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MAL'CEV PRODUCTS OF VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

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Abstract

Whilst the Mal'cev product of completely regular varieties need not again be a variety, it is shown that in many important instances a variety is in fact obtained. However, unlike the product of group varieties this product is nonassociative.

Two important operators introduced by Reilly are studied in the context of Mal'cev products. These operators are shown to generate from any given variety one of the networks discovered by Pastijn and Trotter, enabling identities to be provided for the varieties in the network. In particular the join $\mathbf{O} \vee \mathbf{BG}$ of the varieties of orthogroups and of bands of groups is determined, answering a question of Petrich.

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Completely regular semigroups are semigroups which are unions of groups. They form a variety $\mathbf{CR}$ of unary semigroups, determined by the identities $xx^{-1}x = x$, $(x^{-1})^{-1} = x$ and $xx^{-1} = x^{-1}x$.

Given subvarieties $\mathbf{U}$ and $\mathbf{V}$ of $\mathbf{CR}$, their *Mal'cev product* $\mathbf{U} \circ \mathbf{V}$ consists of those completely regular semigroups S which possess a congruence ρ whose quotient S/ρ belongs to $\mathbf{V}$ and which is *over* $\mathbf{U}$, that is, each class $e\rho (e^2 = e)$ belongs to $\mathbf{U}$. (This is a specialization of Mal'cev's original definition [5].) In general $\mathbf{U} \circ \mathbf{V}$ is a quasivariety.

Two such products were considered (in different terms) by Reilly [10]: for any subvariety $\mathbf{V}$ of $\mathbf{CR}$, he put $\mathbf{V}^{+}\mathbf{G} \circ \mathbf{V}$ and $\mathbf{V}^{P} = \mathbf{B} \circ \mathbf{V}$, where $\mathbf{G}$ and $\mathbf{B}$ denote the varieties of groups and of bands, respectively. The former is always a variety, whereas $\mathbf{G}^{P}$, for example, is not; it consists of all 'E-unitary' completely regular semigroups, whose homomorphic closure is the variety $\mathbf{O}$ of *orthogroups* (orthodox completely regular semigroups).

The main results of this paper are as follows. Let $\mathbf{S}$ denote the variety of *semilattices*. If $\mathbf{S} \subseteq \mathbf{U}$ but $\mathbf{S} \subsetneq \mathbf{V}$, then $\mathbf{U} \circ \mathbf{V}$ is never a variety, unless it is degenerately so, that is, unless $\mathbf{U} \circ \mathbf{V} = \mathbf{U}$. If $\mathbf{S} \subseteq \mathbf{V}$, then $\mathbf{U} \circ \mathbf{V} = (\mathbf{U} \cap \mathbf{CS}) \circ \mathbf{V}$, where $\mathbf{CS}$ is the variety of *completely simple semigroups*. Since $\mathbf{S} \subsetneq \mathbf{U}$ if and only if $\mathbf{U} \subseteq \mathbf{CS}$, it follows that these are the only cases of interest. (This is Theorem 3.1). If $\mathbf{V} \subseteq \mathbf{CS}$ also, then (Theorem 4.1) $\mathbf{U} \circ \mathbf{V}$ is always a variety (again in $\mathbf{CS}$). If $\mathbf{V} \subsetneq \mathbf{CS}$, then (Theorem 5.1) $\mathbf{U} \circ \mathbf{V}$ is a variety whenever $\mathbf{U} \subseteq \mathbf{RG}$, the variety of *rectangular groups* (that is, completely simple orthogroups). The most difficult case is $\mathbf{RG} \circ \mathbf{V}$ itself. Whether $\mathbf{U} \circ \mathbf{V}$ is *always* a variety when $\mathbf{V} \subsetneq \mathbf{CS}$ we do not know.

From Theorem 5.1 it follows that $\mathbf{V}^P = \mathbf{RB} \circ \mathbf{V}$ is always a variety when $\mathbf{V} \subsetneq \mathbf{CS}$. Thus the example $\mathbf{G}^P$ is the exception rather than the rule. Another interesting case is that of $\mathbf{RG} \circ \mathbf{B}$: *bands of rectangular groups* form a variety. This variety is used to show that the Mal′cev product is *nonassociative* even when all products are varieties (Proposition 6.6).

In the last sections the operators $(-)^+$ and $(-)^P$ are studied in more depth. In Section 6, simple identities for $\mathbf{V}^P$ are given in terms of those for $\mathbf{V}$. Identities for $\mathbf{V}^+$ are given as an alternative to those given by Reilly.

Another purpose of this paper is to correlate these operators with 'networks' of subvarieties of $\mathbf{CR}$ which were obtained by Pastijn and Trotter [7] by considering fully invariant congruences on the free completely regular semigroup F on a countably infinite set of generators. From any such congruence ρ (with associated variety $\mathbf{V}_\rho$) one such descending network is obtained by iterating the operations $\rho \to \rho_{\min}$ and $\rho \to \rho^{\min}$ (see Section 7) and taking intersections.

We show in Section 7 that, for $\rho \subseteq \mathscr{D}$, (that is, for $\mathbf{V}_\rho \supseteq \mathbf{S}$), we have $\mathbf{V}_{\rho_{\min}} = \mathbf{V}_\rho^+$ and $\mathbf{V}_{\rho^{\min}} = \mathbf{V}_\rho^P$, and we provide identities for each variety in the corresponding network of varieties *ascending* from $\mathbf{V}_\rho$; the joins in the network (corresponding to the intersections mentioned above) are also described. In particular, the join $\mathbf{O} \vee \mathbf{BG}$ of the varieties of orthogroups and of *bands of groups*, which Pastijn and Trotter showed belongs to the network ascending from $\mathbf{S}$, is just $\mathbf{O}^+ \cap \mathbf{BG}^P$, and consists precisely of the *subdirect products* of orthogroups and bands of groups, the defining identities being a consequence of the earlier results. This answers a question of Petrich [8].

1. Completely regular semigroups

In the sequel 'completely regular' will generally be abbreviated to 'c.r.'.

Let S be a c.r. semigroup and let $x \in S$. Then x^{-1} is the inverse of x in the subgroup H_x. The set of idempotents of S will be denoted E_S. Other notation and terminology will generally follow Howie [3].

The lattice of subvarieties of any subvariety **V** of **CR** will be denoted $\mathscr{L}(\mathbf{V})$. Various varieties were defined in the introduction and others will be defined later when needed. We briefly discuss here the subvarieties of **RG**, the variety of *rectangular groups* (completely simple orthogroups), for future reference. The varieties of *left zero* semigroups, of *right zero* semigroups and of *rectangular bands* are denoted **LZ**, **RZ**, and **RB**, respectively.

The lattice $\mathscr{L}(\mathbf{RG})$ is the direct product of $\mathscr{L}(\mathbf{RB})$ and $\mathscr{L}(\mathbf{G})$, according to Figure 1. For any variety **H** of groups, the varieties **LtH**, **RtH** and **RH** consist of those *left groups*, *right groups* and *rectangular groups* whose subgroups belong to **H**; **T** denotes the trivial variety.

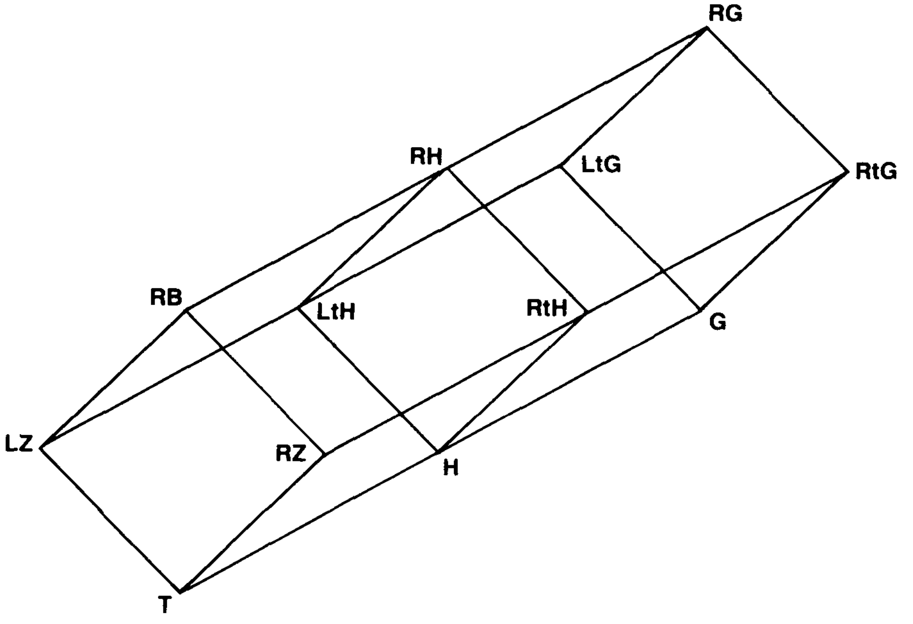

FIGURE 1

Defining identities for many subvarieties of **CR** may be found in [8], to which the reader is referred; [7] contains a fairly comprehensive bibliography.

In the remainder of this section we briefly discuss congruences on regular and c.r. semigroups. Let S be a regular semigroup. Its lattice of congruences will be denoted $\Lambda(S)$. The *trace* $\operatorname{tr}\rho$ of a congruence ρ is its restriction to E_S; its *kernel* $\ker\rho$ is the union of the classes $e\rho$, $e \in E_S$. Two important properties a congruence ρ may possess are:

(i) ρ is *idempotent separating*, that is $\operatorname{tr}\rho = \iota$ (the identical relation), and

(ii) ρ is *idempotent pure*, that is, $\ker\rho = E_S$.

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The largest idempotent separating congruence μ may be described as the largest congruence contained in Green's relation $\mathscr{H}$:

$$\mu = \{(a, b) \in S \times S: xay\mathscr{H}xby \text{ for all } x,\ y \text{ in } S^1\},$$

or, upon specializing the characterization by Hall [1, Theorem 5] to a c.r. semigroup, by:

$$\mu = \{(a, b) \in S \times S: a^0 = b^0 \text{ and } a^{-1}ea = b^{-1}eb \text{ for all } e \text{ in } E_S,\ e \leqslant a^0\}.$$

The largest idempotent pure congruence τ is the 'syntactic' congruence on E_S:

$$\tau = \{(a, b) \in S \times S: xay \in E_S \text{ if and only if } xby \in E_S, \text{ for all } x,\ y \text{ in } S^1\}.$$

Whilst this description is not very useful in practice, the following simple description of $\tau \cap \mathscr{D}$ on a c.r. semigroup will be used in Section 6.

LEMMA 1.1. *Let S be a c.r. semigroup. Then*

$$\tau \cap \mathscr{D} = \{(a, b) \in \mathscr{D}: (xay)(xby)^{-1} \in E_S \textit{ for all } x,\ y \textit{ in } S\}.$$

PROOF. We first prove that for any $\mathscr{D}$-related elements a and b of S, if a, ab^{-1} and $a^0b^0 \in E_S$, then $b \in E_S$. For we may write $b = b(ab^{-1}) = ba^0b^{-1}$ (since $ab^{-1} \in E_S \cap L_b$), whence $b^2 = ba^0b^{-1}ba^0b^{-1} = ba^0b^0a^0b^{-1} = b$ (since $a^0b^0 \in E_S \cap R_a$).

Now let $(a, b) \in \mathscr{D}$, with $(xay)(xby)^{-1} \in E_S$ for all x, y in S. Let $u, v \in S^1$ and suppose that $uav \in E_S$. We will show that $ubv \in E_S$. Put $c = uav$ and $d = ubv$ ($c\mathscr{D}d$). Then

$$(d^0cd^0)d^{-1} = (d^0cd^0)(d^0dd^0)^{-1} = (d^0uavd^0)(d^0ubvd^0)^{-1} \in E_S,$$

since d^0u, $vd^{-1} \in S$. But $d^0cd^0\mathscr{H}d$ (since $D_c = D_d$ is completely simple), so $d = d^0cd^0$ and $d^0 = dd^{-1} = d^0(cd^{-1})$, whence $cd^{-1} \in E_S$. Similarly

$$(cd^{-1})(c^0d^0)^{-1} = (c^0cd^{-1})(c^0dd^{-1})^{-1} = (c^0uavd^{-1})(c^0ubvd^{-1})^{-1} \in E_S,$$

since c^0u, $vd^{-1} \in S$. Again $cd^{-1}\mathscr{H}c^0d^0$, whence $c^0d^0 = cd^{-1} \in E_S$. By the first paragraph of the proof, $d \in E_S$, as required. A similar argument shows that if $ubv \in E_S$, then $uav \in E_S$. Hence $(a, b) \in \tau \cap \mathscr{D}$.

Conversely, let $(a, b) \in \tau \cap \mathscr{D}$ and let x, $y \in S$. Since τ is a congruence, we have $xay\tau xby$ and $(xay)(xby)^{-1}\tau(xby)^0 \in E_S$, whence, since τ is idempotent pure, $(xay)(xby)^{-1} \in E_S$.

The next, well known, result will be used frequently.

LEMMA 1.2. *Let S be a regular semigroup and let α, $\beta \in \Lambda(S)$, $\alpha \subseteq \beta$. Then* $\operatorname{tr}\alpha = \operatorname{tr}\beta$ [$\ker\alpha = \ker\beta$] *if and only if β/α is idempotent separating* [*idempotent pure*] *on S/α.*

When S is c.r. and $\rho \in \Lambda(S)$, then $\ker\rho = \{x\colon x\rho x^0\}$. Thus ρ is idempotent pure if and only if $\rho \cap \mathscr{H} = \iota$. The following technical lemma, false for congruences in general, will be needed in the sequel.

LEMMA 1.3. *Let S be a c.r. semigroup and let $\alpha, \beta \in \Lambda(S)$, $\beta \subseteq \tau \cap \mathscr{D}$. Then* $\ker(\alpha \vee \beta) = \ker\alpha$.

PROOF. Suppose that $x \in \ker(\alpha \vee \beta)$, that is, $(x, x^0) \in \alpha \vee \beta$, and put $e = x^0$. There is a sequence $x = x_0, x_1, \ldots, x_n = e$ with each $(x_{i-1}, x_i) \in \alpha \cup \beta$ and (since $x = exe$) with each $x_i \in eSe$. The proof is by induction on n.

For $n = 1$, either $x\alpha e$ or $x\beta e$, the latter case yielding $x = e$, since β is idempotent pure.

Assume that the existence of any shorter such sequence implies $x\alpha x^0$. Now if $x\beta x_1$, then $x_1 \in D_e \cap eSe = H_e$, so $x = x_1$ and the induction hypothesis applies. Hence we may suppose that $x\alpha x_1$ and, similarly, that $x_{n-1}\alpha e$. Now $x_1^0\alpha e\alpha x_{n-1}$, so applying the induction hypothesis to the sequence $x_1, x_1^0x_2x_1^0, \ldots, x_1^0x_{n-1}x_1^0, x_1^0$ yields $x_1\alpha x_1^0$, whence $x\alpha x_1^0\alpha e$.

2. Mal'cev products

For any variety $\mathbf{A}$ of universal algebras Mal'cev [5, Chapter 32] defined the product $\mathbf{U} \circ \mathbf{V}$ of subvarieties $\mathbf{U}$ and $\mathbf{V}$ of $\mathbf{A}$ to consist of those algebras A in $\mathbf{A}$ which possess a congruence ρ *over* $\mathbf{U}$, (that is, each ρ-class which is a subalgebra belongs to $\mathbf{U}$), such that A/ρ belongs to $\mathbf{V}$. (This specializes to the definition in the introduction for c.r. semigroups.) He showed that the product of (quasi-) varieties is always a quasi-variety (that is, defined by implications) and is thus closed under subalgebras and direct products. Thus in any particular instance only homomorphic closure need be shown in order to prove that the product is again a variety. For varieties of groups, for example, this is a consequence of 'congruence permutability' (see Lemma 2.3 below).

We first present some general remarks on products of universal algebras. For any subvariety $\mathbf{V}$ of $\mathbf{A}$, $\rho_{\mathbf{V}}$ will denote the least $\mathbf{V}$-congruence on any $A \in \mathbf{A}$. It is obvious that $\mathbf{U} \circ \mathbf{V} = \{A \in \mathbf{A}\colon \rho_{\mathbf{V}} \text{ is over } \mathbf{U}\}$.

LEMMA 2.1. *Let* $\mathbf{U}$, $\mathbf{V}$ *be subvarieties of* $\mathbf{A}$, *let* $A \in \mathbf{U} \circ \mathbf{V}$, *and let* ρ *be a congruence on* A. *If* α *is a [the least]* $\mathbf{V}$*-congruence on* A, *then* $(\alpha \vee \rho)/\rho$ *is a [the least]* $\mathbf{V}$*-congruence on* A/ρ. *Thus if* $(\alpha \vee \rho)/\rho$ *is over* $\mathbf{U}$, *then* $A/\rho \in \mathbf{U} \circ \mathbf{V}$.

PROOF. The unbracketed statement is immediate from the isomorphism theorem:

$$(A/\rho)/(\alpha \vee \rho/\rho) \cong A/\alpha \vee \rho \cong (A/\alpha)/(\alpha \vee \rho/\alpha) \in \mathbf{V}.$$

If α is the *least* $\mathbf{V}$-congruence on A, and if π is the congruence on A, containing ρ, which induces the least $\mathbf{V}$-congruence on A/ρ, then $\alpha \subseteq \pi$, since $A/\pi \cong (A/\rho)/(\pi/\rho) \in \mathbf{V}$, so that $(\alpha \vee \rho)/\rho \subseteq \pi/\rho$. Equality follows from the previous paragraph.

We now concentrate on subvarieties of **CR** (although some of the following results may be generalized to 'polar' varieties of universal algebras (see [5])).

LEMMA 2.2. *Let S be a c.r. semigroup and let α, $\rho \in \Lambda(S)$, $\rho \subseteq \alpha$. For any idempotent e of S, the class $(e\rho)(\alpha/\rho)$ of S/ρ is the quotient of the class $e\alpha$ modulo the restriction of ρ. Thus if α is over* $\mathbf{U}$, *then so is α/ρ.*

Hence if $\mathbf{U}, \mathbf{V} \in \mathscr{L}(\mathbf{CR})$, *then* $\mathbf{U} \circ \mathbf{V}$ *is closed under quotients modulo congruences contained in $\rho_{\mathbf{V}}$.*

PROOF. The statements in the first paragraph are obvious. If $S \in \mathbf{U} \circ \mathbf{V}$, and if $\rho \subseteq \rho_{\mathbf{V}}$ on S, then, since $\rho_{\mathbf{V}}$ is over $\mathbf{U}$, so is $\rho_{\mathbf{V}}/\rho$, and so by the previous lemma we have $S/\rho \in \mathbf{U} \circ \mathbf{V}$.

The general form of the next lemma is proven in [5]. We sketch the proof for completeness.

LEMMA 2.3. *Suppose that ρ and α are permuting congruences on the c.r. semigroup S, and let $e \in E_S$. Then $(e\rho)(\alpha \vee \rho)/\rho \cong e(\rho \cap \alpha)(\alpha/\rho \cap \alpha)$. Thus if α is over* $\mathbf{U}$, *then so is $(\alpha \vee \rho)/\rho$.*

Hence if $\mathbf{U}, \mathbf{V} \in \mathscr{L}(\mathbf{CR})$, *then* $\mathbf{U} \circ \mathbf{V}$ *is closed under quotients modulo congruences which permute with $\rho_{\mathbf{V}}$.*

PROOF. Suppose that α and ρ permute, that is, $\alpha \vee \rho = \alpha\rho = \rho\alpha$. If $(x\rho, e\rho) \in (\alpha \vee \rho)/\rho$, then $(x, e) \in \alpha \vee \rho$, and so $x\rho a \alpha e$ for some a in S. The assignment $x\rho \to a(\rho \cap \alpha)$ may be verified to be an isomorphism between the specified classes.

If α is over $\mathbf{U}$ then, by the previous lemma, so is $\alpha/\rho \cap \alpha$, whence $(\alpha \vee \rho)/\rho$ is also, via the isomorphism.

The final statement again follows from Lemma 2.1.

The lemma will be applied in the following situation.

LEMMA 2.4. *Let S be a c.r. semigroup. Then any idempotent separating congruence permutes with any congruence contained in $\mathscr{D}$.*

PROOF. Suppose that ρ is idempotent separating, that is, $\rho \subseteq \mathscr{H}$, and suppose that $\alpha \subseteq \mathscr{D}$. Let $x\rho a\alpha b$ in S. Put $e = x^0 (= a^0)$ and $f = b^0$. Since $a\alpha b$, it follows that $e\alpha f$, and since ρ, $\alpha \subseteq \mathscr{D}$, it follows that each element belongs to the completely simple subsemigroup D_x. Hence

$$b = bb^0 = b(faf)^{-1}(faf)\rho b(faf)^{-1}(fxf)\alpha a(eae)^{-1}(exe) = aa^{-1}x = x.$$

Therefore $\rho\alpha \subseteq \alpha\rho$, and so ρ and α permute.

COROLLARY 2.5. *Let* $\mathbf{U}$ *and* $\mathbf{V} \in \mathscr{L}(\mathbf{CR})$, $\mathbf{U} \subseteq \mathbf{CS}$. *Then* $\mathbf{U} \circ \mathbf{V}$ *is closed under idempotent separating quotients.*

PROOF. If $S \in \mathbf{U} \circ \mathbf{V}$, then $\rho_{\mathbf{V}}$ is over $\mathbf{U}$, so $\rho_{\mathbf{V}} \subseteq \mathscr{D}$.

Next we prove another general lemma.

LEMMA 2.6. *Let S be a c.r. semigroup and let α, $\rho \in \Lambda(S)$, with $\alpha \subseteq \mathscr{D}$ and $\rho \cap \mathscr{D} = \iota$. For any finite set $\{x_1, \ldots, x_n\}$ of $\alpha \vee \rho$-related elements of S, there is a set $\{a_1, \ldots, a_n\}$ of α-related elements of S such that $x_i \rho a_i$, $1 \leqslant i \leqslant n$. If any x_i is idempotent then a_i is idempotent.*

PROOF. The proof is by induction on n. The case $n = 2$ is equivalent to $\alpha \vee \rho = \rho\alpha\rho$, so it is sufficient to show that $\alpha\rho\alpha \subseteq \rho\alpha\rho$. Suppose that $x\alpha a\rho b\alpha y$, and set $c = (xbx)^0 x (xbx)^0$, $d = (yay)^0 y (yay)^0$. Then $c\rho(xax)^0 x (xax)^0 = x$ (since $x\mathscr{D}a$), and $d\rho y$, similarly. Note that c, $d \in D_{xy}$. Now $c\alpha(aba)^0 a(aba)^0 = (bab)^0 b(bab)^0 \alpha d$, the equality following from triviality of ρ on D_{xy}.

Suppose now that $n \geqslant 2$ and that the statement is true for all sets with fewer than n elements. Let $\{x_1, \ldots, x_n\} \subseteq x_1(\alpha \vee \rho)$. Applying the hypothesis to the subset $\{x_2, \ldots, x_n\}$ yields a set $\{y_2, \ldots, y_n\}$ of α-related (and thus $\mathscr{D}$-related) elements such that $x_i \rho y_i$, $2 \leqslant i \leqslant n$; and from the set $\{x_1, x_2\}$ is obtained $\{z_1, z_2\}$, with $x_1 \rho z_1 \alpha z_2 \rho x_2$.

Now for each i, $2 \leqslant i \leqslant n$, we have $z_1 \alpha z_2 \rho y_2 \alpha y_i$, so from the first paragraph of the proof there exist c_i, d_i in $D_{z_1 y_2}$ such that $z_1 \rho c_i \alpha d_i \rho y_i$. Since all the c_i's are ρ- and $\mathscr{D}$-related, they are in fact equal. Putting $a_1 = c_2$ and $a_i = d_i$, $2 \leqslant i \leqslant n$, yields an appropriate set of α-related elements.

Finally, if any $x_i = x_i^0$, then $a_i \rho a_i^0$, whence $a_i = a_i^0$.

COROLLARY 2.7. *Let* $\mathbf{U}$ *and* $\mathbf{V} \in \mathscr{L}(\mathbf{CR})$, $\mathbf{U} \subseteq \mathbf{CS}$. *Then* $\mathbf{U} \circ \mathbf{V}$ *is closed under quotients modulo congruences disjoint from* $\mathscr{D}$. *Hence* $\mathbf{U} \circ \mathbf{V}$ *is a variety if and only if it is closed under quotients modulo congruences contained in* $\mathscr{D}$.

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PROOF. Let $S \in \mathbf{U} \circ \mathbf{V}$ and put $\alpha = \rho_{\mathbf{V}} \subseteq \mathscr{D}$. If $\rho \in \Lambda(S)$, with $\rho \cap \mathscr{D} = \iota$, then by the lemma any finitely generated c.r. subsemigroup of an $(\alpha \vee \rho)/\rho$-class of S/ρ is the image of a c.r. subsemigroup of some α-class of S, under the natural homomorphism. Thus since α is over $\mathbf{U}$, so is $(\alpha \vee \rho)/\rho$, and Lemma 2.1 completes the proof of the first statement.

The second now follows from the isomorphism

$$S/\rho \cong (S/\rho \cap \mathscr{D})/(\rho/\rho \cap \mathscr{D}),$$

since $\mathscr{D}$ on S induces $\mathscr{D}$ on $S/\rho \cap \mathscr{D}$.

Combining the results of this section yields the following technical, but specific, criterion for closure, which will be applied in Sections 4–6. The restriction $\mathbf{U} \subseteq \mathbf{CS}$ is unimportant, as Theorem 3.1 shows.

PROPOSITION 2.8. *Let* $\mathbf{U}$ *and* $\mathbf{V} \in \mathscr{L}(\mathbf{CR})$, $\mathbf{U} \subseteq \mathbf{CS}$. *Then* $\mathbf{U} \circ \mathbf{V}$ *is a variety if and only if for each* $S \in \mathbf{U} \circ \mathbf{V}$, *for each* $\mathscr{D}$*-class* D *of* S, *and for each congruence* $\rho \subseteq \mathscr{D}$ *whose restriction to* D *is idempotent pure*, $(\rho_{\mathbf{V}} \vee \rho)/\rho$ *is over* $\mathbf{U}$ *on* D/ρ.

PROOF. Necessity follows from Lemma 2.1. Conversely, let $S \in \mathbf{U} \circ \mathbf{V}$, put $\alpha = \rho_{\mathbf{V}} (\subseteq \mathscr{D})$ and suppose that $\rho \in \Lambda(S)$. By Corollary 2.7 it may be assumed that $\rho \subseteq \mathscr{D}$. To show that $(\alpha \vee \rho)/\rho$ is over $\mathbf{U}$ on S/ρ, it is therefore sufficient to show the same on D/ρ for any $\mathscr{D}$-class D of S. For such a D, let F be the 'filter' it generates, that is, $F = \{x \in S\colon D_x \geqslant D\}$. Since the restriction of α to F is again the least $\mathbf{V}$-congruence on F and since $e(\alpha \vee \rho) \subseteq F$ for any $e \in D$, we may assume in fact that $S = F$ or, equivalently, that D is the minimum $\mathscr{D}$-class of S.

It is well known that $\mathscr{H}$ on D extends to the congruence $\bar{\mathscr{H}} = \mathscr{H} \cup \iota$ on S. Since $\bar{\mathscr{H}}$ is idempotent separating, so is $\rho \cap \bar{\mathscr{H}}$, whence, by Corollary 2.5, we have $S/\rho \cap \bar{\mathscr{H}} \in \mathbf{U} \circ \mathbf{V}$. But $S/\rho \cong (S/\rho \cap \bar{\mathscr{H}})/(\rho/(\rho \cap \bar{\mathscr{H}}))$, where $\rho/\rho \cap \bar{\mathscr{H}}$ is idempotent pure on the $\mathscr{D}$-class $D/\rho \cap \bar{\mathscr{H}}$ (since $\ker \rho \cap D = \ker(\rho \cap \bar{\mathscr{H}}) \cap D$, where $\ker \bar{\mathscr{H}}$ contains D). If we apply the hypothesis to this $\mathscr{D}$-class of $S/\rho \cap \bar{\mathscr{H}}$, we see that the *induced* congruence $(\rho_{\mathbf{V}} \vee (\rho/\rho \cap \bar{\mathscr{H}}))/(\rho/\rho \cap \bar{\mathscr{H}})$ is over $\mathbf{U}$ on $(S/\rho \cap \bar{\mathscr{H}})/(\rho/\rho \cap \bar{\mathscr{H}})$, where $\rho_{\mathbf{V}} = (\alpha \vee (\rho \cap \bar{\mathscr{H}}))/(\rho \cap \bar{\mathscr{H}})$, by Lemma 2.1, so that $\rho_{\mathbf{V}} \vee (\rho/\rho \cap \bar{\mathscr{H}}) = (\alpha \vee \rho)/\rho \cap \bar{\mathscr{H}}$. When we identify $(S/\rho \cap \bar{\mathscr{H}})/(\rho/\rho \cap \bar{\mathscr{H}})$ with S/ρ, the quotient congruence above becomes $(\alpha \vee \rho)/\rho$.

Note that the restriction of $\rho_{\mathbf{V}}$ to D will *not*, in general, be the least $\mathbf{V}$-congruence on D itself.

Finally we specialize a result from the proof of [5, Chapter 32, Theorem 8].

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RESULT 2.9. *For any* $\mathbf{U}$, $\mathbf{V}$ *and* $\mathbf{W}$ *in* $\mathscr{L}(\mathbf{CR})$, *we have*

$$\mathbf{U}\circ(\mathbf{V}\circ\mathbf{W}) \subseteq (\mathbf{U}\circ\mathbf{V})\circ\mathbf{W}.$$

Sufficient criteria for associativity are given in the theorem cited. In Proposition 6.6 we shall see that the Mal'cev product is not associative on $\mathscr{L}(\mathbf{CR})$, even when all partial products are again varieties.

3. Some special cases

First we show that only products whose first factor is a variety of completely *simple* semigroups are of interest.

THEOREM 3.1. *Let* $\mathbf{U}, \mathbf{V} \in \mathscr{L}(\mathbf{CR})$.

(i) *If* $\mathbf{V} \subsetneqq \mathbf{CS}$, *then* $\mathbf{U}\circ\mathbf{V} = (\mathbf{CS}\cap\mathbf{U})\circ\mathbf{V}$ *for any* $\mathbf{U}$.

(ii) *If* $\mathbf{V} \subseteq \mathbf{CS}$ *and* $\mathbf{U} \subsetneqq \mathbf{CS}$, *then* $\mathbf{U}\circ\mathbf{V}$ *cannot be a variety except in the degenerate instances when* $\mathbf{U}\circ\mathbf{V} = \mathbf{U}$.

PROOF. (i) If $\mathbf{V} \subsetneqq \mathbf{CS}$, then $\mathbf{S} \subseteq \mathbf{V}$, so on any c.r. semigroup S, $\rho_{\mathbf{V}} \subseteq \mathscr{D} = \rho_{\mathbf{S}}$. Thus if $S \in \mathbf{U}\circ\mathbf{V}$, so that $\rho_{\mathbf{V}}$ is over $\mathbf{U}$, then $\rho_{\mathbf{V}}$ is over $\mathbf{CS}\cap\mathbf{U}$.

(ii) If $\mathbf{U} \subsetneqq \mathbf{CS}$, then, again, $\mathbf{S} \subseteq \mathbf{U}$. In particular, $\mathbf{U}$ contains the two-element semilattice $Y = \{0,1\}$, $0 < 1$. Suppose that $T \in \mathbf{U}\circ\mathbf{V}$, $T \notin \mathbf{U}$. Then $Y \times T \in \mathbf{U}\circ\mathbf{V}$ and consists of the two $\mathscr{D}$-classes $\{0\}\times T$ and $\{1\}\times T$. Now the Rees quotient A modulo the ideal $\{0\}\times T$ does not belong to $\mathbf{U}$, since T does not, and the only $\mathbf{V}$-congruence on A is the universal congruence. Thus $A \notin \mathbf{U}\circ\mathbf{V}$, and so $\mathbf{U}\circ\mathbf{V}$ is not a variety.

In the next two sections we shall consider separately the cases $\mathbf{U}, \mathbf{V} \subseteq \mathbf{CS}$ and $\mathbf{U} \subseteq \mathbf{CS}$, $\mathbf{V} \subsetneqq \mathbf{CS}$.

Next we consider some products $\mathbf{U}\circ\mathbf{V}$, $\mathbf{U} \subseteq \mathbf{CS}$, for which $\mathbf{U}$ and $\mathbf{V}$ are particularly special. First note that for any variety $\mathbf{H}$ of groups, $\mathbf{H}\circ\mathbf{RB}$ consists of all completely simple semigroups whose subgroups belong to $\mathbf{H}$. It is easily seen to be a variety, which we shall denote $\mathbf{CS(H)}$.

For any variety $\mathbf{U} \subseteq \mathbf{CS}$, $\mathbf{U}\circ\mathbf{S}$ consists of all c.r. semigroups whose $\mathscr{D}$-classes belong to $\mathbf{U}$. Reilly [10, Theorem 4.4] showed that $\mathbf{U}\circ\mathbf{S}$ is a variety. In particular, $\mathbf{CR(H)} = \mathbf{CS(H)}\circ\mathbf{S}$ is the variety of all c.r. semigroups whose subgroups belong to $\mathbf{H}$, and $\mathbf{CS}\circ\mathbf{S}$ is, of course, just $\mathbf{CR}$. (Thus $\mathbf{CS}\circ\mathbf{V} = \mathbf{CR}$ whenever $\mathbf{S} \subseteq \mathbf{V}$.)

Next we treat the cases $\mathbf{U} = \mathbf{G}$ and $\mathbf{U} = \mathbf{RtG}$ (and also $\mathbf{U} = \mathbf{LtG}$ by duality). It is obvious that a congruence ρ on a c.r. semigroup is over $\mathbf{G}$ or $\mathbf{RtG}$ if and only if ρ is contained in $\mathscr{H}$ or $\mathscr{R}$, respectively. Since $\mathscr{H}$ and $\mathscr{R}$ are equivalences they contain greatest congruences $\mathscr{H}^{\flat} (= \mu)$ and $\mathscr{R}^{\flat}$ respectively. Thus the following descriptions apply.

LEMMA 3.2. *Let* $\mathbf{V} \in \mathscr{L}(\mathbf{CR})$. *Then*
(i) $\mathbf{G} \circ \mathbf{V} = \{S: \rho_{\mathbf{V}} \subseteq \mathscr{H}\} = \{S: S/\mu \in \mathbf{V}\}$, *and*
(ii) $\mathbf{RtG} \circ \mathbf{V} = \{S: \rho_{\mathbf{V}} \subseteq \mathscr{R}\} = \{S: S/\mathscr{R}^{b} \in \mathbf{V}\}$.

As defined in the introduction, $\mathbf{G} \circ \mathbf{V} = \mathbf{V}^{+}$. That $\mathbf{V}^{+}$ is a variety was shown by Reilly [10, Lemma 3.2].

PROPOSITION 3.3. *Let* $\mathbf{V} \in \mathscr{L}(\mathbf{CR})$. *Then* $\mathbf{RtG} \circ \mathbf{V}$, $\mathbf{LtG} \circ \mathbf{V}$ *and* $\mathbf{G} \circ \mathbf{V}$ *are always varieties.*

PROOF. It suffices to prove that $\mathbf{RtG} \circ \mathbf{V}$ is a variety, since $\mathbf{G} = \mathbf{RtG} \cap \mathbf{LtG}$. In view of Lemma 2.1 and the above discussion, it is enough to show that for any congruence $\alpha \subseteq \mathscr{R}$ on a c.r. semigroup S, we have $(\alpha \vee \rho)/\rho \subseteq \mathscr{R}$ on S/ρ for any congruence ρ on S.

So let $x, y \in S$, with $(x\rho, y\rho) \in (\alpha \vee \rho)/\rho$, that is, $x(\alpha \vee \rho)y$. Then there is a sequence $x = x_0, x_1, \ldots, x_n = y$ of elements of S with each $(x_{i-1}, x_i) \in \alpha \cup \rho$. Now if $x_{i-1}\alpha x_i$, then $x_{i-1}\mathscr{R}x_i$ and $(x_{i-1}\rho)\mathscr{R}(x_i\rho)$ in S/ρ. Otherwise $x_{i-1}\rho = x_i\rho$, so $(x\rho)\mathscr{R}(y\rho)$ in S/ρ, as required.

Properties of congruences over groups, left groups, right groups and completely simple semigroups are discussed for regular semigroups in general in [6].

Similar behaviour is provided by $\mathbf{V}^{P} = \mathbf{B} \circ \mathbf{V}$ ($= \mathbf{RB} \circ \mathbf{V}$ when $\mathbf{S} \subseteq \mathbf{V}$, which by Theorem 3.1 is the only case of interest). A congruence $\rho \subseteq \mathscr{D}$ is over $\mathbf{RB}$ if and only if it is idempotent pure, and there is a largest such congruence $\tau \cap \mathscr{D}$. Thus when $\mathbf{S} \subseteq \mathbf{V}$, we have $\mathbf{V}^{P} = \{S: S/\tau \cap \mathscr{D} \in \mathbf{V}\}$. Properties of $\mathbf{V}^{+}$ and $\mathbf{V}^{P}$ will be studied further in Sections 6 and 7.

In general there may be no greatest congruence over $\mathbf{U}$ on a c.r. semigroup. For example, let S be a completely simple semigroup. Then $\mathscr{L}$ and $\mathscr{R}$ are each over $\mathbf{RG}$ on S, but their join, the universal congruence on S, is not over $\mathbf{RG}$ if S is not itself in $\mathbf{RG}$. We will show, however, that $\mathbf{RG} \circ \mathbf{V}$ *is* a variety whenever $\mathbf{S} \subseteq \mathbf{V}$.

4. Products of completely simple varieties

Throughout this section 'completely simple' will be abbreviated to 'c.s.'.

THEOREM 4.1. *Let* $\mathbf{U}$ *and* $\mathbf{V} \in \mathscr{L}(\mathbf{CS})$. *Then* $\mathbf{U} \circ \mathbf{V}$ *is again a variety* (*in* $\mathscr{L}(\mathbf{CS})$).

PROOF. By Proposition 2.8 it is sufficient to consider closure under idempotent pure quotients. So let $S \in \mathbf{U} \circ \mathbf{V}$, put $\alpha = \rho_{\mathbf{V}}$ and let ρ be an idempotent pure congruence on S. By Lemma 1.3, $\ker(\alpha \vee \rho) = \ker \alpha$. Thus for each e in E_S, we have $e(\alpha \vee \rho) \cap H_e = \ker(\alpha \vee \rho) \cap H_e = \ker\alpha \cap H_e = e\alpha \cap H_e$. Hence if α is over $\mathbf{CS(H)}$ for some variety $\mathbf{H}$ of *groups*, then so is $\alpha \vee \rho$ and therefore also $(\alpha \vee \rho)/\rho$. Hence $\mathbf{CS(H)} \circ \mathbf{V}$ is a variety.

Next, if $\mathbf{U}$ is any subvariety of $\mathbf{CS}$ and $\mathbf{H} = \mathbf{U} \cap \mathbf{G}$, then $\mathbf{H} \subseteq \mathbf{U} \subseteq \mathbf{CS(H)}$ and $\mathbf{G} \cap \mathbf{CS(H)} = \mathbf{H}$, so by modularity of $\mathscr{L}(\mathbf{CS})$ (see, for example, [2, Corollary 3.6]), we have $\mathbf{U} = (\mathbf{U} \vee \mathbf{G}) \cap \mathbf{CS(H)}$, whence $\mathbf{U} \circ \mathbf{V} = (\mathbf{U} \vee \mathbf{G}) \circ \mathbf{V} \cap \mathbf{CS(H)} \circ \mathbf{V}$.

Without loss of generality it may be assumed, therefore, that $\mathbf{U}$ contains $\mathbf{G}$. The cases $\mathbf{U} = \mathbf{G}$, $\mathbf{U} = \mathbf{LtG}$ and $\mathbf{U} = \mathbf{RtG}$ having been treated in Proposition 3.3, it may be further assumed that $\mathbf{U}$ contains $\mathbf{RG}$.

Now if $\mathbf{V} \subsetneqq \mathbf{G}$, then either $\mathbf{LZ} \subseteq \mathbf{V}$ or $\mathbf{RZ} \subseteq \mathbf{V}$, whence either $\alpha \subseteq \mathscr{R}$ or $\alpha \subseteq \mathscr{L}$ and $\mathbf{U} \circ \mathbf{V} = (\mathbf{U} \cap \mathbf{RtG}) \circ \mathbf{V}$ or $(\mathbf{U} \cap \mathbf{LtG}) \circ \mathbf{V}$, respectively. Thus we may assume that $\mathbf{V} \subseteq \mathbf{G}$. But then for any e in E_S, we have $e\alpha = \ker\alpha = \ker(\alpha \vee \rho) = e(\alpha \vee \rho)$. Thus $\alpha \vee \rho$, and therefore $(\alpha \vee \rho)/\rho$, is over $\mathbf{U}$, so $S/\rho \in \mathbf{U} \circ \mathbf{V}$, from Lemma 2.1.

5. Products $\mathbf{U} \circ \mathbf{V}$ with $\mathbf{U} \subseteq \mathbf{CS}$ and $\mathbf{S} \subseteq \mathbf{V}$

In view of Theorem 3.1 and the results of Section 4, this is the remaining situation of interest. Whether *every* such product yields a variety we do not know. The main result is the following, the most difficult case being $\mathbf{U} = \mathbf{RG}$ itself.

THEOREM 5.1. *The product* $\mathbf{U} \circ \mathbf{V}$ *is a variety whenever* $\mathbf{U} \subseteq \mathbf{RG}$ *and* $\mathbf{S} \subseteq \mathbf{V}$, *and thus whenever* $\mathbf{U} \subseteq \mathbf{O}$ *and* $\mathbf{S} \subseteq \mathbf{V}$.

PROOF. We apply Proposition 2.8. Let $S \in \mathbf{U} \circ \mathbf{V}$, put $\alpha = \rho_{\mathbf{V}}$ $(\subseteq \mathscr{D})$, let D be a $\mathscr{D}$-class of S and let $\rho \in \Lambda(S)$ be contained in $\mathscr{D}$ and idempotent pure on D. It remains to show that $(\alpha \vee \rho)/\rho$ is over $\mathbf{U}$ on D/ρ. Arguments similar to those of the previous section permit the assumption that $\mathbf{G} \subseteq \mathbf{U}$. (We omit the details.) The cases $\mathbf{U} = \mathbf{LtG}$, $\mathbf{U} = \mathbf{RtG}$ and $\mathbf{U} = \mathbf{G}$ were treated in Proposition 3.3. Only the case $\mathbf{U} = \mathbf{RG}$ therefore remains. This case is an immediate consequence of Proposition 5.3 below, which depends on the following lemma.

LEMMA 5.2. *Let D be a completely simple semigroup and suppose that $A_1, A_2, \ldots, A_n$ are mutually disjoint orthodox subsemigroups of D with the properties*:

(i) $\{R \in D/\mathscr{R}\colon R \cap A_i \neq \varnothing\} = \{R \in D/\mathscr{R}\colon R \cap A_j \neq \varnothing\}$ *for all* i, j;

(ii) *For each* i, $1 \leqslant i \leqslant n$, *there exist* $a_i \in A_i$ *and* $b_i \in A_{i+1}$ *such that* $L_{a_i} \cup L_{b_i}$ *is orthodox*.

Then D is orthodox, that is, a rectangular group.

PROOF. We will prove this for $n = 2$ only, since a simple induction then establishes the general result.

Let $e \in A_1 \cap E_D$, let $f \in A_2 \cap E_D$, and, using (i), choose idempotents g and h in $R_f \cap L_{a_1}$ and $R_f \cap L_{b_1}$, respectively. Then $ef = eghf$. Since A_1 is orthodox, eg is an idempotent (in L_{a_1}) whence, by (ii), egh is an idempotent (in L_{b_1}). By (i) again, A_2 meets the $\mathscr{H}$-class H_{egh}, and so contains egh. Hence $eghf \in E_D$, as required.

This lemma was proved jointly with T. E. Hall.

PROPOSITION 5.3. *Let D be a completely simple semigroup and let α, $\rho \in \Lambda(D)$, with α over* **RG** *and ρ idempotent pure. Then $\alpha \vee \rho$ is over* **RG**, *whence $(\alpha \vee \rho)/\rho$ is also.*

PROOF. The proof is based on the observation that if $a\rho b$, then $L_a \cup L_b$ is orthodox. For since $a^0 \rho b^0$, we may assume that a, $b \in E_D$, so if $e \in L_a$ and $f \in L_b$ are idempotents, then $ef = eafb\rho eabfb = eab\rho ea = e$, whence $ef \in E_D$ since ρ is idempotent pure.

Now let $(e, f) \in \alpha \vee \rho$, e, $f \in E_D$. If $(e, f) \notin \alpha$, then there is a sequence $e = e_0 \alpha f_0 \rho e_1 \alpha f_1 \cdots e_n \alpha f_n = f$ of idempotents of D. For each j, $0 \leqslant j \leqslant n$, put $R_j = \bigcup\{(e_j e_i)^0\alpha \colon 1 \leqslant i \leqslant n\}$. If R is an $\mathscr{R}$-class of D, then $R \cap (e_j e_i)^0\alpha = (R \cap (e_j e_k)^0\alpha)(e_j e_i)^0$ for all j, k (since $e_j e_i \mathscr{R} e_i$ for all j). From the above sequence is obtained the new sequence $(e_j e_0)^0\alpha(e_j f_0)^0\rho(e_j e_1)^0\alpha(e_j f_1)^0 \cdots (e_j e_n)^0\alpha(e_j f_n)^0$. Thus the distinct classes $(e_j e_i)^0\alpha$ satisfy the hypotheses of the lemma, whence R_j is orthodox for each j.

Note that e_j, $f_j \in R_j$ for each j. By applying the dual of the lemma to the distinct sets among the R_j's, a similar argument shows that $\bigcup\{R_j \colon 0 \leqslant j \leqslant n\}$ is orthodox. Thus ef $(= e_0 f_n)$ is idempotent.

One case of particular interest is $\mathbf{V} = \mathbf{B}$. In that case we write **BU** for $\mathbf{U} = \mathbf{B}$, the variety of *bands of* **U**-*semigroups*. Thus $\mathbf{B}(-)$ is a well defined operator on $\mathscr{L}(\mathbf{O})$. (But note that $\mathbf{BU} = \mathbf{B}(\mathbf{U} \cap \mathbf{CS})$). For example, **BG** $(= \mathbf{B}^+)$ is the well known variety of *bands of groups*. The following corollary was discovered jointly with Hall and motivated the study of the general case.

COROLLARY 5.4. *Bands of rectangular groups form a variety.*

The $\mathbf{B}(-)$ operator will be discussed further at the end of the next section.

6. The operators $(-)^+$ and $(-)^P$

Recall that for any subvariety $\mathbf{V}$ of $\mathbf{CR}$, we have $\mathbf{V}^+ = \{S\colon S/\mu \in \mathbf{V}\} = \mathbf{G} \circ \mathbf{V}$ and $\mathbf{V}^P = \{S\colon S/\tau \in \mathbf{V}\} = \mathbf{B} \circ \mathbf{V}$. When $\mathbf{S} \subseteq \mathbf{V}$, we have $\mathbf{V}^P = \{S\colon S/\tau \cap \mathscr{D} \in \mathbf{V}\}$, and so it follows from Theorem 5.1 that $\mathbf{V}^P$ is a variety. The following direct proof of this fact is included for its simplicity.

PROPOSITION 6.1. *Let* $\mathbf{S} \subseteq \mathbf{V}$. *Then* $\mathbf{V}^P$ *is a variety.*

PROOF. Let $S \in \mathbf{V}^P$ (so that $\rho_{\mathbf{V}} \subseteq \tau \cap \mathscr{D}$) and let $\rho \in \Lambda(S)$. By Lemma 1.3, $\ker(\rho_{\mathbf{V}} \vee \rho) = \ker \rho$, so $(\rho_{\mathbf{V}} \vee \rho)/\rho$ is idempotent pure on S/ρ. An application of Lemma 2.1 completes the proof.

Suppose that $\mathbf{V}$ is determined by the identities $u_i = v_i$, $i \in I$, where $u_i = u_i(x_1, \dots, x_n)$ and $v_i = v_i(x_1, \dots, x_n)$. Then we write $\mathbf{V} = [u_i = v_i\colon i \in I]$. In the sequel any new variables $x, y, \dots$ appearing in identities will be presumed different from $x_1, \dots, x_n$.

RESULT 6.2. [10, Theorem 3.9]. *If* $\mathbf{V} = [u_i = v_i\colon i \in I]$, *then* $\mathbf{V}^+ = [u_i^0 = v_i^0, (xu_iy)^0 = (xv_iy)^0\colon i \in I]$.

This result is based on the description of μ as $\mathscr{H}^\flat$. Identities for $\mathbf{RtG} \circ \mathbf{V}$ and $\mathbf{LtG} \circ \mathbf{V}$ may be provided by using similar descriptions of $\mathscr{R}^\flat$ and $\mathscr{L}^\flat$. (See Section 3.) Identities for $\mathbf{V}^+$ will now be derived from the following alternative description of μ. Whilst each identity is more complicated, the set has the virtue of having the same cardinality as the defining set of $\mathbf{V}$. Thus $\mathbf{V}^+$ is determined by at most as many identities as is $\mathbf{V}$.

PROPOSITION 6.3. *Let S be a c.r. semigroup. Then* $\mu = \{(a, b)\colon a^{-1}(a^0ea^0)^0a = b^{-1}(b^0eb^0)^0b$ *for all e in* $E_S\}$.

PROOF. Denote the relation on the right by π. We use the description

$$\mu = \left\{(a, b)\colon a^0 = b^0 \text{ and } a^{-1}ea = b^{-1}eb \text{ for all } e \in E_S,\ e \leqslant a^0\right\}$$

from Section 1.

If $a\mu b$ and $e \in E_S$, then $a^{-1}(a^0ea^0)^0a\mu b^{-1}(b^0eb^0)^0b$, and so these elements are $\mathscr{H}$-related. But each is an idempotent, so they are equal and $(a, b) \in \pi$.

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Conversely, let $(a, b) \in \pi$. First, put $e = a^0$. Then $a^0 = a^{-1}a^0a = b^{-1}(b^0a^0b^0)^0b$, so $a^0 = b^0a^0$. Similarly, putting $e = b^0$ yields $a^{-1}(a^0b^0a^0)^0a = b^0$, so $b^0 = b^0a^0 = a^0$.

Next if $f \in E_S$ and $f \leqslant a^0$, then $f = (a^0fa^0)^0 = (b^0fb^0)^0$, so $a^{-1}fa = b^{-1}fb$. Thus $a\mu b$, as required.

The following corollary is obtained in the same way as Result 6.2.

COROLLARY 6.4. *If* $\mathbf{V} = [u_i = v_i\colon i \in I]$, *then*

$$\mathbf{V}^+ = \left[u_i^{-1}\left(u_i^0x^0u_i^0\right)^0u_i = v_i^{-1}\left(v_i^0x^0v_i^0\right)^0v_i\colon i \in I\right].$$

Now identities for $\mathbf{V}^P$ will be given (for $\mathbf{S} \subseteq \mathbf{V}$) based on the description of $\tau \cap \mathscr{D}$ in Lemma 1.1. Again the details are omitted. (That '$u_i\mathscr{D}v_i$' follows from the fact that if $\mathbf{S} \subseteq \mathbf{V}$, then u_i and v_i involve the same variables.) In the sequel, an identity of the form $u = u^0$ will be expressed as '$u \in E$'.

COROLLARY 6.5. *Let* $\mathbf{V} = [u_i = v_i\colon i \in I]$. *Then* $\mathbf{V}^P = [(xu_iy)(xv_iy)^{-1} \in E\colon i \in I]$.

Thus $\mathbf{V}^P$, too, is determined by at most as many identites as is $\mathbf{V}$.

Many further properties, particularly of the $(-)^+$ operator, are discussed in [10]. We complete this section by considering the relationship between the $(-)^P$ operator and the $\mathbf{B}(-)$ operator introduced in the previous section. Specializing Result 2.9 yields $\mathbf{RB} \circ (\mathbf{V} \circ \mathbf{B}) \subseteq (\mathbf{RB} \circ \mathbf{V}) \circ \mathbf{B}$, that is, $(\mathbf{BV})^P \subseteq \mathbf{BV}^P$ for any variety $\mathbf{V}$. We now present an example to show that equality need not hold, even if $\mathbf{S} \subseteq \mathbf{V} \subseteq \mathbf{O}$. We choose $\mathbf{V} = \mathbf{SG}$, the variety of *semilattices of groups*. Note that $\mathbf{B}(\mathbf{SG}) = \mathbf{BG}$ and that $(\mathbf{SG})^P = \mathbf{O}$ (see Section 8), so that $\mathbf{B}(\mathbf{SG})^P = \mathbf{BO} = \mathbf{BRG}$.

PROPOSITION 6.6. *The varieties* $(\mathbf{BSG})^P (= (\mathbf{BG})^P)$ *and* $\mathbf{B}(\mathbf{SG})^P (= \mathbf{BRG})$ *are distinct. Hence the Mal'cev product is not associative on the lattice of varieties containing* $\mathbf{S}$.

PROOF. Let $D = \mathscr{M}(G; I, I; P)$, where $I = \{0, 1\}$, $G = \{e, a\}$ and $P = \left(\begin{smallmatrix} e & e \\ e & a \end{smallmatrix}\right)$. Then $D \notin \mathbf{RG}$. Let S be the ideal extension of D by the group $\{1, x\}$, where 1 is the identity for S, and where

$$(i, g, \lambda)x = (i, g, \lambda + 1 (\mathrm{mod}\, 2)),$$

$$x(i, g, \lambda) = \begin{cases} (i, g, \lambda) & \text{if } i = 0, \\ (i, ag, \lambda) & \text{if } i = 1, \end{cases}$$

for $g \in G$, $i, \lambda \in I$.

Then $\mathscr{R}$ is a $\mathbf{B}$-congruence on S over $\mathbf{RtG}$, so $S \in \mathbf{RG} \circ \mathbf{B} = \mathbf{BRG} = \mathbf{B(SG)}^P$, but $\mathscr{L}$ is not a congruence since $(1, x) \in \mathscr{L}$ and $((1, e, 1) \cdot 1, (1, e, 1) \cdot x) \notin \mathscr{L}$. Thus $S \notin \mathbf{BG}$. But since D is nonorthodox, we have $\tau \cap \mathscr{D} = \iota$ on S, so no idempotent pure quotient of S is a band of groups. Hence $S \notin (\mathbf{BG})^P = (\mathbf{BSG})^P$.

Thus the variety $\mathbf{BRG}$ (see Corollary 5.4) does not consist merely of idempotent pure 'co-extensions' of bands of groups.

7. The Pastijn-Trotter network

Let F denote the free c.r. semigroup on a countably infinite set. It is well known that there is an anti-isomorphism $\rho \to \mathbf{V}_\rho$ between the lattice of fully invariant congruences on F and the lattice $\mathscr{L}(\mathbf{CR})$. In [7] Pastijn and Trotter derived from any given fully invariant congruence ρ two networks of such congruences, one ascending, the other descending, from ρ. The more interesting seems to be the latter (since it generates larger varieties from smaller). In this section the associated network of *varieties* will be correlated with the operators $(-)^+$ and $(-)^P$, yielding identities for the varieties and enabling certain joins to be described in more detail.

Let ρ be a fully invariant congruence on F and denote by $\rho_{\min}$ $[\rho^{\min}]$ the least congruence on F whose trace [kernel] is that of ρ. In [7, Theorem 3.6] $\rho_{\min}$ and $\rho^{\min}$ are shown to be fully invariant. The network in question consists of the descending sequences $\rho_{\min}, (\rho_{\min})^{\min}, \dots$ and $\rho^{\min}, (\rho^{\min})_{\min}, \dots,$ together with their intersections. That a sublattice with ρ as maximum element is obtained is a consequence of the following.

RESULT 7.1 ([7, Theorem 3.4]). *Let S be a regular semigroup and let $\rho \in \Lambda(S)$. Then $\rho_{\min} \cap \rho^{\min} = (\rho^{\min})_{\min} \vee (\rho_{\min})^{\min}$. If ρ is fully invariant on F, then, as intervals in the lattice of fully invariant congruences on F, we have the relation*

$$[\rho_{\min} \cap \rho^{\min}, \rho] \cong [\rho_{\min}, \rho] \times [\rho^{\min}, \rho].$$

(*See Figure* 2.)

For any variety $\mathbf{V}$ there is, therefore, a corresponding network of varieties *ascending* from $\mathbf{V}$. The reader is referred to [7] for further details. The connection with the operators $(-)^+$ and $(-)^P$ is now made clear.

PROPOSITION 7.2. *Let ρ be a fully invariant congruence on F and $\mathbf{V}_\rho$ the associated variety. Then*

(i) $\mathbf{V}_{\rho_{\min}} = \mathbf{V}_\rho^+$, *and, if* $\rho \subseteq \mathscr{D}$ (*that is,* $\mathbf{S} \subseteq \mathbf{V}$), *then*
(ii) $\mathbf{V}_{\rho^{\min}} = \mathbf{V}_\rho^P$.

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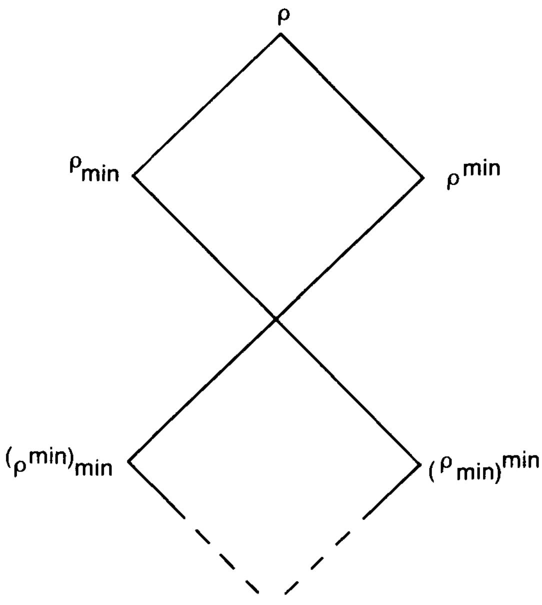

FIGURE 2

PROOF. Let $F' = F/\rho_{\min}$. Now $F' \in \mathbf{V}_\rho^+$, since $\rho/\rho_{\min}$ is idempotent separating (Lemma 1.2), and $F'/(\rho/\rho_{\min}) \cong F/\rho \in \mathbf{V}_\rho$. Thus $\mathbf{V}_{\rho_{\min}} \subseteq \mathbf{V}_\rho^+$.

Conversely, since $\mathbf{V}_\rho^+$ is a variety, $\mathbf{V}_\rho^+ = \mathbf{V}_\pi$ for some fully invariant congruence π on F. Let $T = F/\pi$. Then the induced congruence ρ/π on T is the least $\mathbf{V}_\rho$-congruence on T and is therefore idempotent separating, that is $\operatorname{tr}\rho = \operatorname{tr}\pi$. Hence $\rho_{\min} \subseteq \pi$ and $T \in \mathbf{V}_{\rho_{\min}}$, whence $\mathbf{V}_\rho^+ \subseteq \mathbf{V}_{\rho_{\min}}$.

A similar argument, using the fact that $\mathbf{V}^P$ *is* a variety when $\mathbf{S} \subseteq \mathbf{V}$, proves (ii).

We shall consider only varieties containing $\mathbf{S}$. The operators yield the network shown in Figure 3. By Result 7.1,

$$\mathbf{V}^P \vee \mathbf{V}^+ = (\mathbf{V}^+)^P \cap (\mathbf{V}^P)^+, \quad \text{and} \quad [\mathbf{V}, \mathbf{V}^P \vee \mathbf{V}^+] \cong [\mathbf{V}, \mathbf{V}^P] \times [\mathbf{V}, \mathbf{V}^+].$$

Clearly the identities obtained for $\mathbf{V}^P$ and $\mathbf{V}^+$ in the previous section enable identities to be found for *every* variety in this network. In fact the varieties in the generating sequences may be defined by at most as many identities as define $\mathbf{V}$, and those found as joins by at most twice as many.

In the next result we use the simpler identities of Result 6.2 together with those of Corollary 6.4.

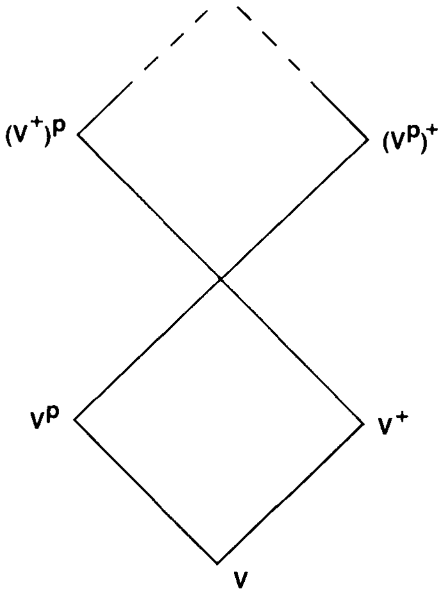

FIGURE 3

PROPOSITION 7.3. *Let* $\mathbf{V} = [u_i = v_i \colon i \in I]$ *contain* $\mathbf{S}$. *Then*
(1) $(\mathbf{V}^P)^+ = [(a(xu_iy)(xv_iy)^{-1}b)^0 = (a((xu_iy)(xv_iy)^{-1})^0b)^0 \colon i \in I]$, *and*
(2) $(\mathbf{V}^+)^P = [(xu_i^0y)(xv_i^0y)^{-1} \in E, (x(au_ib)^0y)(x(av_ib)^0y)^{-1} \in E \colon i \in I]$.

PROOF. The other identities for $(\mathbf{V}^P)^+$ obtained from Result 6.2 are trivially valid.

The joins $\mathbf{V}^+ \vee \mathbf{V}^P$ are of particular interest, since many joins of well known varieties may be represented in this fashion (see [7] and [8]). Recall that for varieties $\mathbf{A}$ and $\mathbf{B}$ in general, $\mathbf{A} \vee \mathbf{B}$ consists of all *homomorphic images* of subdirect products of members of $\mathbf{A}$ and $\mathbf{B}$. Only in very special situations does $\mathbf{A} \vee \mathbf{B}$ consist of subdirect products alone.

THEOREM 7.4. *Let* $\mathbf{V} = [u_i = v_i \colon i \in I]$. *The following are equivalent for a c.r. semigroup* S:

(i) $S \in \mathbf{V}^+ \vee \mathbf{V}^P$;

(ii) S *is a subdirect product of a member of* $\mathbf{V}^+$ *with a member of* $\mathbf{V}^P$;

(iii) S *satisfies the identites* (1) *and* (2) *above*.

PROOF. The equivalence of (i) and (iii) follows from the equality of $\mathbf{V}^+ \vee \mathbf{V}^P$ with $(\mathbf{V}^+)^P \cap (\mathbf{V}^P)^+$.

Now let $S \in \mathbf{V}^{+} \vee \mathbf{V}^{P}$. Then $S/\tau \in \mathbf{V}^{+}$ and $S/\mu \in \mathbf{V}^{P}$. But $\mu \cap \tau = \iota$, so S is a subdirect product of S/μ and S/τ. That (ii) implies (i) is trivial.

8. The join $\mathbf{O} \vee \mathbf{BG}$

Theorem 7.4 is now specialized to answer, in two different ways, the question posed by Petrich [8]: What is the join $\mathbf{O} \vee \mathbf{BG}$? That this join belongs to the network ascending from $\mathbf{S}$ (or from $\mathbf{OBG}$), was observed by Pastijn and Trotter [7]: as noted there and in [10], we have $\mathbf{S}^{+} = \mathbf{SG}$, $\mathbf{S}^{P} = \mathbf{B}$, $\mathbf{B}^{+} = \mathbf{BG}$ and $(\mathbf{SG})^{P} = \mathbf{O}$; moreover, $\mathbf{OBG}$ is the variety of *orthodox bands of groups*.

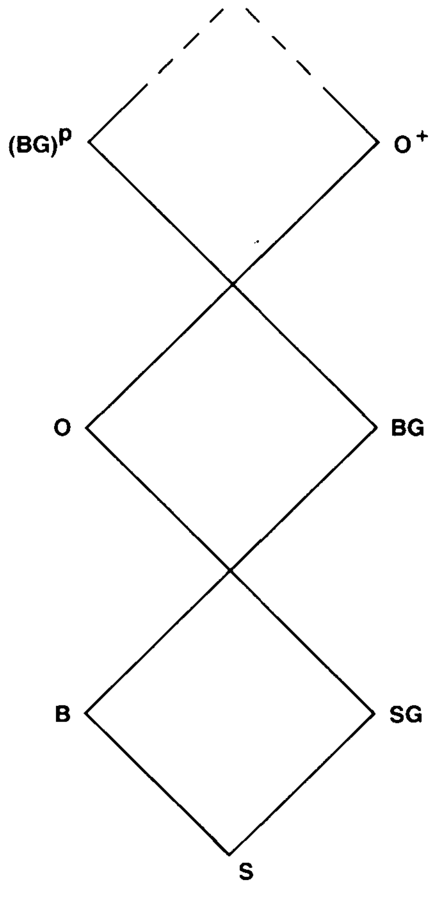

FIGURE 4

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The variety $\mathbf{O}^+$ was introduced in different terms by Hall and the author [2, Lemma 5.2] as '$\mathbf{I}$'; alternative descriptions were found in [10, Proposition 3.5, Corollary 3.10] and [7, Theorem 5.15]:

RESULT 8.1. *The variety* $\mathbf{O}^+ = [(xa^0b^0y)^0 = (x(a^0b^0)^0y)^0]$ *and consists of those c.r. semigroups whose idempotent-generated subsemigroups are bands of groups.*

The identity above follows from Result 6.2. Similarly, applying Corollary 6.7 to $\mathbf{BG}(=\mathbf{B}^+) = [(xay)^0 = (xa^0y)^0]$ yields $(\mathbf{BG})^P = [(c(xay)^0d)(c(xa^0y)^0d)^{-1} \in E]$. It would be useful to have a description of $(\mathbf{BG})^P$ akin to that of $\mathbf{O}^+$. From Proposition 6.6 and the remarks prior to it, we have $(\mathbf{BG})^P \subset \mathbf{BRG}$, but the containment is strict.

Applying Theorem 7.4, with $\mathbf{V} = \mathbf{OBG}$ $(\mathbf{V}^+ = \mathbf{BG}, \mathbf{V}^P = \mathbf{O})$, yields the following descriptions of $\mathbf{O} \vee \mathbf{BG}$.

THEOREM 8.2. *The following are equivalent for a c.r. semigroup* S:

(i) $S \in \mathbf{O} \vee \mathbf{BG}$;

(ii) S *is a subdirect product of an orthogroup and a band of groups*;

(iii) S *satisfies the identities*

$$(xa^0b^0y)^0 = \left(x(a^0b^0)^0y\right)^0 \quad \textit{and} \quad \left(c(xay)^0d\right)\left(c(xa^0y)^0d\right)^{-1} \in E.$$

We remark that since $\mathbf{B} = [x^2 = x]$ and $\mathbf{SG} = [x^0y^0 = y^0x^0]$, each 'generating' variety of the above network is defined by a single identity, and each join by at most two.

Various other similar situations are described in [7]. In particular, setting $\mathbf{V} = \mathbf{NB}$ (*normal bands*) yields $\mathbf{V}^+ = \mathbf{NBG}$ (*normal bands of groups*) and $\mathbf{V}^P = \mathbf{B}$. Thus $\mathbf{NBG} \vee \mathbf{B} = (\mathbf{NBG})^P \cap \mathbf{BG}$. As shown in [2] and [10], $(\mathbf{NBG})^P = \mathbf{PO}$, the variety of *pseudo-orthodox* (now more commonly called '*locally orthodox*') c.r. semigroups. Thus $(\mathbf{CS} \vee \mathbf{B} =)\mathbf{NBG} \vee \mathbf{B} = \mathbf{PO} \cap \mathbf{BG}$, and identities are easily derived. This join was found by Hall and the author [2], and by Rasin [9].

Replacing $\mathbf{NB}$ by $\mathbf{ONBG}$ $(= \mathbf{O} \cap \mathbf{NBG})$ yields the join $(\mathbf{CS} \vee \mathbf{O} =)\ \mathbf{NBG} \vee \mathbf{O} = \mathbf{PO} \cap \mathbf{O}^+$, where $\mathbf{O}^+$ was described above. Again identities follow. This join was determined in [2, Theorem 5.3]; identities were found by Reilly [10, Corollary 5.3], who also described the lattice $\mathscr{L}(\mathbf{CS} \vee \mathbf{O})$.

In each of these cases the join consists precisely of the subdirect products of semigroups from the specified varieties.

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