|_{A_{G\cdot\tau}}) \leq \dim_{\mathrm{Rok}}(\alpha|_{H_\tau})$.

In order to check (1), let $g, k \in G$, $j = 0, \ldots, d$, let $\ell = 0, \ldots, m$, and let $x \in A_{g^{-1}} \cap F_\ell$. We need to show that

$$\|\alpha_g(f_k^{(j)}x) - f_{gk}^{(j)}\alpha_g(x)\| < \varepsilon. \tag{4.1}$$

Since by the decomposition property (specifically, condition (b) in Definition 4.2), the element $x \in A_{g^{-1}} \cap F_\ell$ is zero (and hence the inequality holds trivially) whenever $g^{-1} \notin x_\ell^{-1}\tau$, it suffices to assume that there exist unique $h \in H_\tau$ and $r = 0, \ldots, m$ such that $g = x_r^{-1}hx_\ell$. By construction, we have $f_k^{(j)} = 0$ unless $k \in \tau^{-1}$, and similarly $f_{gk}^{(j)} = 0$ unless $gk \in \tau^{-1}$. We accordingly divide the proof into three cases.

Case 1. $k \notin \tau^{-1}$, so that $f_k^{(j)} = 0$. We claim that $f_{gk}^{(j)}\alpha_g(x) = 0$. Arguing by contradiction, assume that the product $f_{gk}^{(j)}\alpha_g(x)$, which belongs to $A_{gk} \cap A_\tau$, is non-zero. By the intersection property, we must have $gk \in \tau$ and thus there are unique $s = 0, \ldots, m$ and $h_1 \in H_\tau$ with $gk = x_s^{-1}h_1$. Since $f_{gk}^{(j)}\alpha_g(x)$, which belongs to $A_{x_r^{-1}\tau} \cap A_{x_s^{-1}\tau}$, is nonzero, the decomposition property implies that $r = s$. Thus

$$k = g^{-1}x_s^{-1}h_1 = x_t^{-1}hx_rx_s^{-1}h_1 = x_\ell^{-1}hh_1 \in x_\ell^{-1}H \subseteq \tau^{-1},$$

contradicting our assumption. This verifies (4.1) in this case.

Case 2. $gk \notin \tau^{-1}$, so that $f_{gk}^{(j)} = 0$. We claim that $\alpha_g(f_k^{(j)}x) = 0$. Arguing by contradiction, assume that the product $f_k^{(j)}x$, which belongs to $A_k \cap A_{x_\ell^{-1}\tau}$, is non-zero. By the intersection property, we must have $k \in x_\ell^{-1}\tau$, and thus there is $h_2 \in H_\tau$ with $k = x_\ell^{-1}h_2$. Thus

$$gk = x_r^{-1}hx_\ell x_\ell^{-1}h_2 = x_r^{-1}hh_2 \in x_r^{-1}H \subseteq \tau^{-1},$$

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contradicting our assumption. This verifies (4.1) in this case.

Case 3. $gk, k \in \tau^{-1}$. Then there exists $\widetilde{h} \in H_\tau$ with $k = x_\ell^{-1}\widetilde{h}$, so that $gk = x_r^{-1}h\widetilde{h}$. Thus,

$$\|\alpha_g(f_k^{(j)}x) - f_{gk}^{(j)}\alpha_g(x)\| = \| \underbrace{\alpha_{x_r^{-1}hx_\ell}(\alpha_{x_\ell^{-1}}(\xi_{\widetilde{h}}^{(j)})x)}_{\approx_\varepsilon\ \alpha_{x_r^{-1}}(\xi_{h\widetilde{h}}^{(j)})\alpha_g(x)} - \alpha_{x_r^{-1}}(\xi_{h\widetilde{h}}^{(j)})\alpha_g(x)\| < \varepsilon,$$

verifying (4.1) also in this case.

We turn to condition (2) in Definition 2.1 Let $g, k \in G$ with $g \neq k$, let $j = 0, \ldots, d$, let $\ell = 0, \ldots, m$, and let $a \in F_\ell$. We need to show that $f_g^{(j)} f_k^{(j)} a$ has norm at most ε. We may assume that there exist $r, s = 0, \ldots, m$ and $h_1, h_2 \in H_\tau$ with $g = x_r^{-1}h_1$ and $k = x_s^{-1}h_2$ (or else either $f_g^{(j)} = 0$ or $f_k^{(j)} = 0$). Additionally, since the product $f_g^{(j)} f_k^{(j)} a$ belongs to $A_g \cap A_k \cap A_{x_\ell^{-1}\tau}$, we may assume that $r = s = \ell$ (or else $f_g^{(j)} f_k^{(j)} a = 0$). In this case, we have

$$f_g^{(j)} f_k^{(j)} a = \alpha_{x_\ell^{-1}}(\xi_{h_1}^{(j)})\alpha_{x_\ell^{-1}}(\xi_{h_2}^{(j)})a = \alpha_{x_\ell^{-1}}(\underbrace{\xi_{h_1}^{(j)}\xi_{h_2}^{(j)}\alpha_{x_\ell}(a)}_{\approx_\varepsilon\ 0}) \approx_\varepsilon 0,$$

as desired. In order to check condition (3), let $\ell = 0, \ldots, m$ and $a \in F_\ell$. Using at the second step that a product of the form $\alpha_{x_r^{-1}}(\xi_h^{(j)})a$ is zero unless $r = \ell$, we have

$$\begin{aligned}
\sum_{j=0}^{d}\sum_{g\in G} f_g^{(j)} a &= \sum_{j=0}^{d}\sum_{h\in H_\tau}\sum_{r=0}^{m} \alpha_{x_r^{-1}}(\xi_h^{(j)})a \\
&= \sum_{j=0}^{d}\sum_{h\in H_\tau} \alpha_{x_\ell^{-1}}(\xi_h^{(j)})a \\
&= \sum_{j=0}^{d}\sum_{h\in H_\tau} \alpha_{x_\ell^{-1}}(\xi_h^{(j)}\alpha_{x_\ell}(a)) \\
&= \alpha_{x_\ell^{-1}}\left(\sum_{j=0}^{d}\sum_{h\in H_\tau} \xi_h^{(j)}\alpha_{x_\ell}(a)\right) \approx_\varepsilon \alpha_{x_\ell}(a),
\end{aligned}$$

as desired. Finally, to check condition (4), let $g \in G$, $j = 0, \ldots, d$, $\ell = 0, \ldots, m$, and $a \in F_\ell$ be given. Since $f_g^{(j)} = 0$ unless $g \in \tau^{-1}$, we may assume that there are $h \in H_\tau$ and $s = 0, \ldots, m$ such that $g = x_s^{-1}h$. Since the products $f_g^{(j)}a$ and $af_g^{(j)}$ are both zero unless $g \in x_\ell^{-1}\tau$, we may assume that $s = \ell$. For $b \in F$, we have

$$\begin{aligned}
\|(f_g^{(j)}a - af_g^{(j)})b\| &= \|(\alpha_{x_\ell^{-1}}(\xi_h^{(j)})a - a\alpha_{x_\ell^{-1}}(\xi_h^{(j)}))b\| \\
&= \|(\xi_h^{(j)}\alpha_{x_\ell}(a) - \alpha_{x_\ell}(a)\xi_h^{(j)})\alpha_{x_\ell}(b)\| < \varepsilon.
\end{aligned}$$

This completes the proof that $\dim_{\mathrm{Rok}}(\alpha|_{A_{G\cdot\tau}}) \leq \dim_{\mathrm{Rok}}(\alpha|_{H_\tau})$.

Next, we show $\dim_{\mathrm{Rok}}(\alpha|_{H_\tau}) \leq \dim_{\mathrm{Rok}}(\alpha|_{A_{G\cdot\tau}})$. Set $d = \dim_{\mathrm{Rok}}(\alpha|_{A_{G\cdot\tau}})$ and assume that $d < \infty$. Fix an approximate unit $(e_\lambda)_{\lambda\in\Lambda}$ of A_τ and let $\pi : A_{G\cdot\tau} \to A_\tau$ be the quotient map given by $\pi(x) = \lim_\lambda xe_\lambda$ for all $x \in A_{G\cdot\tau}$. (The map π does not depend on the approximate identity, but this is not relevant to us.) Note that

$$\pi(x)a = xa \tag{4.2}$$

for all $x \in A_{G\cdot\tau}$ and all $a \in A_\tau$. Let $F \subseteq A_\tau$ be a finite subset and let $\varepsilon > 0$. We may assume without loss of generality that F is H_τ-invariant and consists of contractions. For every right coset $q \in H_\tau \setminus G$, let $g_q \in G$ satisfy $H_\tau g_q = q$. Set $\varepsilon_0 = \varepsilon/[G : H_\tau]^2$ and let $\xi_g^{(j)} \in A_{G\cdot\tau} \cap A_g$, for $g \in G$ and $j = 0, \ldots, d$, be Rokhlin towers for $\alpha|_{A_{G\cdot\tau}}$ with respect to (F, ε_0).

For $h \in H_\tau$ and $j = 0, \ldots, d$, we set

$$f_h^{(j)} = \sum_{q\in H_\tau\setminus G} \pi(\xi_{hg_q}^{(j)}) \in A_\tau.$$

We claim that these positive contractions witness that $\dim_{\mathrm{Rok}}(\alpha|_{H_\tau}) \leq d$ for (F, ε).

Let $h_1, h_2 \in H_\tau$, let $j = 0, \ldots, d$ and let $a \in F$ be given. Then

$$\begin{aligned}
\|\alpha_{h_1}(f_{h_2}^{(j)})a - f_{h_1h_2}a\| &\leq \sum_{q\in H_\tau\setminus G} \|\alpha_{h_1}(\pi(\xi_{h_2g_q}^{(j)}))a - \xi_{h_1h_2g_q}^{(j)}a\| \\
&\overset{(4.2)}{=} \sum_{q\in H_\tau\setminus G} \|\alpha_{h_1}(\xi_{h_2g_q}^{(j)}\alpha_{h_1^{-1}}(a)) - \xi_{h_1h_2g_q}^{(j)}a\| \\
&\leq [G : H_\tau]\varepsilon_0 \leq \varepsilon,
\end{aligned}$$

thus establishing condition (1) in Definition 2.1. In order to prove (2), let $h_1, h_2 \in H_\tau$ with $h_1 \neq h_2$, let $j = 0, \ldots, d$, and let $a \in F$. Then

$$\|f_{h_1}^{(j)} f_{h_2}^{(j)} a\| = \left\| \sum_{p,q\in H_\tau\setminus G} \xi_{h_1g_q}^{(j)}\xi_{h_2g_p}^{(j)}a \right\| \leq [G : H_\tau]^2\varepsilon_0 = \varepsilon,$$

using at the second step that $h_1g_q \neq h_2q_p$ for all $p, q \in H_\tau \setminus G$. To check (3), let $a \in F$. Then

$$\begin{aligned}
\sum_{j=0}^{d}\sum_{h\in H_\tau} f_h^{(j)}a &= \sum_{j=0}^{d}\sum_{h\in H_\tau}\sum_{q\in H_\tau\setminus G} \pi(\xi_{hg_q}^{(j)})a \\
&\overset{(4.2)}{=} \sum_{j=0}^{d}\sum_{h\in H_\tau}\sum_{q\in H_\tau\setminus G} \xi_{hg_q}^{(j)}a \\
&= \sum_{j=0}^{d}\sum_{g\in G} \xi_g^{(j)}a \approx_\varepsilon a.
\end{aligned}$$

Finally, to check (4), let $h \in H_\tau$, $j = 0, \ldots, d$, and $a, b \in F$ be given. Then

$$\|(f_h^{(j)}a - af_h^{(j)})b\| = \left\| \sum_{q\in H_\tau\setminus G} (\xi_{hg_q}^{(j)}a - a\xi_{hg_q}^{(j)})b \right\| \leq [G : H_\tau]\varepsilon_0 \leq \varepsilon,$$

as required. This shows that $\dim_{\mathrm{Rok}}(\alpha|_{H_\tau}) \leq d$, and completes the proof. □

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4.2. *Crossed products and fixed point algebras.* Our next result shows that a number of properties are preserved by formation of crossed products and fixed point algebras by partial actions with finite Rokhlin dimension. The kind of properties preserved in this setting are more restrictive than in the global setting, particularly since the properties in question must pass to extensions. For *unital* partial actions, we will show in Theorem 4.10 that even more properties are preserved.

Recall that if α is a partial action of a finite group G on a C^*-algebra A, then its *crossed product* $A \rtimes_\alpha G$ is the set of all formal linear combinations of elements of the form $a_g u_g$, where $g \in G$ and $a_g \in A_g$, subject to the relations

$$a_g u_g b_h u_h = \alpha_g(\alpha_{g^{-1}}(a_g) b_h) u_{gh} \quad \text{and} \quad (a_g u_g)^* = \alpha_{g^{-1}}(a_g^*) u_{g^{-1}}.$$

We consider $A \rtimes_\alpha G$ with its greatest C^*-norm, which is not hard to see exists.

Moreover, its *fixed point algebra* A^α is defined as

$$A^\alpha = \{x \in A \colon \alpha_g(x a_{g^{-1}}) = x \alpha_g(a_{g^{-1}}) \text{ for all } g \in G \text{ and all } a_{g^{-1}} \in A_{g^{-1}}\}.$$

THEOREM 4.7. *Let G be a finite group, and let $d \in \mathbb{N}$. Let $\mathbf{P}$ be a property for C^*-algebras which is preserved by:*

(E) *passage to ideals, quotients and extensions;*

(M) *Morita equivalence;*

(C) *crossed products by global actions of G with $\dim_{\mathrm{Rok}} \leq d$.*

Let A be a unital C^*-algebra, and let α be a partial action of G on A. If $\dim_{\mathrm{Rok}}(\alpha) \leq d$ and A satisfies $\mathbf{P}$, then so do $A \rtimes_\alpha G$ and A^α. In particular, the following hold.

(1) *If $\dim_{\mathrm{Rok}}(\alpha) < \infty$ and $\dim_{\mathrm{nuc}}(A) < \infty$, then $\dim_{\mathrm{nuc}}(A^\alpha), \dim_{\mathrm{nuc}}(A \rtimes_\alpha G) < \infty$. Indeed,*

$$\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq (|G| - 1)(\dim_{\mathrm{Rok}}(\alpha) + 1)(\dim_{\mathrm{nuc}}(A) + 1) + \dim_{\mathrm{nuc}}(A).$$

(2) *If $\dim_{\mathrm{Rok}}(\alpha) < \infty$ and $\mathrm{dr}(A) < \infty$, then $\mathrm{dr}(A^\alpha), \mathrm{dr}(A \rtimes_\alpha G) < \infty$. Indeed,*

$$\mathrm{dr}(A \rtimes_\alpha G) \leq (|G| - 1)(\dim_{\mathrm{Rok}}(\alpha) + 1)(\mathrm{dr}(A) + 1) + \mathrm{dr}(A).$$

A similar statement is true for $\dim_{\mathrm{Rok}}^{\mathrm{c}}$. In particular, the following hold.

(3) *If $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$ and $\mathrm{sr}(A) < \infty$, then $\mathrm{sr}(A^\alpha), \mathrm{sr}(A \rtimes_\alpha G) < \infty$. Indeed,*

$$\mathrm{sr}(A \rtimes_\alpha G) \leq \frac{|G|(\mathrm{sr}(A) + \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) + 3) - 2}{2}.$$

(4) *If $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$ and $\mathrm{RR}(A) < \infty$, then $\mathrm{RR}(A^\alpha), \mathrm{RR}(A \rtimes_\alpha G) < \infty$.*

(5) *Let $\mathcal{D}$ be a strongly self-absorbing C^*-algebra. If $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$ and A is $\mathcal{D}$-absorbing, then A^α and $A \rtimes_\alpha G$ are $\mathcal{D}$-absorbing as well.*

Proof. Let $\mathbf{P}$ be a property as in the statement, let A be a C^*-algebra satisfying $\mathbf{P}$, and let α be a partial action of G on A with $\dim_{\mathrm{Rok}}(\alpha) \leq d$. By Theorem 4.5, there there are canonical equivariant extensions

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0, \tag{4.3}$$

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for $2 \leq k \leq |G|$, satisfying the following properties:

(D.1) $A^{(|G|)} = A$ and $\alpha^{|G|} = \alpha$;

(D.2) $\delta^{(k)}$ has the k-decomposition property;

(D.3) $\alpha^{(1)}$ has the 1-decomposition property.

In particular, each $A^{(k)}$ is a quotient of A, and each $D^{(k)}$ is an ideal of a quotient of A. By (E), all of these C^*-algebras satisfy **P**. By repeatedly applying Proposition 2.6, we deduce that

$$\dim_{\mathrm{Rok}}(\alpha^{(k)}) \leq d \quad \text{and} \quad \dim_{\mathrm{Rok}}(\delta^{(k)}) \leq d \tag{4.4}$$

for all $k = 2, \ldots, |G|$, while $\dim_{\mathrm{Rok}}(\alpha^{(1)}) = 0$ by (D.3) and Example 2.2. For $k = 2, \ldots, |G|$, apply crossed products to (4.3) to get the extension

$$0 \longrightarrow D^{(k)} \rtimes_{\delta^{(k)}} G \longrightarrow A^{(k)} \rtimes_{\alpha^{(k)}} G \longrightarrow A^{(k-1)} \rtimes_{\alpha^{(k-1)}} G \longrightarrow 0. \tag{4.5}$$

Claim. $A^{(k)} \rtimes_{\alpha^{(k)}} G$ satisfies **P** for all $k = 1, \ldots, |G|$. We prove this by induction on k. Since $A^{(1)} \rtimes_{\alpha^{(1)}} G = A^{(1)}$ by Example 2.2 and $A^{(1)}$ is a quotient of A, this follows from (E). Assume we have proved it for $k-1$, and let us prove it for k. Since **P** passes to extensions by (E), the exact sequence in (4.5) implies that it suffices to show that $D^{(k)} \rtimes_{\delta^{(k)}} G$ satisfies **P**. Combining (D.2) and Theorem C in [2], it follows that $D^{(k)} \rtimes_{\delta^{(k)}} G$ is isomorphic to a finite direct sum of algebras of the form $M_{m_\tau}(D_\tau^{(k)} \rtimes_{\delta_\tau^{(k)}} H_\tau)$, for $m_\tau = k/|H_\tau| \leq |G|$ and $\tau \in \mathcal{T}_k(G)$, where $D_\tau^{(k)}$ is an ideal in $D^{(k)}$, H_τ is a subgroup of G, and $\delta_\tau^{(k)}$ is the *global* action obtained as the restriction of $\delta^{(k)}$ to H_τ and to $D_\tau^{(k)}$. Thus, by (M) it suffices to show that $D_\tau^{(k)} \rtimes_{\delta_\tau^{(k)}} H_\tau$ satisfies **P** for every $\tau \in \mathcal{T}_k(G)$. Using Theorem 4.6 at the first step, we have

$$\dim_{\mathrm{Rok}}(\delta_\tau^{(k)}) \leq \dim_{\mathrm{Rok}}(\delta^{(k)}) \overset{(4.4)}{\leq} d.$$

Moreover, $D_\tau^{(k)}$ satisfies **P** by (E), since it is an ideal in $D^{(k)}$. It follows from (C) that $D_\tau^{(k)} \rtimes_{\delta_\tau^{(k)}} H_\tau$ satisfies **P**, as desired. This proves the claim.

Since $A \rtimes_\alpha G$ equals $A^{(|G|)} \rtimes_{\alpha^{(|G|)}} G$ by (D.1), this proves the first assertion in the theorem. Note that an identical argument applies to $\dim_{\mathrm{Rok}}^{\mathrm{c}}$ in place of $\dim_{\mathrm{Rok}}$. Moreover, the argument for fixed point algebras is analogous, by applying fixed point algebras to the extensions in (4.4), using Theorem 4.5 in [2] instead of Theorem C there, and using the fact that fixed point algebras and crossed products are Morita equivalent in the global case. We omit the details.

The properties listed in conditions (1)–(5) are well known to satisfy (E) and (M); see [3, 20, 22, 25]. Finally, they also satisfy (C) by Corollary 4.25 in [14] and Theorem 3.20 in [13].

For the estimates in (1), (2), and (3), one combines the estimates from Corollary 4.25 in [14] and Theorem 3.20 in [13] with the known estimates for $\dim_{\mathrm{nuc}}$, dr, or sr of an extension, or of $A \otimes M_n$, and applies these a total of $|G| - 1$ times to the extensions in (4.5). We omit the details. $\square$

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We note in passing that if α is a partial action of a finite group G on A, and A satisfies **P**, then $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} G$ also satisfies **P**. Indeed, A is Morita equivalent to an ideal I of $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} G$ whose linear orbit under the bidual action $\hat{\hat{\alpha}}$ is all of $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} G$ ([1, Theorem 6.1]). Since **P** is preserved by Morita equivalence and by passage to ideals, quotients and extensions, then $\hat{\hat{\alpha}}_g(I)$, and each sum $\hat{\hat{\alpha}}_g(I) + \hat{\hat{\alpha}}_{g'}(I)$, also satisfy **P**. So $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} G = \sum_{g\in G} \hat{\hat{\alpha}}_g(I)$ satisfies **P**.

The fact that the preservation results for A^α and $A \rtimes_\alpha G$ from Theorem 4.7 look identical suggests that there may be a tighter connection between these algebras. Indeed, in the global setting we have $A^\alpha \sim_M A \rtimes_\alpha G$ whenever G is abelian and $\dim_{\mathrm{Rok}}(\alpha) < \infty$; see Corollary 1.18 in [13]. The situation for partial actions is much more complicated, and there exist partial actions of finite groups with finite Rokhlin dimension such that A^α is not Morita equivalent to $A \rtimes_\alpha G$.

Example 4.8. Set $X = (0, 2]$ and $U = (0, 1) \cup (1, 2) \subseteq X$, and let $\sigma \in \mathrm{Homeo}(U)$ be given by $\sigma(x) = x + 1 \bmod 2$ for all $x \in U$. Let α be the partial action of $G = \mathbb{Z}_2 = \{-1, 1\}$ on $A = C_0(X)$ induced by σ. This action is considered in Example 5.2 of [2], where it is shown that $C_0(X)^{\mathbb{Z}_2}$ is not Morita equivalent to $C_0(X) \rtimes_\alpha \mathbb{Z}_2$.

We claim that $\dim_{\mathrm{Rok}}(\alpha) < \infty$. Let $F \subseteq C_0(X)$ be a finite set, and let $\varepsilon > 0$. For $\delta > 0$, let $f_\delta \in C_0(X)$ and $e_\delta \in C_0(U)$ be given by

$$f_\delta(x) = \begin{cases} 1 & \text{if } x \geq \delta, \\ \text{linear} & \text{otherwise} \end{cases} \quad \text{and}$$

$$e_\delta(x) = \begin{cases} 1 & \text{if } x \in [\delta, 1-\delta] \cup [1+\delta, 2-\delta], \\ \text{linear} & \text{otherwise.} \end{cases}$$

Find $\delta \in (0, 1/4)$ such that $\|e_\delta a - a\| < \varepsilon$ for all $a \in F \cap C_0(U)$ and $\|f_\delta a - a\| < \varepsilon$ for all $a \in F$. Set

$$f_{-1}^{(0)} = e_\delta|_{(0,1)}, \quad f_1^{(0)} = e_\delta|_{(1,2]}, \quad f_{-1} = (f_\delta - e_\delta)_{(0,1)}, \quad \text{and } f_1^{(1)} = (f_\delta - e_\delta)_{(1,2]}.$$

It is easy to check that the above positive contractions are Rokhlin towers for (F, ε), thus showing that $\dim_{\mathrm{Rok}}(\alpha) \leq 1$, as desired. (In fact, one can also show that $\dim_{\mathrm{Rok}}(\alpha) \neq 0$, so that $\dim_{\mathrm{Rok}}(\alpha)$ is actually 1.)

4.3. *Unital partial actions.* For unital partial actions, a different approach can be used to obtain information about the crossed product, leading to a result which is stronger than Theorem 4.7, in that here we only demand that the property in question pass to direct sums and summands, as opposed to general extensions and ideals; see Theorem 4.10. Most significantly, this implies that the UCT is preserved in this setting; note that the UCT is not in general known to pass to ideals or quotients and hence does not satisfy condition (E) in Theorem 4.7. The same applies to several properties of the K-theory. Additionally, we isolate the case of Rokhlin dimension zero, for which considerably more can be said with this approach.

In contrast to Example 4.8, we show next that A^α and $A \rtimes_\alpha G$ are always Morita equivalent whenever $\dim_{\mathrm{Rok}}(\alpha) < \infty$ and α is unital. Our proof is very different from

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the arguments used in [13] and has the advantage of being applicable to global actions of groups that are not necessarily abelian, thus obtaining new information even in the global setting.

In the next proof, for $\xi \in A \rtimes_\alpha G$ we write $\xi(g) \in A_g$, $g \in G$, for the coefficient of u_g in ξ, so that $\xi = \sum_{g\in G} \xi(g)u_g$.

THEOREM 4.9. *Let α be a unital partial action of a finite group G on a C^*-algebra A with* $\dim_{\mathrm{Rok}}(\alpha) < \infty$. *Then A^α is Morita equivalent to $A \rtimes_\alpha G$.*

Proof. Set $x_\alpha = \sum_{g\in G} 1_g$. Then x_α is central and positive, and since $x_\alpha \geq 1$, it is invertible. Moreover, x_α belongs to A^α, since for $h \in G$ we have

$$\alpha_h(x_\alpha 1_{h^{-1}}) = \sum_{g\in G} \alpha_h(1_g 1_{h^{-1}}) = \sum_{g\in G} 1_{hg}1_h = x_\alpha 1_h,$$

where at the second step we used Remark 2.3. We define a Hilbert A^α-$A \rtimes_\alpha G$-bimodule structure on A as follows. For $a \in A^\alpha$, $x \in A$ and $\xi \in A \rtimes_\alpha G$, set $a \cdot x = ax$ (the product is taken in A) and $x \cdot \xi = \sum_{g\in G} \alpha_{g^{-1}}(x\xi(g))$. The inner products are given by

$${}_{A^\alpha}\langle x, y\rangle = \sum_{g\in G} \alpha_g(xy^*1_{g^{-1}}) \quad \text{and} \quad \langle x, y\rangle_{A\rtimes_\alpha G} = \sum_{g\in G} x^*\alpha_g(y1_{g^{-1}})u_g$$

for $a \in A^\alpha$, $x, y \in A$ and $\xi \in A \rtimes_\alpha G$. (One readily checks that ${}_{A^\alpha}\langle x, y\rangle$ belongs to A^α.) The properties of the inner products are easily verified. For example, for $x \in A$ we have

$$x_\alpha \sum_{g\in G} x^*\alpha_g(x1_{g^{-1}})u_g = \bigg(\sum_{g\in G} \alpha_g(x1_{g^{-1}})u_g\bigg)^* \bigg(\sum_{g\in G} \alpha_g(x1_{g^{-1}})u_g\bigg) \geq 0,$$

so $\langle x, x\rangle_{A\rtimes_\alpha G} = \sum_{g\in G} x^*\alpha_g(x1_{g^{-1}})u_g \geq 0$, with equality exactly when $x = 0$. One can also easily check the properties of a bimodule. For example, given $\xi \in A \rtimes_\alpha G$ and $x, y \in A$, we have

$$\begin{aligned}
\langle x, y\rangle_{A\rtimes_\alpha G} \cdot \xi &= \sum_{g\in G} x^*\alpha_g(y1_{g^{-1}})u_g \sum_{h\in G} \xi(h)u_h \\
&= \sum_{g,h\in G} \alpha_g(\alpha_{g^{-1}}(x^*\alpha_g(y1_{g^{-1}}))\xi(h))u_{gh} \\
&= \sum_{g,h\in G} \alpha_g(\alpha_{g^{-1}}(x^*1_g)y\xi(h))u_{gh} \\
&= \sum_{g,h\in G} x^*\alpha_g(y\xi(h)1_{g^{-1}})u_{gh} \\
&= \sum_{g,h\in G} x^*\alpha_{gh^{-1}}(y\xi(h)1_{hg^{-1}})u_g \\
&= \sum_{g,h\in G} x^*\alpha_g(\alpha_{h^{-1}}(y\xi(h))1_{g^{-1}})u_g \\
&= \langle x, y \cdot \xi\rangle_{A\rtimes_\alpha G}.
\end{aligned}$$

The other properties can be shown similarly. Moreover, for $x \in A^\alpha$, we have

$$_{A^\alpha}\langle x, x_\alpha^{-1}\rangle = \sum_{h\in G} \alpha_h(x x_\alpha^{-1} 1_{h^{-1}}) = \sum_{h\in G} x x_\alpha^{-1} 1_h = x.$$

In particular, A is a full left A^α-module. Finally, we claim that A is a full right $A \rtimes_\alpha G$-module. For this, it suffices to show that given $t \in G$, $x \in A_t$ and $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and $a_1, b_1, \ldots, a_k, b_k \in A$ such that $x u_t \approx_\varepsilon \sum_{\ell=1}^k \langle a_\ell, b_\ell\rangle_{A\rtimes_\alpha G}$. Set $d = \dim_{\mathrm{Rok}}(\alpha)$ and $\varepsilon_0 = \varepsilon/2(d+1)|G|$, and use Proposition 2.4 to find $f_g^{(j)} \in A_g$, for $g \in G$ and $j = 0, \ldots, d$, satisfying:

(a) $\alpha_g(f_h^{(j)} 1_{g^{-1}}) = f_{gh}^{(j)} 1_g$ for all $g, h \in G$;

(b) $f_g^{(j)} f_h^{(j)} \approx_{\varepsilon_0} 0$ for all $j = 0, \ldots, d$ and all distinct $g, h \in G$;

(c) $\sum_{g\in G} \sum_{j=0}^d f_g^{(j)} \approx_{\varepsilon_0} 1$.

Using at the last step that $x \in A_t$, we get

$$\begin{aligned}
\sum_{g\in G}\sum_{j=0}^{d} \langle f_{tg}^{(j)\,1/2} x^*, f_g^{(j)\,1/2}\rangle_{A\rtimes_\alpha G} &= \sum_{g,h\in G}\sum_{j=0}^{d} x f_{tg}^{(j)\,1/2} \alpha_h(f_g^{(j)\,1/2} 1_{h^{-1}}) u_h \\
&\overset{(a)}{=} \sum_{g,h\in G}\sum_{j=0}^{d} x f_{tg}^{(j)\,1/2} f_{hg}^{(j)\,1/2} 1_h u_h \\
&\overset{(b)}{\approx}_{\varepsilon/2} \sum_{g\in G}\sum_{j=0}^{d} x f_{tg}^{(j)} 1_t u_t \\
&\overset{(c)}{\approx}_{\varepsilon_0} x 1_t u_t = x u_t,
\end{aligned}$$

as desired. We conclude that A is an A^α-$A \rtimes_\alpha G$-imprimitivity bimodule, so these C^*-algebras are Morita equivalent, and the proof is complete. □

THEOREM 4.10. *Let G be a finite group, and let $d \in \mathbb{N}$. Let $\boldsymbol{P}$ be a property for C^*-algebras which is preserved by:*

(S) *passage to direct sums and summands;*

(M) *Morita equivalence;*

(C) *crossed products by global actions of G with $\dim_{\mathrm{Rok}} \leq d$.*

Let A be a unital C^*-algebra, and let α be a unital partial action of G on A. If $\dim_{\mathrm{Rok}}(\alpha) \leq d$ and A satisfies $\boldsymbol{P}$, then so do $A \rtimes_\alpha G$ and A^α. A similar statement is true if $\dim_{\mathrm{Rok}}$ is replaced everywhere by $\dim_{\mathrm{Rok}}^{\mathrm{c}}$. In particular, the following hold.

(1) *If $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$ and A satisfies the UCT, then so do $A \rtimes_\alpha G$ and A^α.*

(2) *If $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$ and $K_*(A)$ is either trivial, free, torsion-free, or finitely generated, then the same holds for $A \rtimes_\alpha G$ and A^α.*

When $\dim_{\mathrm{Rok}}(\alpha) = 0$, it follows that the properties listed in the main theorem of [8] *pass from A to A^α and $A \rtimes_\alpha G$. This includes having real rank zero, having stable rank one, being an AF/approximately interval (AI)/approximately circle ($A\mathbb{T}$)-algebra, being purely infinite, the order on projections being determined by traces, being weakly semiprojective, and having K-groups which are either trivial, free, torsion-free, or finitely generated.*

Proof. By Theorem 4.9 and condition (M), it suffices to show the result for $A \rtimes_\alpha G$. Let (B, β) denote the globalization of (A, α).

Claim. There exist central projections $p_1, \ldots, p_{|G|} \in A$ such that $B \cong p_1 A \oplus \cdots \oplus p_{|G|} A$. Since A is a unital ideal in B and $B = \sum_{g \in G} \beta_g(A)$, it suffices to show the following: if I, J are unital ideals in a C^*-algebra, then $I + J \cong pI \oplus J$ for some central projection $p \in I$. This follows by taking $p = 1_I - 1_I 1_J$ and letting $\psi \colon I + J \to pI \oplus J$ be $\psi(x) = (px, 1_J x)$ for all $x \in I + J$. We omit the easy details.

Assume that A satisfies **P** and that $\dim_{\mathrm{Rok}}(\alpha) \leq d$. By the claim above and (S), it follows that B satisfies **P**. By Theorem 3.4, we have $\dim_{\mathrm{Rok}}(\beta) = \dim_{\mathrm{Rok}}(\alpha) \leq d$, and hence by (C) the global crossed product $B \rtimes_\beta G$ satisfies **P**. Finally, since $B \rtimes_\beta G \sim_M A \rtimes_\alpha G$ by Theorem 4.18 in [2], the result follows from (M).

For the properties listed in (1) and (2), the preservation conditions (S) and (M) are well-known, while (C) is part of Theorem 3.20 in [13]. The properties mentioned in the last part of the statement are also known to satisfy (S) and (M), and for $d = 0$ they also satisfy (C) by the main result in [8]. □

Remark 4.11. In the context of the above theorem, one can obtain bounds for $\dim_{\mathrm{nuc}}$, dr, sr, and RR of $A \rtimes_\alpha G$ and A^α that are much better than the ones in Theorem 4.7, particularly since they do not depend on the cardinality of G. For example, one gets $\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq (\dim_{\mathrm{Rok}}(\alpha) + 1)(\dim_{\mathrm{nuc}}(A) + 1) - 1$.

5. *Topological partial actions*

In this section, we study the Rokhlin dimension of topological partial actions. In Theorem 5.10, we will show that a topological partial action $G \curvearrowright X$ on a finite-dimensional space X has finite Rokhlin dimension *if and only if* it is free. The case of global actions is implicit in [17], and it is an easy consequence of the existence of local cross-sections for the quotient map $\pi \colon X \to X/G$. However, the proof in the partial setting is considerably more complicated, since even for free partial actions there may not exist local cross-sections for π. The proof in our context is quite involved and will occupy the entire section.

Definition 5.1. Let X be a topological space. A topological partial action of a discrete group G on X is given by a pair $((X_g)_{g \in G}, (\theta_g)_{g \in G})$, consisting of open subsets $X_g \subseteq X$ and homeomorphisms $\theta_g \colon X_{g^{-1}} \to X_g$, satisfying

(1) $X_1 = X$ and $\theta_1 = \mathrm{id}_X$; and
(2) $\theta_g \circ \theta_h \subseteq \theta_{gh}$ for all $g, h \in G$.

We say that θ is *free* if $\theta_g(x) \neq x$ for all $g \in G \setminus \{1_G\}$ and all $x \in X_{g^{-1}}$.

Next, we show that finite Rokhlin dimension implies freeness in the above setting.

PROPOSITION 5.2. *Let X be a locally compact Hausdorff space, let G be a finite group, let θ be a topological partial action of G on X, and denote by α the induced partial action of G on $C_0(X)$. If $\dim_{\mathrm{Rok}}(\alpha) < \infty$, then θ is free.*

Proof. Set $d = \dim_{\mathrm{Rok}}(\alpha)$, and assume that $d < \infty$. Let $g \in G \setminus \{1\}$ be given. Arguing by contradiction, assume that $F_g = \{x \in X_{g^{-1}} : \theta_g(x) = x\}$ is not empty, and fix $x \in F_g$. Note

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that x belongs to $X_g \cap X_{g^{-1}}$. Let $a \in C_0(X_{g^{-1}})$ be a positive contraction satisfying $a(x) = 1$. Choose $\varepsilon > 0$ with $\varepsilon < (1/2(|G|(d+1)+1)^2)$, and let $f_h^{(j)} \in C_0(X_h)$, for $h \in G$, and $j = 0, \ldots, d$ be Rokhlin towers for $(\{a, \alpha_g(a)\}, \varepsilon)$. Given $h \in G$ and $j = 0, \ldots, d$, we have:

$$\|f_h^{(j)}\alpha_g(f_h^{(j)}a)\|_\infty \leq \|f_h^{(j)}(\underbrace{\alpha_g(f_h^{(j)}a) - f_{gh}^{(j)}\alpha_g(a)}_{\approx_\varepsilon 0})\|_\infty + \|\underbrace{f_h^{(j)} f_{gh}^{(j)}}_{\approx_\varepsilon 0}\alpha_g(a)\|_\infty < 2\varepsilon.$$

Evaluating at x, we get

$$(f_h^{(j)}\alpha_g(f_h^{(j)}a))(x) = f_h^{(j)}(x) f_h^{(j)}(\theta_{g^{-1}}(x)) a(\theta_{g^{-1}}(x)) < 2\varepsilon.$$

Since $\theta_{g^{-1}}(x) = x$ and $a(x) = 1$, we deduce that

$$f_h^{(j)}(x) < \sqrt{2\varepsilon}, \tag{5.1}$$

for all $h \in G$ and $j = 0, \ldots, d$. Moreover, condition (3) from Definition 2.1 gives

$$\left|\sum_{j=0}^{d}\sum_{h\in G} f_h^{(j)}(x)a(x) - a(x)\right| \overset{a(x)=1}{=} \left|\sum_{j=0}^{d}\sum_{h\in G} f_h^{(j)}(x) - 1\right| < \varepsilon,$$

so $\sum_{j=0}^{d}\sum_{h\in G} f_h^{(j)}(x) > 1 - \varepsilon$. Using this at the second step, we get

$$|G|(d+1)\sqrt{2\varepsilon} \overset{(5.1)}{>} \sum_{j=0}^{d}\sum_{g\in G} f_g^{(j)}(x) > 1 - \varepsilon \geq 1 - \sqrt{2\varepsilon}.$$

This contradicts the choice of ε, showing that F_g is empty. Thus θ is free. $\square$

The rest of the section will be devoted to proving that free partial actions have finite Rokhlin dimension. The general strategy is as follows.

Step 1: Show that free *decomposable* partial actions have finite Rokhlin dimension.

Step 2: Show that an extension of topological partial actions with finite Rokhlin dimension again has finite Rokhlin dimension.

Step 3: Use the equivariant decomposition into successive extensions from Theorem 4.5 to conclude that the given action has finite Rokhlin dimension.

Using the results we proved in §4, we can easily establish the first step.

PROPOSITION 5.3. *Let G be a finite group, let $n \in \{1, \ldots, |G|\}$, let X be a locally compact space with* $\dim(X) < \infty$, *and let σ be a partial action of G on X with the n-decomposition property (see Definition 4.2). Denote by α the induced partial action of G on $C_0(X)$. If σ is free, then* $\dim_{\mathrm{Rok}}(\alpha) \leq \dim(X)$.

Proof. Fix $\tau \in \mathcal{T}_n(G)$. Then X_τ is an open subset of X and thus $\dim(X_\tau) \leq \dim(X)$. By part (2) of Theorem 5.4 in [2], the restricted global action of H_τ on X_τ is free. Using Proposition 2.11 in [17] at the first step, it follows that

$$\dim_{\mathrm{Rok}}(\alpha|_{H_\tau}) \leq \dim(X_\tau) \leq \dim(X).$$

Thus, the result follows from Theorem 4.6. $\square$

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Step 2 is considerably more complicated. Roughly speaking, one needs to lift Rokhlin towers from the quotient to the algebra, while at the same time respecting the domains of the partial action. If we did not care about respecting the domains (such as in the global case), the result would be an immediate consequence of the fact that the cone of $\mathbb{C}^n$ is projective. In our setting, the greatest difficulty lies in showing that this can be done in a way compatible with the domains, and for this we will need some lemmas that allow us to assume that the Rokhlin towers and their lifts are orthogonal; see Lemma 5.4 and Lemma 5.5. It is unclear whether these results hold without assuming that the algebra is commutative.

First, we show that for partial actions on commutative unital C^*-algebras, the elements of each Rokhlin tower can be assumed to be exactly orthogonal.

LEMMA 5.4. *Let α be a partial action of a finite group G on a unital, commutative C^*-algebra A, and let $d \in \mathbb{N}$. Then $\dim_{\mathrm{Rok}}(\alpha) \le d$ if and only if for all $\varepsilon > 0$ and every finite subset $F \subseteq A$ there exist positive contractions $f_g^{(j)} \in A_g$, for $g \in G$ and $j = 0, \ldots, d$, satisfying:*

(1) $\|\alpha_g(f_h^{(j)} a) - f_{gh}^{(j)} \alpha_g(a)\| < \varepsilon$ *for all* $a \in F \cap A_{g^{-1}}$;

(2) $f_g^{(j)} f_h^{(j)} = 0$ *for all* $g, h \in G$ *with* $g \neq h$ *and for all* $j = 0, \ldots, d$;

(3) $\|\sum_{g\in G} \sum_{j=0}^{d} f_g^{(j)} - 1\| < \varepsilon$.

Proof. It is clear that any action satisfying conditions (1), (2), and (3) in the statement has Rokhlin dimension at most d (the approximate commutation condition from Definition 2.1 is vacuous since A is commutative). We therefore prove the non-trivial implication. Let $\varepsilon > 0$ and let a finite subset $F \subseteq A$ be given. Without loss of generality, we assume that F consists of positive contractions.

Fix $\varepsilon_0 < 1$ such that $(d+2)|G|^2\sqrt{\varepsilon_0} < \varepsilon$, and choose positive contractions $x_g^{(j)} \in A_g$, for $g \in G$ and $j = 0, \ldots, d$, satisfying:

(a) $\|\alpha_g(x_h^{(j)} a) - x_{gh}^{(j)} \alpha_g(a)\| < \varepsilon_0$ for all $a \in F \cap A_{g^{-1}}$;

(b) $\|x_g^{(j)} x_h^{(j)}\| < \varepsilon_0$ for all $g, h \in G$ with $g \neq h$ and for all $j = 0, \ldots, d$;

(c) $\|\sum_{g\in G} \sum_{j=0}^{d} x_g^{(j)} - 1\| < \varepsilon_0$.

For $g \in G$ and $j = 0, \ldots, d$, set $y_g^{(j)} = \sum_{h \in G\setminus\{g\}} x_h^{(j)}$ and

$$f_g^{(j)} = (x_g^{(j)} - y_g^{(j)})_+.$$

Since A is commutative, we have $f_g^{(j)} \le x_g^{(j)}$ and hence $f_g^{(j)}$ is a positive contraction in A_g. We will show that these elements satisfy the conditions in the statement. We begin by noting that the $f_g^{(j)}$ are very close to the $x_g^{(j)}$.

Claim. We have $\|x_g^{(j)} - f_g^{(j)}\| < |G|\sqrt{\varepsilon_0}$ for all $g \in G$ and $j = 0, \ldots, d$. To see this, since A is abelian, it is enough to prove that $|x_g^{(j)}(t) - f_g^{(j)}(t)| \le |G|\sqrt{\varepsilon_0}$ for every $t \in \mathrm{Spec}(A)$. Fix $t \in \mathrm{Spec}(A)$. We divide the proof into two cases.

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Suppose that $x_g^{(j)}(t) \geq y_g^{(j)}(t)$. Then $f_g^{(j)}(t) = x_g^{(j)}(t) - y_g^{(j)}(t)$ and hence

$$x_g^{(j)}(t) - f_g^{(j)}(t) = y_g^{(j)}(t) = \sum_{h \in G\setminus\{g\}} x_h^{(j)}(t). \tag{5.2}$$

For $h \neq g$, we have $x_h^{(j)}(t) \leq y_g^{(j)}(t) \leq x_g^{(j)}(t)$, and thus

$$x_h^{(j)}(t)^2 \leq x_h^{(j)}(t) x_g^{(j)}(t) < \varepsilon_0. \tag{5.3}$$

We conclude that

$$|x_g^{(j)}(t) - f_g^{(j)}(t)| \overset{(5.2)}{\leq} \sum_{h \in G\setminus\{g\}} |x_h^{(j)}(t)| \overset{(5.3)}{<} (|G|-1)\sqrt{\varepsilon_0} < |G|\sqrt{\varepsilon_0}.$$

Suppose that $x_g^{(j)}(t) \leq y_g^{(j)}(t)$. Then $f_g^{(j)}(t) = 0$ and hence $x_g^{(j)}(t) - f_g^{(j)}(t) = x_g^{(j)}(t)$. Moreover,

$$x_g^{(j)}(t)^2 \leq \left(\sum_{h \in G\setminus\{g\}} x_h^{(j)}(t) \right) x_g^{(j)}(t) \overset{\text{(b)}}{\leq} (|G|-1)\varepsilon_0. \tag{5.4}$$

Thus,

$$|x_g^{(j)}(t) - f_g^{(j)}(t)| = |x_g^{(j)}(t)| \overset{(5.4)}{\leq} \sqrt{(|G|-1)\varepsilon_0} < |G|\sqrt{\varepsilon_0}.$$

This proves the claim.

We check that the positive contractions $f_g^{(j)}$ for $g \in G$ and $j = 0, \ldots, d$ satisfy conditions (1), (2), and (3) in the statement. Conditions (1) and (3) follow immediately by combining the claim above with conditions (a) and (c), respectively, so we only check (2). Let $g, h \in G$ with $g \neq h$ and let $j = 0, \ldots, d$. Using the inequalities $x_g^{(j)} \leq y_h^{(j)}$ and $x_h^{(j)} \leq y_g^{(j)}$ (which are valid since $g \neq h$) at the second step, and the identity $(z-w)_+(w-z)_+ = 0$ at the last step, we get

$$\begin{aligned} f_g^{(j)} f_h^{(j)} &= (x_g^{(j)} - y_g^{(j)})_+ (x_h^{(j)} - y_h^{(j)})_+ \\ &\leq (y_h^{(j)} - y_g^{(j)})_+ (y_g^{(j)} - y_h^{(j)})_+ = 0. \end{aligned}$$

□

The following lemma deals with obtaining domain-respecting orthogonal lifts.

LEMMA 5.5. *Let A be a commutative C^*-algebra, let $n \in \mathbb{N}$, and let $J, A_1, \ldots, A_n$ be ideals in A. We set $B = A/J$ with quotient map $\pi\colon A \to B$, and*

$$B_j = \frac{A_j}{A_j \cap J} \cong \frac{A_j + J}{J}$$

for $j = 1, \ldots, n$. For $j = 1, \ldots, n$, let $x_j \in A_j$ be a positive contraction satisfying $x_j x_k \in J$ for all $k = 1, \ldots, n$ with $j \neq k$. Then there exist pairwise orthogonal positive contractions $y_j \in A_j$, for $j = 1, \ldots, n$, satisfying $\pi(y_j) = \pi(x_j)$.

Proof. We prove the lemma by induction on n, the case $n = 1$ being trivial. Assume that $n = 2$, and let $x_1 \in A_1$ and $x_2 \in A_2$ be positive contractions satisfying $x_1 x_2 \in J$. Set $y_1 = (x_1 - x_2)_+$ and $y_2 = (x_2 - x_1)_+$, which are positive contractions. Since A is abelian, we

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have $y_j \leq x_j$ and hence y_j belongs to A_j for $j = 1, 2$. Moreover, $y_1 y_2 = 0$. Finally, using at the first step that $\pi(x_1)\pi(x_2) = 0$, we get

$$\pi(x_1) = (\pi(x_1) - \pi(x_2))_+ = \pi((x_1 - x_2)_+) = \pi(y_1).$$

Similarly, $\pi(x_2) = \pi(y_2)$. This proves the case $n = 2$ of the statement.

Assume now that we have proved the result for an integer $n \geq 2$, and let us prove it for $n + 1$. Thus, let $A_1, \ldots, A_{n+1}$ be ideals in A, and let $x_1, \ldots, x_{n+1}$ be as in the statement. Set $x = \sum_{j=1}^{n} x_j$ and define $y_{n+1} = (x_{n+1} - x)_+$. Then y_{n+1} is a positive contraction with $y_{n+1} \leq x_{n+1}$, and hence $y_{n+1} \in A_{n+1}$. Moreover, since $\pi(x)\pi(x_{n+1}) = 0$ and using the same argument as in the case $n = 2$, it follows that $\pi(y_{n+1}) = \pi(x_{n+1})$.

For $j = 1, \ldots, n$, set $z_j = (x_j - x_{n+1})_+$, which is a positive contraction in A_j with $z_j \leq x_j$. Since $\pi(x_j)\pi(x_{n+1}) = 0$ for $j \leq n$, it follows that $\pi(z_j) = \pi(x_j)$. In particular, for $1 \leq j \neq k \leq n$ we have $\pi(z_j z_k) = 0$, that is, $z_j z_k \in J$. By the inductive step applied to $z_1, \ldots, z_n$, there exist orthogonal positive contractions $y_j \leq z_j$, for $j = 1, \ldots, n$, satisfying $\pi(y_j) = \pi(z_j)$. Then $y_j \leq x_j$ and $\pi(y_j) = \pi(x_j)$ for $j = 1, \ldots, n + 1$. It remains to prove that the $y_1, \ldots, y_{n+1}$ are pairwise orthogonal. By construction, the contractions $y_1, \ldots, y_n$ are pairwise orthogonal, so it suffices to check orthogonality with y_{n+1}. For $j = 1, \ldots, n$, we have

$$y_j y_{n+1} \leq z_j y_{n+1} = (x_j - x_{n+1})_+ (x_{n+1} - x)_+ \leq (x_j - x_{n+1})_+ (x_{n+1} - x_j)_+ = 0,$$

using at the third step that A is commutative and $x \geq x_j$. □

We are now ready to prove that an extension of partial actions on commutative C^*-algebras with finite Rokhlin dimension again has finite Rokhlin dimension. For use in this proof, we recall the following standard fact. Let A be a unital C^*-algebra, let J be an ideal in A, let $\pi \colon A \to A/J$ denote the quotient map, let $(e_\lambda)_{\lambda \in \Lambda}$ be any approximate identity for J, and let $a \in A$. Then $\|\pi(a)\| = \limsup_{\lambda \in \Lambda} \|a(1 - e_\lambda)\|$.

PROPOSITION 5.6. *Let G be a finite group, let A be a commutative unital C^*-algebra, let α be a partial action of G on A, and let J be an α-invariant ideal in A with a G-invariant approximate identity. Denote by $\overline{\alpha}$ the induced partial action of G on A/J as in Proposition 2.6. Then*

$$\dim_{\mathrm{Rok}}(\alpha) \leq \dim_{\mathrm{Rok}}(\alpha|_J) + \dim_{\mathrm{Rok}}(\overline{\alpha}) + 1.$$

Proof. Set $d_1 = \dim_{\mathrm{Rok}}(\alpha|_J)$ and $d_2 = \dim_{\mathrm{Rok}}(\overline{\alpha})$. Without loss of generality, we assume that $d_1, d_2 < \infty$. Let $\varepsilon > 0$ and let $F \subseteq A$ be a finite subset. We abbreviate $B = A/J$ and $B_g = A_g/(A_g \cap J)$ for all $g \in G$. We denote by $\pi \colon A \to B$ the canonical quotient map. We use Lemma 5.4 to choose positive contractions $x_g^{(j)} \in B_g$, for $g \in G$ and $j = 0, \ldots, d_2$, satisfying:

(B.1) $\|\overline{\alpha}_g(x_h^{(j)} b) - x_{gh}^{(j)} \overline{\alpha}_g(b)\| < \varepsilon$ for all $g, h \in G$, for all $j = 0, \ldots, d_2$, and for all $b \in \pi(F) \cap B_{g^{-1}}$;

(B.2) $x_g^{(j)} x_h^{(j)} = 0$ for all $g, h \in G$ with $g \neq h$ and for all $j = 0, \ldots, d_2$;

(B.3) $\|\sum_{g \in G} \sum_{j=0}^{d_2} x_g^{(j)} - 1\| < \varepsilon$.

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Fix $j = 0, \ldots, d_2$. Use Lemma 5.5 to find mutually orthogonal positive contractions $y_g^{(j)} \in A_g$, for $g \in G$, satisfying $\pi(y_g^{(j)}) = x_g^{(j)}$. Using that J has a G-invariant approximate identity, find a positive contraction $q \in J$ satisfying:

(a) $\alpha_g(qx) = q\alpha_g(x)$ for all $x \in A_{g^{-1}} \cap J$;

(b) $\|(1 - \sum_{g\in G} \sum_{j=0}^{d} y_g^{(j)})(1-q)\| < \varepsilon$;

(c) $\|(\alpha_g(y_h^{(j)}a) - y_{gh}^{(j)}\alpha_g(a))(1-q)\| < \varepsilon$ for all $a \in A_{g^{-1}} \cap F$.

Claim. $\alpha_g(qa) = q\alpha_g(a)$ *for all* $a \in A_{g^{-1}}$. To prove this, let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for $A_{g^{-1}} \cap J$. Then $(\alpha_g(e_\lambda))_{\lambda\in\Lambda}$ is an approximate identity for $A_g \cap J$. Fix $a \in A_{g^{-1}}$. In the next computation, we use at the third step condition (a) above with $x = ae_\lambda$, and the fact that qa belongs to $A_{g^{-1}} \cap J$ at the fourth step:

$$q\alpha_g(a) = \lim_{\lambda\in\Lambda} q\alpha_g(a)\alpha_g(e_\lambda) = \lim_{\lambda\in\Lambda} q\alpha_g(ae_\lambda) = \lim_{\lambda\in\Lambda} \alpha_g(qae_\lambda) = \alpha_g(qa).$$

This proves the claim.

Set $z_g^{(j)} = y_g^{(j)}(1-q)$ for $g \in G$ and $j = 0, \ldots, d_2$. Then

$$z_g^{(j)} z_h^{(j)} = 0 \tag{5.5}$$

for $g, h \in G$ with $g \neq h$ and for $j = 0, \ldots, d_2$, since $1-q$ is central. Let $g, h \in G$, let $a \in F \cap A_{g^{-1}}$, and let $j = 0, \ldots, d_2$ be given. Using the above claim at the second step, we get

$$\begin{aligned} \|\alpha_g(z_h^{(j)}a) - z_{gh}^{(j)}\alpha_g(a)\| &= \|\alpha_g(y_h^{(j)}a(1-q)) - y_{gh}^{(j)}\alpha_g(a)(1-q)\| \\ &= \|(\alpha_g(y_h^{(j)}a) - y_{gh}^{(j)}\alpha_g(a))(1-q)\| \overset{(c)}{<} \varepsilon. \end{aligned} \tag{5.6}$$

Furthermore,

$$\left\| \sum_{g\in G} \sum_{j=0}^{d_2} z_g^{(j)} - (1-q) \right\| < \varepsilon. \tag{5.7}$$

Set $F_J = \{qa : a \in F\} \cup \{q\} \subseteq J$. Choose positive contractions $\xi_g^{(k)} \in A_g \cap J$, for $g \in G$ and $k = 0, \ldots, d_1$, satisfying:

(J.1) $\|\alpha_g(\xi_h^{(k)}c) - \xi_{gh}^{(k)}\alpha_g(c)\| < \varepsilon$ for all $g, h \in G$, for all $k = 0, \ldots, d_1$, and for all $c \in F_J \cap J_{g^{-1}}$;

(J.2) $\|\xi_g^{(k)}\xi_h^{(k)}c\| < \varepsilon$ for all $g, h \in G$ with $g \neq h$, for all $k = 0, \ldots, d_1$, and for all $c \in F_J$;

(J.3) $\|c - c(\sum_{g\in G} \sum_{k=0}^{d_1} \xi_g^{(k)})\| < \varepsilon$ for all $c \in F_J$.

Set $\eta_g^{(k)} = \xi_g^{(k)}q$ for all $g \in G$ and $k = 0, \ldots, d_1$. For $a \in F \cap A_{g^{-1}}$, we have

$$\|\alpha_g(\eta_h^{(k)}a) - \eta_{gh}^{(k)}\alpha_g(a)\| = \|\alpha_g(\xi_h^{(k)}qa) - \xi_{gh}^{(k)}q\alpha_g(a)\| \overset{(J.1)}{<} \varepsilon, \tag{5.8}$$

because $qa \in J_{g^{-1}} \cap F_J$. Since q is central (because A is commutative), we have

$$\|\eta_g^{(k)}\eta_h^{(k)}\| \leq \|\xi_g^{(k)}\xi_h^{(k)}\| < \varepsilon \tag{5.9}$$

for $g, h \in G$ with $g \neq h$ and for $k = 0, \ldots, d_1$. Further, taking $c = q$ in (J.3) gives

$$\left\| \sum_{g\in G} \sum_{k=0}^{d_1} \eta_g^{(k)} - q \right\| \leq \varepsilon. \tag{5.10}$$

For $g \in G$ and $\ell = 0, \ldots, d_1 + d_2 + 1$, set

$$f_g^{(j)} = \begin{cases} z_g^{(j)} & \text{for } \ell = 0, \ldots, d_2, \\ \eta_g^{(\ell - d_2 - 1)} & \text{for } \ell = d_2 + 1, d_2 + 2, \ldots, d_1 + d_2 + 1. \end{cases}$$

Then $\{f_g^{(j)} \colon g \in G, 0 \leq \ell \leq d_1 + d_2 + 1\}$ are Rokhlin towers for α with respect to $(F, 2\varepsilon)$. Indeed, condition (1) follows from (5.5) and (5.8); condition (2) follows from (5.6) and (5.9); and condition (3) follows from (5.7) and (5.10). $\square$

For technical reasons it will be more convenient to work with unital C^*-algebras. In order to reduce to the unital case, we define the minimal partial unitization of a partial action. This unitization differs from the usual unitization of a global action, since the minimal partial unitization is never a global action, even if the original action was. For a C^*-algebra A, we denote by A^+ the unitization of A, which as a Banach space is isomorphic to $A \oplus \mathbb{C}$.

Definition 5.7. Let G be a locally compact group, let A be a C^*-algebra, and let $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ be a partial action of G on A. We define the *minimal partial unitization* α^+ of α to be the partial action of G on A^+ determined by $A_g^+ = A_g$ and $\alpha_g^+ = \alpha_g$ for all $g \in G \setminus \{1\}$.

For later use, we record the following easy observation.

Remark 5.8. Let G be a finite group, let A be a commutative C^*-algebra, and let α be a partial action of G on A. Then (A, α) is an equivariant ideal in (A^+, α^+), and hence $\dim_{\mathrm{Rok}}(\alpha) \leq \dim_{\mathrm{Rok}}(\alpha^+)$ by Proposition 2.6.

For a partial action θ of a group G on a locally compact space X, one defines its *minimal partial compactification* θ^+ analogously to Definition 5.7, only by compactifying the domain of the identity element and leaving the rest unchanged. It is clear that the partial action of G on $C(X^+)$ induced by θ^+ can be canonically identified by the minimal partial unitization of the partial action induced by θ.

One feature of the minimal partial unitization, in comparison with the unitization in the sense of global actions, is that the minimal partial unitization of a free action on a locally compact space is again free, as we show next. On the other hand, the unitization (in the global sense) of a free action is never free, since the point at infinity is necessarily fixed.

LEMMA 5.9. *Let G be a locally compact group, let X be a locally compact Hausdorff space, and let $\theta = ((X_g)_{g\in G}, (\theta_g)_{g\in G})$ be a partial action of G on X. Then θ is free if and only if θ^+ is free.*

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Proof. It is clear that θ is free if θ^+ is free. Conversely, and arguing by contradiction, let $g \in G \setminus \{1\}$ and $x \in X^+_{g^{-1}}$ satisfy $\theta^+_g(x) = x$. Since $g \neq 1$, this implies that $x \in X_{g^{-1}}$ and $\theta_g(x) = x$, which contradicts the fact that θ is free, as desired. □

We have now arrived at the main result of this section.

THEOREM 5.10. *Let G be a finite group, let X be a locally compact space with* $\dim(X) < \infty$, *let θ be a partial action of G on X, and let α denote the partial action of G on $C_0(X)$ induced by θ. If θ is free, then*

$$\dim_{\mathrm{Rok}}(\alpha) \leq (|G| - 1)(\dim(X) + 1).$$

Proof. We begin with a general fact.

Claim. Let β be a free partial action of G on a unital commutative C^*-algebra B with finite-dimensional spectrum. Suppose that there is an equivariant extension

$$0 \longrightarrow (D, \delta) \longrightarrow (B, \beta) \longrightarrow (C, \gamma) \longrightarrow 0,$$

with δ decomposable and $\dim_{\mathrm{Rok}}(\gamma) < \infty$. Then

$$\dim_{\mathrm{Rok}}(\beta) \leq \dim_{\mathrm{Rok}}(\delta) + \dim_{\mathrm{Rok}}(\gamma) + 1 < \infty.$$

By Proposition 4.4 in [2], (D, δ) admits a G-invariant approximate identity. Since B is unital and commutative, by Proposition 5.6, we have $\dim_{\mathrm{Rok}}(\beta) \leq \dim_{\mathrm{Rok}}(\delta) + \dim_{\mathrm{Rok}}(\gamma) + 1$, so it suffices to show that $\dim_{\mathrm{Rok}}(\delta)$ is finite. Since β is free, it is easy to see that so is δ. Since δ is decomposable, the claim follows from Proposition 5.3.

We now prove the theorem. We will obtain the bound for $\dim_{\mathrm{Rok}}(\alpha^+)$, which is enough by Remark 5.8. Since α^+ is free by Lemma 5.9, upon replacing α with α^+ we may assume that X is compact. By Theorem 6.1 in [2], there are canonical equivariant extensions

$$0 \longrightarrow (D^{(k)}, \delta^{(k)}) \longrightarrow (A^{(k)}, \alpha^{(k)}) \longrightarrow (A^{(k-1)}, \alpha^{(k-1)}) \longrightarrow 0, \tag{5.11}$$

for $2 \leq k \leq |G|$, satisfying the following properties:

(1) $A^{(|G|)} = C(X)$ and $\alpha^{|G|} = \alpha$;

(2) $\delta^{(k)}$ has the k-decomposition property;

(3) $\alpha^{(1)}$ has the 1-decomposition property.

Note that all the partial actions involved are free (this is shown inductively, using that restrictions of free actions are free). Moreover, $A^{(k)}$ is unital and commutative for all $k = 2, \ldots, |G|$. Note that $\alpha^{(1)}$ has Rokhlin dimension zero by Example 2.2. Applying the above claim to the extension in (5.11) with $k = 2$, we get

$$\dim_{\mathrm{Rok}}(\alpha^{(2)}) \leq \dim_{\mathrm{Rok}}(\delta^{(2)}) + 1 < \infty.$$

Continuing inductively, after $|G| - 1$ steps we deduce that

$$\dim_{\mathrm{Rok}}(\alpha) \leq (\dim_{\mathrm{Rok}}(\delta^{(2)}) + 1) + \cdots + (\dim_{\mathrm{Rok}}(\delta^{(|G|)}) + 1) < \infty. \tag{5.12}$$

Suppose now that $\dim(X)$ is finite. For $k = 2, \ldots, |G|$, let Y_k denote the spectrum of $D^{(k)}$. Since $\dim_{\mathrm{Rok}}(\delta^{(k)}) \leq \dim(Y_k)$ by Proposition 5.3, and $\dim(Y_k) \leq \dim(X)$, the dimensional estimate in the statement follows from (5.12). □

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For *clopen* partial actions (that is, partial actions with clopen domains), no dimensionality assumptions on X are needed.

PROPOSITION 5.11. *Let G be a finite group, let X be a (not necessarily finite-dimensional) compact Hausdorff space, let θ be a free clopen partial action of G on X, and let α denote the induced partial action of G on $C(X)$. Then*

$$\dim_{\mathrm{Rok}}(\alpha) < \min\{\dim(X)+1, \infty\}.$$

In particular, $\dim_{\mathrm{Rok}}(\alpha) < \infty$ *even if* $\dim(X) = \infty$.

Proof. Denote by $(C(Y), \beta)$ the globalization of $(C(X), \alpha)$; see Proposition 3.5 in [1], noting that the graph of θ is closed. Note that $\dim_{\mathrm{Rok}}(\alpha) = \dim_{\mathrm{Rok}}(\beta)$ by Theorem 3.4. Let σ denote the action of G on Y induced by β.

We claim that σ is free. Note that $Y = \bigcup_{g\in G} \sigma_g(X)$. Let $g \in G$ and $y \in Y$ satisfy $\sigma_g(y) = y$. Choose $h \in G$ and $x \in X$ such that $y = \sigma_h(x)$, so that $\sigma_{h^{-1}gh}(x) = x$. Thus x belongs to $X \cap \sigma_{h^{-1}gh}(X) = X_{h^{-1}gh}$, which is where $\sigma_{hgh^{-1}}$ agrees with $\theta_{hgh^{-1}}$. Thus $\theta_{hgh^{-1}}(x) = x$, and since θ is free, this implies that $hgh^{-1} = 1_G$ and hence also $g = 1_G$. Hence σ is free.

Since Y is compact, it follows from Theorem 4.2 in [11] that $\dim_{\mathrm{Rok}}(\beta) < \infty$, and thus $\dim_{\mathrm{Rok}}(\alpha) < \infty$.

We complete the proof by showing that $\dim_{\mathrm{Rok}}(\alpha) \leq \dim(X)$. Note that $\dim(Y) = \dim(X)$ (this follows, for example, by the claim in the proof of Theorem 4.10). Using [17, Lemma 1.9] at the second step, we get

$$\dim_{\mathrm{Rok}}(\alpha) = \dim_{\mathrm{Rok}}(\beta) \leq \dim(Y) = \dim(X). \qquad \square$$

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