Can. J. Math., Vol. XXVI, No. 1, 1974, pp. 207–213

ENTIRE FUNCTIONS WITH SOME DERIVATIVES UNIVALENT

S. M. SHAH AND S. Y. TRIMBLE

1. Introduction. This paper is a continuation of the author's previous work, [**6**; **7**], on the relationship between the radius of convergence of a power series and the number of derivatives of the power series which are univalent in a given disc.

In particular, let D be the open disc centered at 0, and let f be regular there. Suppose that $\{n_p\}_{p=1}^{\infty}$ is a strictly-increasing sequence of positive integers such that each $f^{(n_p)}$ is univalent in D. Let R be the radius of convergence of the power series, centered at 0, that represents f. In [**7**], we investigated the connection between R and $\{n_p\}_{p=1}^{\infty}$. We showed that, in general,

$$\liminf_{p\to\infty} (n_1 n_2 \ldots n_p)^{1/n_p} \leqq 4R. \tag{1.1}$$

In the present paper, we both improve and simplify (1.1). We show that if $\lim_{p\to\infty} n_p/n_{p+1} = 1$ or if $n_{p+2} - 2n_{p+1} + n_p = o(n_p)$, then f is entire.

We shall use the following result from univalent function theory. If f is defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

and if it is univalent in D, then there is a constant, C, such that

$$|a_k| \leqq Ck, \tag{1.2}$$

for $k = 2, 3, \ldots$. The Bieberbach conjecture is that $C = 1$. It has long been known that $C \leqq e$ [**2**, pp. 10–11; **5**, p. 218]. Recently, it has been shown that $C \leqq \sqrt{7/6}$ [**1**]. For our results, the exact value of C is immaterial, so we shall not assign a particular value to it. Often, we shall work with the derivatives of a function, F, which is defined by

$$F(z) = \sum_{k=0}^{\infty} A_k z^k.$$

In this case, if $F^{(n)}$ is univalent in D, then (1.2) becomes

$$(n + k)!|A_{n+k}| \leqq Ck(k!)(n + 1)!|A_{n+1}| \tag{1.3}$$

for $k = 2, 3, \ldots$.

Finally, we remark that we shall often be dealing with subsequences. For ease of notation, we shall write $a(n_p)$ and $n(p_k)$ for a_{n_p} and n_{p_k}.

Received September 6, 1972 and in revised form, October 25, 1972.

https://doi.org/10.4153/CJM-1974-020-4 Published online by Cambridge University Press

2. Relations between R and $\{n_p\}_{p=1}^{\infty}$. In a previous result, (1.1), the constant, 4, appears on one side of the inequality. In our present work, we shall encounter other such constants. In order to eliminate these constants from our final conclusions, we need the following lemma.

LEMMA 1. *Let $\{n_p\}_{p=1}^{\infty}$ be a strictly-increasing sequence of non-negative integers. Let*

$$\alpha = \liminf_{p\to\infty} \frac{n_{p-1}}{n_p} .$$

(1) *If* $\limsup_{p\to\infty}(n_{p-1}/n_p) < 1$, *then* $\lim_{p\to\infty}(p/n_p) = 0$.

(2) *Suppose that* $0 < \alpha < 1$ *and that* $\epsilon > 0$. *Then there is a subsequence,* $\{n(p_k)\}_{k=1}^{\infty}$, *of* $\{n_p\}_{p\to\infty}^{\infty}$ *such that* $\alpha - \epsilon \leqq \liminf_{k\to\infty}[n(p_{k-1})/(np_k)]$ *and* $\limsup_{k\to\infty}[n(p_{k-1})/n(p_k)] < 1$.

Proof. The proof of (1) is straightforward. To prove (2), let μ be chosen so that $\alpha < 1 - \mu < 1$ and $\mu\alpha < \epsilon$. Let $n(p_1) = n_1$. Assume that $n(p_k)$ has been chosen. Let $n(p_{k+1})$ be the smallest integer in the sequence, $\{n_p\}_{p=1}^{\infty}$, such that

$$\frac{n(p_k)}{n(p_{k+1})} \leqq 1 - \mu.$$

Clearly, $\limsup_{k\to\infty} n(p_k)/n(p_{k+1}) < 1$. Suppose that, for some k,

$$\frac{n(p_k)}{n(p_{k+1})} \leqq \alpha - \epsilon. \tag{2.1}$$

Now the definition of $n(p_{k+1})$ implies that

$$\frac{n(p_k)}{n(p_{k+1} - 1)} > 1 - \mu.$$

Hence,

$$\frac{n(p_{k+1} - 1)}{n(p_{k+1})} < \frac{\alpha - \epsilon}{1 - \mu} .$$

Using the definition of μ, it follows that, if (2.1) were true for an infinite number of k, then

$$\liminf_{k\to\infty} \frac{n(p_{k+1} - 1)}{n(p_{k+1})} < \alpha.$$

This is impossible. Hence, (2.1) can be true for only a finite number of k. This implies that $\alpha - \epsilon \leqq \liminf_{k\to\infty} n(p_k)/n(p_{k+1})$. The proof of the lemma is complete.

The following lemma is of a technical nature and will be useful in simplifying a certain inequality in the proof of our theorem.

LEMMA 2. *Let* $\{n_p\}_{p=1}^{\infty}$ *be a strictly increasing sequence of non-negative integers. Then, for each* $p \geqq 2$,

$$\prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)^{1/(n_p+2)} < (1 + n_p/p)^{p/n_p} \leqq 2.$$

Proof. First, note that for $c > 0$ and $y > 0$, it follows that $\log y \leqq cy - 1 - \log c$. So, for $c > 0$,

$$\log \prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)^{1/(n_p+2)} \leqq \frac{1}{n_p + 2} \sum_{j=1}^{p-1} [c(n_{j+1} - n_j + 1) - 1 - \log c]$$

$$< c + \frac{p}{n_p} (c - 1 - \log c).$$

Let $c = p/(n_p + p)$. Using the fact that $(1 + 1/x)^x$ increases with x for $0 < x \leqq 1$, the conclusion follows.

We remark that if $n_p = p$, then the left-hand-side of the inequality in the conclusion of Lemma 2 is just $2^{(p-1)/(p+2)}$. Hence, the number, 2, on the right-hand-side of this inequality cannot in general be reduced in size.

THEOREM. *Let f be regular in D. Let* $\{n_p\}_{p=1}^{\infty}$ *be a strictly increasing sequence of non-negative integers such that* $f^{(n_p)}$ *is univalent in D. Suppose that f is defined by a power series, expanded about* 0, *with a radius of convergence, R. Let* $\alpha = \liminf_{p\to\infty} n_p/n_{p+1}$.

(1) *If* $\alpha = 1$, *then* $R = \infty$, i.e., *f is entire.*

(2) *If* $0 < \alpha < 1$, *then* $R \geqq \alpha^{\alpha/(\alpha-1)}/(1 - \alpha)$.

(3) *If* $\alpha = 0$, *then* $R \geqq 1$.

(While part 3 is obvious, it is included for completeness and to emphasize the fact that $\lim_{\alpha\to 0^+} \alpha^{\alpha/(\alpha-1)}/(1 - \alpha) = 1$.)

Proof. Suppose that

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Using (1.3) it follows that, for $p = 1, 2, \ldots$ and for $k = 2, 3, \ldots$,

$$|a(n_p + k)| \leqq \frac{Ckk!(n_p + 1)!}{(n_p + k)!} |a(n_p + 1)|. \tag{2.2}$$

Let $k = n_{p+1} - n_p + 1$ and induct on p. Then for $p = 2, 3, \ldots$,

$$|a(n_p + 1)| \leqq \frac{C^{p-1}D^*}{(n_p + 1)!} \prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)(n_{j+1} - n_j + 1)!,$$

where $D^* = (n_1 + 1)!\ |a(n_1 + 1)|$. Combining this with (2.2), we conclude that, for $2 \leqq p$ and for $2 \leqq k \leqq n_{p+1} - n_p + 1$,

$$|a(n_p + k)| \leqq \frac{C^p D^* kk!}{(n_p + k)!} \prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)(n_{j+1} - n_j + 1)!. \tag{2.3}$$

https://doi.org/10.4153/CJM-1974-020-4 Published online by Cambridge University Press

There are two positive numbers, A and B, such that, if n is a positive integer larger than 1, then [**4**, p. 183]

$$An^{\frac{1}{2}}(n/e)^n < n! < Bn^{\frac{1}{2}}(n/e)^n.$$

Using this on the right-hand-side of (2.3), taking the $(n_p + k)$th root of both sides of the resulting inequality, and applying Lemma 2 to part of right-hand-side of this, it follows that for $2 \leqq p$ and for $2 \leqq k \leqq n_{p+1} - n_p + 1$,

$$(2.4) \quad |a(n_p + k)|^{1/(n_p+k)} < \left[\frac{D^*B^pC^pe^{n_1+1-p}}{A}\left(\frac{k^3}{n_p + k}\right)^{\frac{1}{2}}\right]^{1/(n_p+k)} \times \left(1 + \frac{n_p}{p}\right)^{5p/2n_p}\left[\frac{k^{k/(n_p+k)}}{n_p + k}\prod_{j=1}^{p-1}(n_{j+1} - n_j + 1)^{(n_{j+1}-n_j)/(n_p+k)}\right].$$

Now we maximize the expression in the second set of brackets on the right-hand-side of (2.4). Let

$$a = \prod_{j=1}^{p-1}(n_{j+1} - n_j + 1)^{n_{j+1}-n_j}$$

and let $b = n_p$. Define ϕ for $x > 0$ by

$$\phi(x) = \frac{(ax^x)^{1/(b+x)}}{b + x}.$$

Then $a^{1/b}$ is the only value of x for which ϕ' vanishes. Further, this yields a minimum for ϕ. It follows that the maximum of ϕ on any closed interval occurs at one of the endpoints. Hence, for $2 \leqq p$ and for $2 \leqq k \leqq n_{p+1} - n_p + 1$,

$$\frac{k^{k/(n_p+k)}}{n_p + k}\prod_{j=1}^{p-1}(n_{j+1} - n_j + 1)^{(n_{j+1}-n_j)/(n_p+k)}$$
$$\leqq \max\left\{\frac{4^{1/(n_p+2)}}{n_p + 2}\prod_{j=1}^{p-1}(n_{j+1} - n_j + 1)^{(n_{j+1}-n_j)/(n_p+2)}, \frac{(n_{p+1} - n_p + 1)^{1/(n_{p+1}+1)}}{n_{p+1} + 1}\prod_{j=1}^{p}(n_{j+1} - n_j + 1)^{(n_{j+1}-n_j)/(n_{p+1}+1)}\right\}.$$

Making obvious simplifications in this, and then using it on the right-hand-side of (2.4), we have that, if $2 \leqq p$ and if $2 \leqq k \leqq n_{p+1} - n_p + 1$,

$$|a(n_p + k)|^{1/(n_p+k)} < \left[\frac{D^*B^pC^pe^{n_1+1-p}}{A}\left(\frac{k^3}{n_p + k}\right)^{\frac{1}{2}}\right]^{1/(n_p+k)}\left(1 + \frac{n_p}{p}\right)^{5p/2n_p}$$
$$(2.5) \quad \times \max\left\{\frac{4^{1/n_p}}{n_p}\prod_{j=1}^{p-1}(n_{j+1} - n_j + 1)^{(n_{j+1}-n_j)/n_p}, \frac{n_{p+1}^{1/n_{p+1}}}{n_{p+1}}\prod_{j=1}^{p}(n_{j+1} - n_j + 1)^{(n_{j+1}-n_j)/n_{p+1}}\right\}.$$

Finally, letting $\beta = \limsup_{p\to\infty} p/n_p$ and using (2.5), we have that

$$(2.6)\quad \frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k} = \limsup \{|a(n_p + k)|^{1/(n_p+k)} : 2 \leqq p,$$

$$2 \leqq k \leqq n_{p+1} - n_p + 1\}$$

$$\leqq K \limsup_{p\to\infty} \frac{1}{n_p} \prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)^{(n_{j+1}-n_j)/n_p},$$

where

$$K = \begin{cases} 1, & \beta = 0 \\ \left[\frac{BC}{e}\left(1 + \frac{1}{\beta}\right)^{5/2}\right]^{\beta}, & \beta > 0. \end{cases}$$

To get the right-hand-side of (2.6) in terms of α, we need the following inequality [**3**, p. 7]: If $\{x_p\}_{p=1}^{\infty}$ is a strictly-increasing sequence of positive numbers such that $\lim_{p\to\infty} x_p = \infty$, and if $\{y_p\}_{p=1}^{\infty}$ is a sequence of real numbers, then

$$\limsup_{p\to\infty} \frac{y_p}{x_p} \leqq \limsup_{p\to\infty} \frac{y_p - y_{p-1}}{x_p - x_{p-1}}.$$

For our application, let $x_p = n_p$ and let

$$y_p = \sum_{j=1}^{p-1} [(n_{j+1} - n_j) \log (n_{j+1} - n_j + 1)] - n_p \log n_p.$$

Then

$$\frac{y_p - y_{p-1}}{x_p - x_{p-1}} = \log\left(1 - \frac{n_{p-1} - 1}{n_p}\right) + \frac{n_{p-1}/n_p}{1 - n_{p-1}/n_p} \log \frac{n_{p-1}}{n_p}.$$

Now the second expression on the right-hand-side of this equation is a decreasing function of n_{p-1}/n_p. Further, $\lim_{x\to 1^-}(x \log x)/(1 - x) = -1$ and $\lim_{x\to 0^+}(x \log x)/(1 - x) = 0$. Hence,

$$\limsup_{p\to\infty} \frac{y_p}{x_p} \leqq \begin{cases} -\infty, & \alpha = 1 \\ \log(1 - \alpha) + \frac{\alpha}{1-\alpha}\log \alpha, & 0 < \alpha < 1 \\ 0, & \alpha = 0. \end{cases}$$

Since

$$\frac{y_p}{x_p} = \log \frac{1}{n_p} \prod_{j=1}^{p-1} (n_{j+1} - n_j + 1)^{(n_{j+1}-n_j)/n_p},$$

it follows from (2.6) that

$$1/R \leqq \begin{cases} 0, & \alpha = 1 \\ K(1 - \alpha)\alpha^{\alpha/(1-\alpha)}, & 0 < \alpha < 1 \\ K, & \alpha = 0. \end{cases}$$

To complete the proof, we must show that we can always assume that $K = 1$. Lemma 1 is used to do this. First of all, note that the only condition on the sequence, $\{n_p\}_{p=1}^{\infty}$, was that each $f^{(n_p)}$ be univalent in D. Any subsequence of $\{n_p\}_{p=1}^{\infty}$ clearly satisfies the same condition.

If $\alpha = 0$, it is obvious that we may let $K = 1$. Suppose $0 < \alpha < 1$. Let $0 < \epsilon < \alpha$. The second part of Lemma 1 shows that there is a subsequence, $\{n(p_k)\}_{k=1}^{\infty}$, such that $\alpha - \epsilon \leqq \liminf_{k\to\infty}[n(p_{k-1})/n(p_k)]$ and

$$\limsup_{k\to\infty}[n(p_{k-1})/n(p_k)] < 1.$$

The first part of that lemma then implies that $\lim_{k\to\infty} k/n(p_k) = 0$. Applying what we have just proved to this subsequence, $\{n(p_k)\}_{k=1}^{\infty}$, it follows that

$$1/R \leqq (1 - \alpha + \epsilon)(\alpha - \epsilon)^{(\alpha-\epsilon)/(1-\alpha+\epsilon)}.$$

Since this is true for all sufficiently small ϵ, the theorem is established.

Our theorem is stated so as to get information about R when α is known. However, if R is known, then the theorem can be used to obtain a bound on α. For example, we have the following.

COROLLARY 1. *Let f and $\{n_p\}_{p=1}^{\infty}$ be defined as in the theorem. Suppose that f cannot be extended to a function regular in a larger disc centered at* 0. *Then*

$$\liminf_{p\to\infty} n_{p-1}/n_p = 0.$$

Note that Theorem 6 in [**7**] shows that there do exist functions which satisfy the hypothesis of this corollary. Further, these functions show that the inequality in part (3) of the conclusion of the present theorem is sharp.

If $\lim_{p\to\infty} n_{p-1}/n_p = 1$, then our result immediately shows that f is entire. From this, it follows that, if $n_p - n_{p-1} = o(n_p)$, then f is also entire. This is an improvement to the corollary to Theorem 3 in [**7**]. In fact, to insure that f is entire, it is enough to put this condition on the second differences of the n_p.

COROLLARY 2. *Assume that f and $\{n_p\}_{p=1}^{\infty}$ satisfy the hypotheses of the theorem. If*

$$n_{p+2} - 2n_{p+1} + n_p = o(n_p),$$

then f is entire.

Proof. We rewrite the hypothesis in three ways:

$$n_{p+1} + n_{p-1} = [2 + o(1)]n_p. \tag{2.7}$$

$$n_{p-2} - 2n_{p-1} = [-1 + o(1)]n_p. \tag{2.8}$$

$$n_{p+2} - 2n_{p+1} = [-1 + o(1)]n_p. \tag{2.9}$$

Multiplying (2.7) by 2 and adding it to (2.8), we have

$$n_{p-2} + 2n_{p+1} = [3 + o(1)]n_p.$$

https://doi.org/10.4153/CJM-1974-020-4 Published online by Cambridge University Press

This implies the following:

(A) If $b \leqq \liminf_{p\to\infty} n_{p-2}/n_p$, then $\limsup_{p\to\infty} n_{p+1}/n_p \leqq (3-b)/2$.

Multiplying (2.7) by 2 and adding it to (2.9), we have

$$n_{p+2} + 2n_{p-1} = [3 + o(1)]n_p.$$

Hence,

$$3 \geqq \limsup_{p\to\infty} \frac{n_p}{n_{p-2}} + 2 \liminf_{p\to\infty} \frac{n_p}{n_{p+1}} = \frac{1}{\liminf_{p\to\infty} \frac{n_{p-2}}{n_p}} + \frac{2}{\limsup_{p\to\infty} \frac{p+1}{n_p}}.$$

This implies the following:

(B) If $\limsup_{p\to\infty} n_{p+1}/n_p \leqq a$, then $\liminf_{p\to\infty} n_{p-2}/n_p \geqq a/(3a-2)$.

Next, we construct two sequences. Let $b_0 = 0$. Assume that b_m has been defined. Let $a_m = (3 - b_m)/2$ and let $b_{m+1} = a_m/(3a_m - 2)$. Using an induction argument, (A) and (B) show that $\liminf_{p\to\infty} n_{p-2}/n_p \geqq b_m$ and that

$$\limsup_{p\to\infty} \frac{n_{p+1}}{n_p} \leqq a_m \tag{2.10}$$

for all m. Further, the definitions of a_m and b_m imply that

$$a_m = \frac{4a_{m-1} - 3}{3a_{m-1} - 2}. \tag{2.11}$$

Now (2.10) allows us to conclude that $a_m \geqq 1$ for all m. This and (2.11) imply that $\{a_m\}_{m=0}^{\infty}$ is a decreasing sequence and that $\lim_{m\to\infty} a_m$ exists. Using the latter fact in (2.11), it follows that $\lim_{m\to\infty} a_m = 1$. Finally, from (2.10), we conclude that $\lim_{p\to\infty} n_{p+1}/n_p = 1$. Hence, f is entire.

References

1. C. H. FitzGerald, *Exponentiation of certain quadratic inequalities for Schlicht functions*, Bull. Amer. Math. Soc. *78* (1972), 209–210.
2. W. K. Hayman, *Multivalent functions* (Cambridge University Press, Cambridge, 1958).
3. E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series*, vol. II (Dover Publications, New York, 1957).
4. D. S. Mitrinovic, *Analytic inequalities* (Springer-Verlag, New York, 1970).
5. Z. Nehari, *Conformal mapping* (McGraw-Hill, New York, 1952).
6. S. M. Shah and S. Y. Trimble, *Univalent functions with univalent derivatives*, Bull. Amer. Math. Soc. *75* (1969), 153–157.
7. ——— *Univalent functions with univalent derivatives, III*, J. Math. Mech. *19* (1969), 451–460.

University of Kentucky,
Lexington, Kentucky;
University of Missouri,
Rolla, Missouri