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Jackson's Theorem for finite products and homomorphic images of locally compact abelian groups

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Let G be a Hausdorff locally compact abelian group. The author has shown (*Bull. Austral. Math. Soc.* 10 (1974), 59-66) that, given $\varepsilon > 0$ and a certain base $\{V_i\}_{i\in I}$ of symmetric open neighbourhoods of zero, the algebra $L^1(G)$ admits a bounded positive approximate unit $\{k_i\}_{i\in I}$ such that for every p-th integrable function f on G,

$$\|k_i*f-f\|_p \le (1+\varepsilon)\omega(p;\, f;\, V_i) \ , \quad i \in I \ ,$$

where $\omega(p;\, f;\, V_i)$ denotes the mean modulus of continuity with exponent p of f. The purpose of this paper is to obtain $\{k_i\}_{i\in I}$ (as above) with a simple dependence of $\operatorname{supp}(\hat{k}_i)$ on V_i; this is achieved for finite products and homomorphic images of groups for which such a simple dependence is known. The results obtained are used to give a simplified proof of the classical Jackson's Theorem for the circle group, and an analogue of this theorem for the a-adic solenoid.

We shall firstly examine extensions of the Corollary on p. 60 of [1]. Throughout this paper λ_G will denote a chosen Haar measure on the locally compact abelian group G. If G_0 is an open subgroup of G then λ_{G_0}

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will just be the restriction of λ_G to G_0. Where no confusion should arise (for example in integrals) the subscript to the Haar measure will be omitted.

We shall say that G has property $P(I, V_i, k_i, \mathrm{T}_i, K)$ if there is a base $\{V_i\}_{i\in I}$ of symmetric open neighbourhoods of zero, and a corresponding family $\{k_i\}_{i\in I}$ of non-negative continuous functions on G such that for each $i \in I$, $\int_G k_i d\lambda = 1$, $\operatorname{supp}(k_i) \subset V_i$ (the open subgroup of G generated by V_i), $\operatorname{supp}(\hat{k}_i) \subset \mathrm{T}_i$, T_i is compact, and

$$\int_{V_i} m_{V_i} k_i d\lambda \le K ;$$

here m_{V_i} is the integer-valued function on V_i defined by

$$m_{V_i}(x) = \min\{m \in \{1, 2, \dots\} : x \in mV_i\} .$$

Using the first part of the proof of [1], Theorem 2, we can readily obtain the following result for finite products of groups:

THEOREM 1. *Suppose* G, G' *have properties* $P(I, V_i, k_i, \mathrm{T}_i, K)$, $P(I, V_i', k_i', \mathrm{T}_i', K')$ *respectively. Then* $G \times G'$ *has property* $P(I, V_i\times V_i', l_i, \mathrm{T}_i'', KK')$, *where*

$$l_i((x, x')) = k_i(x)k_i'(x') , \quad (x, x') \in G \times G' ,$$

and

$$\mathrm{T}_i'' = \{[\gamma, \gamma'] : \gamma \in \mathrm{T}_i, \gamma' \in \mathrm{T}_i'\} ,$$

$$([\gamma, \gamma'](x, x')) = \gamma(x)\gamma'(x') \text{ for } (x, x') \in G\times G') . \qquad //$$

To obtain an analogous result for a homomorphic image of G we require some preliminary work. Let H be a closed subgroup of G and denote by π the natural homomorphism of G onto G/H. For each $i \in I$ it is clear that the restriction $\pi|_{V_i}$ of π to the open subgroup V_i of G is an open continuous homomorphism of V_i onto $\pi(V_i)$. It follows,

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using [2], 5.27, that $\pi(V_i)$ and $V_i/(V_i \cap H)$ are topologically isomorphic, the natural topological isomorphism ν_i being given by

$$\nu_i(x+H) = x + V_i \cap H \ , \quad x \in V_i \ .$$

Haar measure $\lambda_{G/H}$ on G/H will be chosen so that Weil's formula holds (see [3], (28.54) (iii)). If we define Haar measure $\lambda_{V_i/(V_i \cap H)}$ on $V_i/(V_i \cap H)$ by

$$\lambda_{V_i/(V_i \cap H)}(E) = \lambda_{\pi(V_i)}\left[\nu_i^{-1}(E)\right] ,$$

then [5], 7.8, p. 87, shows that λ_{V_i}, $\lambda_{V_i \cap H}$ and $\lambda_{V_i/(V_i \cap H)}$ also satisfy Weil's formula. We shall assume these relations throughout.

The character group $\Gamma_{G/H}$ of G/H is topologically isomorphic with $A(\Gamma_G, H)$ (the annihilator of H in Γ_G) where, to each $\gamma \in A(\Gamma_G, H)$, there corresponds $\gamma^+ \in \Gamma_{G/H}$ such that

$$\gamma^+(x+H) = \gamma(x) \ , \quad x \in G \ .$$

Given $\Xi \subset A(\Gamma_G, H)$ we write

$$\Xi^+ = \{\gamma^+ \in \Gamma_{G/H} : \gamma \in \Xi\} \ .$$

For an open subgroup G_0 of G the restriction map

$$\sigma_{G_0} : \Gamma_G \to \Gamma_{G_0} ,$$

defined by $\sigma_{G_0}(\gamma) = \gamma|_{G_0}$, is an open continuous homomorphism of Γ_G onto the character group of G_0 , with kernel $A(\Gamma_G, G_0)$.

Finally the adjoint $\nu_i^{\sim}$ of ν_i is a map

$$\nu_i^{\sim} : \Gamma_{V_i/(V_i \cap H)} \to \Gamma_{\pi(V_i)} ,$$

given by $\nu_i^{\sim}(\gamma) = \gamma \circ \nu_i$; see [2], (24.37).

With the notation as above we now have:

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THEOREM 2. *Suppose* G *has property* $P(I, V_i, k_i, T_i, K)$. *Then* G/H *has property* $P(I, \pi(V_i), l_i, \Omega_i, K)$, *where*

$$l_i(x+H) = \begin{cases} l_i^! \circ \nu_i(x+H) , & x+H \in \pi(V_i) , \\ 0 , & \textit{otherwise}, \end{cases}$$

$$l_i^!(x+V_i\cap H) = \int_{V_i\cap H} k_i(x+y)d\lambda(y) ,$$

and

$$\Omega_i = \sigma_{\pi(V_i)}^{-1}\left[\nu_i^{\sim}\left\{\left(\left[\sigma_{V_i}(T_i)\cap A\left[\Gamma_{V_i}, V_i\cap H\right]\right]\right)^{+}\right\}\right] .$$

Proof. Clearly $l_i \geq 0$ and $\operatorname{supp}(l_i) \subset \pi(V_i)$ (note that $\pi(V_i)$ is the open subgroup of G/H generated by $\pi(V_i)$). Since $k_i \in L^1(G)$ and $\operatorname{supp}(k_i) \subset V_i$ we have $k_i \in L^1(V_i)$ and, by [3], (28.54) (ii), $l_i^! \in L^1(V_i/(V_i\cap H))$. It follows that $l_i \in L^1(G/H)$ and, appealing to Weil's formula,

$$\begin{aligned} \int_{G/H} l_i d\lambda &= \int_{\pi(V_i)} l_i^! \circ \nu_i d\lambda \\ &= \int_{V_i/(V_i\cap H)} l_i^! d\lambda \\ &= \int_{V_i} k_i d\lambda \\ &= 1 . \end{aligned}$$

It is easy to prove that for all $y \in V_i \cap H$ and $x + V_i \cap H \in V_i/(V_i\cap H)$,

$$m_{\pi(V_i)} \circ \nu_i^{-1}(x+V_i\cap H) \leq m_{V_i}(x+y) < \infty .$$

From this we deduce

$$\begin{aligned}\int_{\pi(V_i)} m_{\pi(V_i)} l_i d\lambda &= \int_{\pi(V_i)} m_{\pi(V_i)} l_i^! \circ \nu_i d\lambda \\ &= \int_{V_i/(V_i\cap H)} m_{\pi(V_i)} \circ \nu_i^{-1}(\dot{x}) \int_{V_i\cap H} k_i(x+y)d\lambda(y)d\lambda(\dot{x}) \\ &\le \int_{V_i/(V_i\cap H)} \int_{V_i\cap H} m_{V_i}(x+y)k_i(x+y)d\lambda(y)d\lambda(\dot{x}) \\ &= \int_{V_i} m_{V_i} k_i d\lambda \\ &\le K .\end{aligned}$$

For $\gamma \in A\left(\Gamma_{V_i}, V_i\cap H\right)$ we know ([3], (31.46) (ii)) that

$$\hat{l}_i^!(\gamma^+) = \hat{k}_i(\gamma) ,$$

whence it follows that

$$\operatorname{supp}\left(\hat{l}_i^!\right) \subset \left(\sigma_{V_i}(\mathrm{T}_i)\cap A\left(\Gamma_{V_i}, V_i\cap H\right)\right)^+ .$$

Also, for $\dot{\gamma} \in \Gamma_{G/H}$, we have

$$\begin{aligned}\hat{l}_i(\dot{\gamma}) &= (l_i^!\circ\nu_i)^\wedge\left(\sigma_{\pi(V_i)}(\dot{\gamma})\right) \\ &= \hat{l}_i^!\left(\nu^{\sim-1}\left(\sigma_{\pi(V_i)}(\dot{\gamma})\right)\right) ,\end{aligned}$$

which, combined with the previous equation, gives

$$\begin{aligned}\operatorname{supp}(\hat{l}_i) &\subset \sigma_{\pi(V_i)}^{-1}\left(\nu_i^\sim\left(\operatorname{supp}\left(\hat{l}_i^!\right)\right)\right) \\ &\subset \sigma_{\pi(V_i)}^{-1}\left[\nu_i^\sim\left(\left(\sigma_{V_i}(\mathrm{T}_i)\cap A\left(\Gamma_{V_i}, V_i\cap H\right)\right)^+\right)\right] .\end{aligned}$$

Now $A\left(\Gamma_{G/H}, \pi(V_i)\right)$ is compact (since $\pi(V_i)$ is open), whence it follows, using [2], (5.24) (a), that Ω_i is compact.

Finally, as $l_i \in L^1(G/H)$ has compactly supported Fourier transform, we have l_i equal almost everywhere to a continuous function. //

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It appears that for groups having property $P(I, V_i, k_i, T_i, K)$ there is a ready analogue of Jackson's Theorem. For each $i \in I$ we let

$$\omega(p; f; V_i) = \sup\{\|\tau_a f - f\|_p : a \in V_i\}$$

denote the mean modulus of continuity with exponent p of $f \in L^p(G)$, and put

$$E_{T_i}(p; f) = \inf\left\{\|f-g\|_p : g \in L^p(G),\ \Sigma(g) \subset T_i\right\} ;$$

(for the definition of the spectrum $\Sigma(g)$ of g, see [1], p. 59).

THEOREM 3. *Let G be a locally compact abelian group having property $P(I, V_i, k_i, T_i, K)$. Then*

(1) $\|k_i*f-f\|_p \le K\omega(p; f; V_i)$,

(2) $E_{T_i}(p; f) \le K\omega(p; f; V_i)$

for every $f \in L^p(G)$ if $p \in [1, \infty)$, or for every bounded uniformly continuous f if $p = \infty$.

Proof. For f as in the statement of the theorem we have

$$k_i * f - f = \int_G (\tau_x f - f) k_i(x) d\lambda(x)$$

(interpreting the right-hand side as a vector-valued integral) and

$$\begin{aligned} \|k_i*f-f\|_p &\le \int_{V_i} \|\tau_x f-f\|_p k_i(x) d\lambda(x) \\ &\le \omega(p; f; V_i) \int_{V_i} m_{V_i} k_i d\lambda \\ &\le K\omega(p; f; V_i) , \end{aligned}$$

thus proving (1); (2) now follows immediately. //

As our first application of the preceding results we give a straightforward proof of Jackson's Theorem for the circle group, which does not rely on the existence of the somewhat complicated kernel of Fejér-Korovkin (see [4], p. 75). With k_n and K (= C) as in [1], Theorem 3, we see that the real line R has property $P\left(\mathsf{N}, \left(-\frac{1}{n}, \frac{1}{n}\right), k_n, [-n, n], K\right)$ (we identify Γ_{R} with R). Theorem 2 then shows that R/Z has property $P\left(\mathsf{N}, \pi\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right), l_n, \Omega_n, K\right)$. Now

$$\Omega_n = \left([-n, n] \cap A(\mathsf{R}, \mathsf{Z})\right)^+ ,$$

so that l_n is a trigonometric polynomial of degree at most n. From Theorem 3 (with a slight change of notation) we obtain

$$E_n(p; f) \leq K\omega\left(p; f; \frac{1}{n}\right)$$

for $f \in L^p(\mathsf{R}/\mathsf{Z})$ if $p \in [1, \infty)$, or for continuous f if $p = \infty$; for $p = \infty$ this is just the classical statement of Jackson's Theorem.

Our other application is for functions on the a-adic solenoid. For this we shall use many of the results in [2], Section 10. Let Δ_{a} denote the 0-dimensional compact Hausdorff abelian group of a-adic integers, where $\mathsf{a} = \left(a_0, a_1, \ldots\right)$ and each a_n is greater than 1. The a-adic solenoid is the compact Hausdorff abelian group defined by

$$\Sigma_{\mathsf{a}} = \left(\mathsf{R}\times\Delta_{\mathsf{a}}\right)/B ,$$

where $B = \{(n, n\mathsf{u})\}_{n=-\infty}^{\infty}$ is an infinite cyclic discrete (closed) subgroup of $\mathsf{R} \times \Delta_{\mathsf{a}}$, and $\mathsf{u} = (1, 0, 0, \ldots)$.

Now R has property $P\left(\mathsf{N}, \left(-\frac{1}{n}, \frac{1}{n}\right), k_n^{(1)}, [-n, n], K\right)$, and the remarks on [1], p. 64, show that Δ_{a} has property $P\left(\mathsf{N}, \Lambda_n, k_n^{(2)}, A\left(\Gamma_{\Delta_{\mathsf{a}}}, \Lambda_n\right), 1\right)$, where Λ_n is the compact open subgroup of Δ_{a} given by

$$\Lambda_n = \left\{x \in \Delta_{\mathsf{a}} : x_k = 0 \quad \text{for} \quad k < n\right\} ,$$

and $k_n^{(2)} = \lambda_{\Delta_{\mathsf{a}}}\left(\Lambda_n\right)^{-1}\xi_{\Lambda_n}$. Theorems 1, 2 now combine to show that Σ_{a} has property $P\left(\mathsf{N}, \pi\left[\left(-\frac{1}{n}, \frac{1}{n}\right)\times\Lambda_n\right], l_n, \Omega_n, K\right)$; here

$$\Omega_n = \sigma_\pi^{-1}(\mathrm{R}\times\Lambda_n)\left[\nu_n^{\sim}\left(\left\{\sigma_{\mathrm{R}\times\Lambda_n}(\Xi_n)\cap A\left(\Gamma_{\mathrm{R}\times\Lambda_n},\ (\mathrm{R}\times\Lambda_n)\cap B\right)\right\}^{+}\right)\right] ,$$

where $\Xi_n = \left\{[\gamma_1, \gamma_2] \in \Gamma_{\mathrm{R}\times\Delta_a} : \gamma_1 \in [-n, n], \gamma_2 \in A\left(\Gamma_{\Delta_a}, \Lambda_n\right)\right\}$. It remains to simplify the above expression for Ω_n .

Firstly observe that for $n \geq 1$,

$$(\mathrm{R}\times\Lambda_n) \cap B = \{(m, m\mathrm{u}) : m \in a_0 a_1 \ldots a_{n-1}\mathrm{Z}\}$$

and

$$\sigma_{\mathrm{R}\times\Lambda_n}(\Xi_n) = \{[\gamma, 0] : \gamma \in [-n, n]\} .$$

Thus

$$\sigma_{\mathrm{R}\times\Lambda_n}(\Xi_n) \cap A\left(\Gamma_{\mathrm{R}\times\Lambda_n},\ (\mathrm{R}\times\Lambda_n)\cap B\right)$$
$$= \left\{\left[\frac{l}{a_0 a_1 \ldots a_{n-1}}, 0\right] : l \in \mathrm{Z} \text{ and } \left[\frac{l}{a_0 a_1 \ldots a_{n-1}}\right] \leq n\right\} .$$

It follows that the members of Ω_n can be identified (as in [2], (25.3)) with rational numbers of the form $\frac{l}{a_0 a_1 \ldots a_{n-1}}$, where l is an integer and $\left|\frac{l}{a_0 a_1 \ldots a_{n-1}}\right| \leq n$. We then have

THEOREM 4. *The algebra* $L^1(\Sigma_a)$ *admits a bounded positive approximate unit* $\{k_n\}_{n=1}^{\infty}$ *such that for each* n , $k_n \in C(\Sigma_a)$, $\hat{k}_n(0) = 1$,

$$\operatorname{supp}(\hat{k}_n) \subset \Omega_n = \left\{\frac{l}{a_0 a_1 \ldots a_{n-1}} : l \in \mathrm{Z} \text{ and } \left|\frac{l}{a_0 a_1 \ldots a_{n-1}}\right| \leq n\right\}$$

and, for some $K > 0$,

(1) $\|k_n * f - f\|_p \leq K\omega\left(p; f; \left(\left[-\frac{1}{n}, \frac{1}{n}\right]\times\Lambda_n\right)+B\right)$,

(2) $E_{\Omega_n}(p; f) \leq K\omega\left(p; f; \left(\left[-\frac{1}{n}, \frac{1}{n}\right]\times\Lambda_n\right)+B\right)$

for every $f \in L^p(\Sigma_a)$ *if* $p \in [1, \infty)$, *or for every continuous* f *if* $p = \infty$. //

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