ON THE INTEGRABILITY AND EXACT SOLUTIONS OF THE NONLINEAR DIFFUSION EQUATION WITH A NONLINEAR SOURCE

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Abstract

The generalized diffusion equation with a nonlinear source term which encompasses the Fisher, Newell–Whitehead and Fitzhugh–Nagumo equations as particular forms and appears in a wide variety of physical and engineering applications has been analysed for its generalized symmetries (isovectors) *via* the isovector approach. This yields a new and exact solution to the generalized diffusion equation. Further applications of group theoretic techniques on the travelling wave reductions of the Fisher, Newell–Whitehead and Fitzhugh–Nagumo equations result in integrability conditions and Lie vector fields for these equations. The Lie group of transformations obtained from the exponential vector fields reduces these equations in generalized form to a standard second-order differential equation of nonlinear type, which for particular cases become the Weierstrass and Jacobi elliptic equations. A particular solution to the generalized case yields the exact solutions that have been obtained through different techniques. The group-theoretic integrability relations of the Fisher and Newell–Whitehead equations have been cross-checked through Painlevé analysis, which yields a new solution to the Fisher equation in a complex-valued function form.

1. Introduction

The importance of group theoretic techniques and Painlevé analysis for solving nonlinear differential equations of many physical and engineering systems has already been highlighted in a number of recent publications. See for example Ovsyannikov [14], Olver [13], Bluman and Kumei [5], Ames and Rogers [2], Edelen [8], Edelen and Wang [9], Hill [11], Ramani *et al.* [15] and Ablowitz and Clarkson [1].

The motivation for the present study had its origin in our attempt to carry over these techniques, either singly or collectively as the case may be, for solving and obtaining the exact solutions of the nonlinear diffusion equation with the source term

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and its symmetry reductions, namely, the second-order nonlinear ordinary differential equations *via* the isovector approach (Edelen [8]) and further group invariance techniques respectively. More specifically, after obtaining the isovectors of the nonlinear diffusion equation with source term in Section 3, we have dealt with its symmetry reductions, invariant solutions and further group invariant solutions of the reductions in the same section for the physically realizable forms of the functions of the dependent variable occurring in it. As a cross check on the results established in Section 3, we have utilized Painlevé analysis to confirm them in Section 4. In particular, Painlevé analysis, when applied to the ordinary differential equations that arise as the travelling-wave reduction of the Fisher and Newell–Whitehead equations, yields the integrability conditions obtained in Section 3 using group theoretic methods. Finally, in Section 5, we have summed up the results of this study.

2. Nonlinear diffusion equation with source term

The generalised diffusion equation, that is, the nonlinear diffusion equation with source term, is written in the form

$$T_t = (D_1(T)T_x)_x + D_2(T), \tag{1}$$

where $D_1(T)$ and $D_2(T)$ are referred to as the diffusivity and source terms. Applications of (1) can be found in diverse fields such as physics (particularly plasma physics, astrophysics, laser and semiconductor physics), population dynamics in biology, and the helium combustion process in chemistry. In particular, when $D_1(T) = aT^m$ and $D_2(T) = bT^n - cT$, where $a - c$ are arbitrary constants and m and n are nonzero constants, (1) represents a population density model (Wilhelmson [18]). For $D_1(T) = a$ constant, $D_2(T) = T(1 - T)$, (1) represents the Fisher equation. Equation (1) becomes the Fitzhugh–Nagumo equation when $D_1(T) = a$ constant and $D_2(T) = T(1 - T)(T - \alpha)$, α is a constant. Further, this possibility reduces to the Newell–Whitehead equation when $\alpha = -1$ (see Sachdeva [16] for complete collection of these equations). In general, (1) represents the nonlinear diffusion equation for the propagation of heat from an instantaneous plane source in an initially cool infinite medium (see Bhutani and Vijayakumar [4]).

Several authors have used group theoretic analysis to examine equations of the form (1). Special mention can be made of Dorodnitsyn [7], who classified the symmetries according to the forms of $D_1(T)$ and $D_2(T)$ he proposed, Nucci and Clarkson [13], who used classical Lie, direct and non-classical methods in their investigations of the Fitzhugh–Nagumo equations for solutions and Arrigo *et al.* [3] for their recent work using a non-classical approach.

In this work, we have utilized the isovector approach [4] to obtain the generalised symmetries of the diffusion equation (1). On using the part of the symmetries which

are the usual classical symmetries (or isovectors), we have obtained some symmetry reductions of (1) for the power-law form of the functions of the dependent variables $D_1(T)$ and $D_2(T)$. One of the reductions in this case yields a new exact solution of (1) for these forms of $D_1(T)$ and $D_2(T)$. A table of symmetries (isovectors) has been given for various choices of the functions $D_1(T)$ and $D_2(T)$ involved in (1). The purpose of this work is to show the presence of exact solutions for the equation of the form

$$T_t = D_0 T_{xx} + D_2(T), \tag{2}$$

where D_0 is the constant diffusion coefficient. More specifically, when $D_2(T) = \pm T \mp T^n$, using group analysis, we have obtained an integrability condition for the travelling-wave reduction of (2), as the only possibility of the symmetries of (2) are the translations in x and t. Under this condition, we have shown that (2) can be reduced to

$$w'' \equiv \pm(1/D_0)w^n, \tag{3}$$

where $w = w(z)$, $z = z(x - ct)$ and c is the wave speed. For $n = 2, 3$, we have shown that (3) (which respectively becomes the Fisher and Newell–Whitehead equations) is reducible to the Weierstrass and Jacobi equations. In addition to the above reductions, we have obtained particular solutions of (2) for the cases that correspond to the Fisher and Newell–Whitehead equations as well as for the general case. We have further shown that these particular solutions lead to all known and available solutions. Painlevé analysis has also been utilized to confirm the integrability properties that have been obtained *via* group analysis. Furthermore, we have obtained the Weierstrass elliptic equation reduction of the Fitzhugh–Nagumo equation

$$T_t = D_0 T_{xx} + T(1 - T)(T - \alpha), \tag{4}$$

where α is a constant, for three choices of α, that is, $\alpha = -1, 2, \frac{1}{2}$ *via* group analysis of the travelling wave reduction of (4). We will see these details in the following sections.

3. Isovector method, isovectors and symmetry reductions and exact solutions of equations (1) and (2)

In order to apply the isovector approach to (1), we rewrite it in the language of exterior differential forms as

$$e_1 = dT - u\,dt - y\,dx, \tag{5}$$

$$e_2 = (u - y^2 D_1'(T) - D_2(T))dx \wedge dt - D_1(T)dy \wedge dt, \tag{6}$$

$$e_3 = -du \wedge dt - dy \wedge dx, \tag{7}$$

where e_1, e_2, e_3 are respectively 1, 2, 2-forms. Here $\wedge$ denotes the exterior product of differential forms and the dash over $D_1(T)$ represents the derivative with respect to T. Further, we have assumed y and u as $y = T_x$ and $u = T_t$ respectively. In order to render the system of forms e_1, e_2, e_3 closed with respect to exterior differentiation, we add de_2 to them. Let $I = e_1, e_2, e_3, de_2$ be the fundamental ideal of the algebra of exterior forms $\Lambda(E)$, where E is the manifold of dimension 5 in the space of variables t, x, T, y, u. Then the ideal I is closed by because $dI \subseteq I$.

Let V be the vector field in the tangent space TE of $E(t, x, T, y, u)$. Then V can be given in its components V^t, V^x, V^T, V^y, V^u as

$$V = V^t \partial t + V^x \partial x + \dot{V}^T \partial T + V^y \partial y + V^u \partial u, \tag{8}$$

where $\partial t = \partial/\partial t$, $\partial x = \partial/\partial x$ etc. The vector field V is an isovector field if and only if (see Edelen [9] for details)

$$L_V I \subseteq I. \tag{9}$$

Using (9) in the forms, namely, the contact 1-form e_1 and the balance 2-form e_2, we rewrite (9) as

$$L_V(e_1) = re_1, \tag{10}$$

$$L_V(e_2) = se_2 + W \wedge e_1 - f de_1, \tag{11}$$

where r, s and f are arbitrary functions of the variables t, x, T, y and y and W are arbitrary 1-forms. Further,

$$L_V(e_1) = d(V \lrcorner e_1) + V \lrcorner de_1, \tag{12}$$

$$L_V(e_2) = d(V \lrcorner e_2) + V \lrcorner de_2, \tag{13}$$

where d denotes the exterior differentiation and $\lrcorner$ denotes inner multiplication of forms. Also, de_2 is a 3-form and it is given as

$$de_3 = du \wedge dx \wedge dt - 2y D_1'(T)dy \wedge dx \wedge dt$$
$$- (y^2 D_1''(T) + D_3'(T))dT \wedge dx \wedge dt - D_1'(T)dT \wedge dy \wedge dt. \tag{14}$$

Further, it can be seen easily that $d(de_3) = 0$. Let

$$V \lrcorner e_1 = G. \tag{15}$$

And we assume that

$$W = A\,dt + B\,dx + C\,du + D\,dy + E\,dT, \qquad (16)$$

where A, B, C, D, E are arbitrary functions of x, t, u, T, y. On making use of (5)–(6), (10), (15) and (16) in equation (12) (we perform expansions under exterior differentiation and inner multiplication and we collect the coefficients of similar 1-forms), we get the following system of equations:

$$V^t = -G_u, \qquad V^x = -G_y, \qquad V^T = G - uG_u - yG_y,$$
$$V^u = G_t + uG_T, \qquad V_y = G_x + yG_T. \qquad (17)$$

On making a similar expansion for the 2-form e_2, we have by using (5)–(7), (14), (16) in (13)

$$G_{uu} = 0, \qquad (18)$$
$$(u - y^2 D_1'(T) - D_2(T))G_{uy} + D_1(T)(G_{ux} + yG_{uT}) = 0, \qquad (19)$$

$$
\begin{aligned}
(uG_T + G_t) &- 2yD_1'(T)(G_x + yG_T) \\
&- (y^2 D_1''(T) + D_2'(T))(G - uG_u - yG_y) \\
&- (u - y^2 D_1'(T) - D_2(T))G_{xy} - D_1(T)(G_{xx} + yG_{xT}) \\
&- y(u - y^2 D_1'(T) - D_2(T))G_{yT} - yD_1(T)(G_{xT} + yG_{TT}) \\
&- (1/D_1(T))(u - y^2 D_1'(T) - D_2(T)^2 G_{yy} \\
&- (D_1'(T)/D_1(T))(u - y^2 D_1'(T) - D_2(T))(G - uG_u - yG_y) \\
&- (u - y^2 D_1'(T)D_2(T))(G_{xy} + G_T + yG_{yT}) = 0. \qquad (20)
\end{aligned}
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On solving (18)–(19), we find

$$G = U_1(x, t, T) + U_2(x, t)y + U_3(t)u. \qquad (21)$$

Further, it can easily be shown that

$$U_1(x, t, T) = f_1(x, t) + f_2(x, t)T. \qquad (22)$$

On making use of equations (21) and (22) in equation (20) and collecting the coefficients of u, y, y^2 and constant terms, we obtain the following system of equations:

$$U_3' - 2U_{2x} - \frac{D_1'}{D_1}(f_1 + f_2 T) = 0, \qquad (23)$$

$$U_{2t} - D_1 U_{2xx} - 2(D_1(f_1 + f_2 T))_{xT} = 0, \qquad (24)$$

$$\left(\frac{D_1'^2}{D_1} - D_1''\right)(f_1 + f_2 T) - D_1' f_2 = 0, \qquad (25)$$

$$-D_2'(f_1 + f_2 T) + f_{1t} + f_{2t}T - D_1(f_{1xx} + f_{2xx}T) + (U_3' + f_2)D_2 = 0. \qquad (26)$$

TABLE 1. a-d and c_1-c_4 are arbitrary constants. Rows 5 and 6 represent the isovectors of the Fisher and Fitzhugh – Nagumo equations respectively.

D_1	D_2	V'	V^x
$b(T+k)^n$	$b_1(T+k)^{n+2c+1}$	$-(n+2c)t+c_1$	$-cx+c_1$
b	$b_1(T+k)^m$	$-(m-1)c_1t+c_2$	$-\left(\frac{m-1}{2}\right)c_1x+c_3$
b	c_1e^{mT}	$-mc_1t+c_2$	$-\frac{m}{2}c_1x+c_3$
b	$c_1T\log T$	c_4	$-\frac{2bc_2}{c_1}e^{c_1t}$
a	$\pm T \mp T^n$	c_2	c_3
d	$T(1-T)(T-\alpha)$	c_2	c_2

V^T	V^u	V^y
$T+k$	$(2c+n+1)T_t$	$(c+1)T_x$
$T+k$	$((m-1)c_1+1)T_t$	$\left(\frac{m-1}{2}c_1+1\right)T_x$
c_1	c_1T_t	$\frac{c_1}{2}T_x$
$(c_2x+c_3)e^{c_1t}T$	$e^{c_1t}[(c_2x+c_3)(bT+T_t)+2c_2bT_x]$	$e^{c_1t}[(c_2(T+xT_x)+c_3T_x]$
0	0	0
0	0	0

From (25) we find that the general form of $D_1(T)$ is

$$D_1(T) = b(T+k)^n, \quad n \neq 0, \quad f_1 = \text{a constant}, \quad f_2 = \text{a constant}$$

Without loss of generality, we can assume that $f_1 = k$ and $f_2 = 1$, where k is arbitrary and n is real. On solving the remaining equations in the system (23)–(26), we find

$$D_2(T) = b_1(T+k)^{n+2c+1}, \quad U_3 = nt + 2ct + c_1, \quad U_2cx + c_2, \quad (27)$$

where b, b_1, c, c_1 and c_2 are arbitrary constants.

On using equations (21), (22) and (27) in (17), we find

$$V' = -(n+2c)t + c_1, \quad V^x = -cx + c_2, \quad V^T = T + k$$
$$V^u = (2c+n+1)[(b(T+k)^nT_x)_x + b_1(T+k)^{n+2c+1}], \quad V^y = (c+1)T_x. \quad (28)$$

For $c = 0, c_1 = c_2 = 0$, we have from (28)

$$z = x, \quad T + k = t^{-1/n}w(z). \quad (29)$$

Equation (29), when applied to (1) for the case under consideration, yields

$$b(w^nw'' + nw^{n-1}w'^2) + (1/n)w + b_1w^{n+1} = 0. \quad (30)$$

Equation (30) can be reduced to a quadrature of the form

$$\int \frac{w^n dw}{\sqrt{c_3 + w^{n+2}/(2n + n^2) - b_1 w^{2n+2}/2(n + 1)}} = (2/b)^{\frac{1}{3}}z + z_0, \qquad (31)$$

where c_3 and z_0 are constants of integrations. When $c_3 = 0$, we get a solution to (1) for this case. It is given as

$$T = \left[\left(\frac{-b_1 n(n + 2)}{2(n + 1)} \right) \left(\frac{\text{sech}[(n/2)(b_1/b(n + 1))^{1/2}(x + z_0)]}{t^{1/2}} \right) \right]^{-2/n} - k. \quad (32)$$

However, for all n and c, we have an ordinary differential equation of order two of the form

$$b(w^n w')' + \frac{c}{n + 2c} zw' + \frac{1}{n + 2c} w + b_1 w^{n+2c+1} = 0 \qquad (33)$$

from (1) under the symmetry transformation

$$z = x/t^{c/(n+2c)}, \qquad T + k = t^{-1/(n+2c)} w(z). \qquad (34)$$

Equation (33) reduces to (30) when $c = 0$. Equation (33) is not solvable in its present form. Further, when $c_2 \neq 0$, $c = 0$, we have the following symmetry transformation:

$$z = k_1 x - \log(t + c_1/n), \quad T + k = \left(t + \frac{c_1}{n} \right)^{1/n} w(z), \quad k_1 = n/c_2. \quad (35)$$

On applying the transformation given in (35) to (1), we get

$$bk_1^2(w^n w')' + w' + (1/n)w + b_1 w^{n+1} = 0. \qquad (36)$$

This equation can be reduced to the Abel equation under $w' = v(w)$ and it is given as

$$vv' + nw^{-1}v^2 + (1/bk_1^2)w^{-n}v + (1/bk_1^2) \left(b_1 w + \frac{1}{n} w^{1-n} \right) = 0. \qquad (37)$$

Equation (37) is a generalised Abel equation, which is difficult to solve. It is worth mentioning that the solution given in (31) represents an exact and new solution to (1) for the forms of $D_1(T)$ and $D_2(T)$ that we have considered. Equation (32) is valid for all n except $n = 0$.

The system of equations (23)–(26) has further been solved for some particular forms of $D_1(T)$ and $D_2(T)$. Consequently, we have arrived at different solutions of U_2 and U_3 with the form of U_1 that is given in (22). On using these values of U_1, U_2 and U_3 in (21) and (17) we get isovectors for these cases. These isovectors have been tabulated in Table 1. From Table 1, we have utilized only the isovectors corresponding

to the travelling wave solutions case, as they are the only possibility for (2) and (3) in the sense of classical symmetries (however, non-classical symmetries exist, see Arrigo *et al.* [3] for details). We have divided the study regarding (2) and (3) into two cases. Case 1 and 2 respectively deal with equations (2) and (3). In these cases we have further used group-invariant techniques to obtain reductions of second-order ordinary differential equations (which are travelling wave reductions of (2) and (3)) to standard forms (Jacobi and Elliptic equations) together with integrability conditions. Consequently, we have solved these reductions for some particular solutions which are new even though they can be written in terms of elliptic functions in general. We shall see these cases in detail in the following paragraphs.

Case 1:   Equation (3)

Since we have only constant isovectors for equation (3) (see row 5 of Table 1), we get $T(x, t) = w(z)$, $z = x - ct$, $c = c_2/c_3$ where c represents the wave speed. On using the transformation given above in (3), we obtain an ordinary differential equation of order two in the form

$$D_0 w'' + cw' + D_2(w) = 0. \tag{38}$$

If we assume the transformations of the infinitesimal type

$$z' = z + \epsilon Z(z, w) + o(\epsilon^2), \tag{39a}$$
$$w' = w + \epsilon W(z, w) + o(\epsilon^2) \tag{39b}$$

keep (38) invariant then the invariant condition reads as

$$N(z, w, w')(W_w - 2Z_z - 3w'Z_w) - N_z Z - N_w W$$
$$- N_{w'}(W_z + w'(W_w - Z_z) - w'^2 Z_w) + W_{zz} + w'(2W_{zw} - Z_{zz})$$
$$+ w'^2(W_{ww} - 2Z_{zw}) - w'^3 Z_{ww} = 0, \tag{40}$$

where $N(z, w, w') = w''$. On substituting (38) in (40), we find by collecting the coefficients of w'^3, w'^2, w' and the constant term and equating them to zero a system of equations

$$Z_{ww} = 0, \tag{41}$$
$$(2c/D_0)Z_w + W_{ww} - 2Z_{zw} = 0, \tag{42}$$
$$(c/D_0)Z_z + 2W_{zw} - Z_{zz} + (3D_2(w)/D_0)Z_w = 0, \tag{43}$$
$$-(D_2(w)/D_0)(W_w - 2Z_z) + (c/D_0)W_z + (D_2'(w)/D_0)W + W_{zz} = 0. \tag{44}$$

This equation looks too complicated to be solved without specific values of $D_2(w)$. So, we consider $D_2(w)$ as $D_2(w) = w - w^n$ and $-w + w^n$. When $D_2(w) = \pm w \mp w^n$,

which coincides with $D_2(T) = \pm T \mp T''$, we find, by solving the system (41)–(44) for $n \neq -3$, that

$$c^2 = \pm \frac{a(n + 3)^2}{2(n + 1)}, \qquad D_0 = a \qquad (45)$$

with

$$Z = k_2(a(n + 3)/c(n - 1))e^{(c(n-1)/a(n+3))z} + k_3, \qquad (46)$$
$$W = (2k_2/(1 - n))e^{(c(n-1)/a(n+3))z}w, \qquad (47)$$

where k_2 and k_3 are arbitrary constants. For $n = -3$, we have only trivial solutions to the system (41)–(44).

Let

$$X_1 = \frac{a(n + 3)}{c(n - 1)}e^{\frac{c(n-1)}{a(n+3)}}\partial z + \frac{2w}{(1 - n)}e^{\frac{c(n-1)}{a(n+3)}z}\partial w, \qquad (48)$$
$$X_2 = \partial z. \qquad (49)$$

From (48) and (49), we have $[X_1, X_2] = ((1 - n)/a(n + 3))cX_1$. This shows that we can use only X_1 for transforming equation (3) for the choice of $D_2(w)$ as shown above. Correspondingly, by using X_1, we get the following change of variables transformation:

$$X = \frac{a(n + 3)}{c(n - 1)}e^{-c(n-1)z/a(n+3)}, \qquad w = e^{-2cz/a(n+3)}Y(X). \qquad (50)$$

On making use of equation in (38) with the choices of $D_2(w)$ that we have assumed for obtaining the generators X_1 and X_2, we find

$$D_0Y'' \mp Y'' = 0, \qquad (51)$$

under the condition given in equation (45). It is worth mentioning that for $n = 2$, (51) becomes a reduction of the Fisher equation with the condition $c^2 = \pm D_0(25/6)$. The corresponding form (51) is the Weierstrass elliptic equation. So the general solution of (51) for $n = 2$ can be written in terms of Weierstrass elliptic functions. Thus the Fisher equation equation is integrable[1]. Similar comments can be made for the case $n = 3$. For $n = 3$, we have from (51) the reduction corresponding to the Newell-Whitehead equation. This case produces the Jacobi elliptic equation. Hence the general solution of the Newell-Whitehead equation can be written in terms of Jacobi elliptic functions. Hence the Newell-Whitehead equation is integrable[1] under

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[1]See Section 4 for the integrability property of the cases $n = 2, 3$

the condition $c^2 = \pm D_0(9/2)$. In general for all n, we can obtain quadrature, which is obvious.

When the constant of integration in the first integral of equation (51) is zero, then we have a particular solution to equation (51) and it is given as

$$Y = \pm \left[\pm \left(\frac{1-n}{2} \right) \left(\pm \frac{2}{D_0(n+1)} \right)^{\frac{1}{2}} X + d_1 \right]^{2/(1-n)}, \tag{52}$$

where d_1 is a constant of integration. The solution given in (52) is not valid for $n = 1$. The negative sign inside the square root is allowable when D_0 assumes negative values. On combining (50) and (51), we get

$$T(x,t) = \frac{\pm e^{\mp \left(\frac{2}{D_0(n+1)} \right)^{1/2} (x-ct)}}{\left[d_1 \pm e^{\mp \left(\frac{1}{2D_0(n+1)} \right)^{1/2} (n-1)(x-ct)} \right]^{2/(n-1)}}, \tag{53}$$

a new solution to equation (2) for the form $D_2(T) = T - T^n$, which corresponds to the choice we made for positive sign in $(\ )^{1/2}$ in equation (52). A similar solution can be obtained for the negative sign in (52), which is complex valued. This complex-valued solution may become real-valued for $D_0 < 0$. The negative sign in (52) corresponds to the case $D_2(T) = -T + T^n$.

An interesting aspect of the solution (53) for $n = 3$ is that it reduces to the known solution reported in Hereman and Takaoka [10] for $-d_1 = e^{\pm d_2}$ and $D_0 = 1$:

$$T(x,t) = \pm(1/2) \left[1 - \tanh(\pm(1/8)^{1/2}(x \mp (9/2)^{1/2}t + d_2)) \right]. \tag{54}$$

Similarly, solution (53) becomes the solution of Wang [17] for $n = 2$ and it is given for $-(1/2)d_1 = e^{\pm d_3}$ and $D_0 = 1$ by

$$T(x,t) = (1/4) \left[1 - \tanh((\pm 1/2)(1/6)^{1/2}(x \mp (25/6)^{1/2}t + d_3)) \right]^2. \tag{55}$$

The general solution in the forms of (54) and (55) from (53) can be written as

$$T(x,t) = \left[(1/2)(1 - \tanh((\pm(1/8D_0(n+1)))^{1/2}(n-1)(x - ct + d_4))) \right]^r, \tag{56}$$

where $r = 2/(n-1)$ and $d_1 = e^{\pm d_4}$. Thus, we have presented an exact solution in the form of travelling waves to the generalised reaction-diffusion equation (2) when $D_2(T) = \pm T \mp T^n$, via group analysis of (38) which is valid only when c, the wave speed, satisfies the condition given in (45). We term this condition as the integrability condition of equation (2) for the choice of $D_2(T)$ mentioned therein.

Case 2:   Equation (4).

As mentioned in Case 1, we obtain only the translations in the x and t directions as isovectors (symmetries) for the functional choice of $D_2(T)$ that corresponds to the Fitzhugh-Nagumo equation. On applying the value $D_2(w) = w(1 - w)(w - \alpha)$, α constant, in (40) and collecting the coefficients of w'^3, w'^2, w' and the constant terms and equating them to zero, we get a system of equations (41)–(44) for $D_2(w) = w(1 - w)(w - \alpha)$. On solving this system, we arrive at the following values of Z and W:

$$Z = (3dk_2/c)e^{(c/3d)z} + k_3, \tag{57}$$
$$W = -k_2 e^{(c/3d)z} w, \tag{58}$$

where $D_0 = d$ a constant and k_2 and k_3 are arbitrary constants. Equations (57) and (58) are valid under the condition

$$2\alpha^2 - 5\alpha + 2 = 0. \tag{59}$$

Obviously, $\alpha = 1/2, 2$ satisfy (59). For $\alpha = -1$, this case reduces to the Newell-Whitehead equation, which we have discussed earlier, when $n \neq 3$. If we write (57) and (58) in the form of vector fields (as in (48) and (49)), we get

$$X_1 = (3d/c)e^{(c/3d)z}\partial z + ((1 + \alpha)/3 - w)e^{(c/3d)z}\partial w, \tag{60}$$
$$X_2 = \partial z. \tag{61}$$

From (60) and (61), we get $[X_1, X_2] = \text{const}.X_1$. This shows that for obtaining the change of variable transformation we can use X_1 alone (Olver [13]). So X_1 yields

$$X = (3d/c)e^{-(c/3d)z}, \qquad Y(X) = e^{(c/3d)z}(w - (1 + \alpha)/3). \tag{62}$$

On applying the transformation given in (62) in (38) for the value of $D_2(w) = w(1 - w)(w - \alpha)$, which corresponds to the Fitzhugh-Nagumo equation, we get

$$Y'' - (1/d)Y^3 = 0. \tag{63}$$

Equation (63) is a possible reduction of the Fitzhugh-Nagumo equation under the change of variable transformation (62) only when $\alpha = 1/2, 2$. This equation is the standard Jacobi elliptic equation. So the solutions to (63) can in general be written in terms of Jacobi elliptic functions (see Arrigo *et al.* [3] for four different solutions for the Jacobi equation). A particular solution to (63) leads to the travelling wave solution to the equation of Fitzhugh-Nagumo of the form

$$T - (1 + \alpha)/3 = \pm(1/2)\left[1 - \tanh((\alpha/8d)^{1/2}(x - ct) + (d_4/2))\right], \tag{64}$$

where $(2/\alpha)^{1/2}e^{\pm d_4} = c_1$ is a constant of integration and $c = (9/4)d$. Here it is important to note that $\alpha = 1/2, 2$. Expression (64) is an exact solution to (4). Thus, we have established the integrability of the Fitzhugh-Nagumo equation for three values of the parameter $\alpha = 1/2, 2$ and -1 via group analysis with exact solution (64) and reduction to the standard Jacobi elliptic equation. In the following section we will show and confirm the integrability conditions of the Fisher and Newell-Whitehead equations via the Painlevé analysis of their travelling wave reductions.

4. Painlevé analysis of the Fisher and Newell-Whitehead equations

As it is very difficult to carry out Painlevé analysis on (2) for the substitution $D_2(T) = \pm T \mp T^n$, we perform the said analysis only for $n = 2$ and 3. Correspondingly, we have divided this section into Cases 1 and 2 respectively dealing with the Painlevé analysis of the Fisher and Newell-Whitehead equations.

Case 1 Painlevé Analysis of the Fisher equation
For $n = 2$ in $D_2(w) = w - w^n$ putting (38), we obtain the travelling wave reduction of the Fisher equation, which is given as

$$D_0 w'' + c w' + w - w^2 = 0. \tag{65}$$

Using the algorithm for the application of Painlevé analysis to equation (65) one can write

$$w(z) = a_0(z - z_0)^p + \sum_{j=1}^{r_s} a_j(z - z_0)^{p+j}, \tag{66}$$

where a_0 and p are determined from the balance of dominant terms, arbitrariness of z_0 corresponds to the root (-1) of the resonance equation and r_s is the largest root of the last equation.

For the equation under consideration z_0 is arbitrary and a_0, p and r_s are given by

$$a_0 = 6D_0, \qquad p = -2, \qquad r_s = 6. \tag{67}$$

Thus, the Laurent series expansion of $w(z)$ as a general solution to equation (65) at the level of $(z - z_0)^4$ can be written as

$$w(z) = 36D_0(z - z_0)^{-2} + a_1(z - z_0)^{-1} + a_2(z - z_0)^0 + a_3(z - z_0)$$
$$+ a_4(z - z_0)^2 + a_5(z - z_0)^3 + a_6(z - z_0)^4. \tag{68}$$

On substituting equation (68) in (65) and equating the coefficients of the various powers of $(z - z_0)$ and equating them to zero, we find that

$$a_0 = 6D_0, \quad a_1 = -(6/5)c, \quad a_2 = (1/50)(25 - c^2/D_0), \quad a_3 = -c^3/250D_0^2$$
$$a_4 = (625D_0^2 - 11c^4)/25000D_0^3, \quad a_5 = (1375D_0^2 c - 790c^5)/75000D_0^4. \tag{69}$$

Analysis suggests that a_6 has to be arbitrary. However, there exists an inconsistency which is expressed as

$$c^2(625D_0^2 - 36c^4) = 0. \tag{70}$$

Equation (70) yields the values of c in terms of D_0, which are given as $c^2 = \pm 25D_0/6$ and $c = 0$ (multiple). This shows that for these values of the wave speed, c, the Fisher equation is integrable. We can see easily that for these values of c we have obtained a two-parameter Lie group, namely, X_1 and X_2 (Section 3). Further, this two-parameter Lie group leads to the reduction of the Fisher equation to the Weierstrass elliptic equation when $c \neq 0$. For $c = 0$, the equation is trivially integrable.

Case 2   Painlevé Analysis for the Newell-Whitehead equation
   Similar analysis (to Case 1) of the travelling wave reduction of the Newell-Whitehead equation

$$w'' + cw' + w(1 - w^2) = 0, \tag{71}$$

when $D_0 = 1$ in (38), results in the Laurent series expansion of the form

$$w(z) = a_0(z - z_0)^{-1} + a_1(z - z_0)^0 + a_2(z - z_0)^1 + a_3(z - z_0)^3 + a_4(z - z_0)^4. \tag{72}$$

On substituting (72) in (71), we find that

$$
\begin{aligned}
a_0 &= \pm 2^{1/2}, \qquad a_1 = \mp(1/18)^{1/2}c, \\
a_2 &= \pm(2/1296)^{1/2}(6 - c^2), \qquad a_3 = \pm 2^{1/2}(9 - 8c^2)/108.
\end{aligned}
\tag{73}
$$

Since a_4 has to be arbitrary, we get a condition on c of the form

$$c^2(2c^2 - 9) = 0. \tag{74}$$

On solving (74), we find the values of c coincide with the one given in (46) for $D_0 = 1$. If we assume $D_0 = -1$, then, after repeating the procedure given in Case 1, we may arrive at a negative value of c^2. Hence, according to the Painlevé analysis, the Newell-Whitehead equation is integrable, as stated in Section 3, for the values of c^2 mentioned above. Recently, Cariello and Tabor [6] obtained the values $c = \pm(9/2)^{1/2}$ for the wave speed. For $c^2 = -25D_0/6$, we can transform the Fisher equation to

$$W'' \pm 6W^2 = 0 \tag{75}$$

under the transformation

$$w(z) = 1 \pm Z^2W(Z), \qquad Z = e^{\mp i(1/6)^{1/2}z}. \tag{76}$$

As mentioned earlier, (75) is the standard Weierstrass elliptic equation. The general solution of the Fisher equation can be written in terms of Weierstrass elliptic functions, as in the case $n = 2$ of (51). The particular solution of (75) leads to a new solution of the Fisher equation of the form

$$T = \frac{c_2^2 \mp 2c_2 e^{\mp i(1/6)^{1/2}(x \pm 5(1/6)^{1/2}t)}}{\left[e^{\mp(1/6)^{1/2}(x \pm i5(1/6)^{1/2}t)} \pm c_2\right]^2},\qquad(77)$$

where c_2 is a constant of integration. Equation (77) is complex-valued, which represents the travelling wave phenomenon.

Recently Nucci and Clarkson [12] performed group theoretic analysis, namely, classical, direct and non-classical methods on the Fitzhugh-Nagumo equation in which they have indicated the same conditions for α. Under those conditions, they have derived exact solutions for the Fitzhugh-Nagumo equation, which we are not able to recover in this paper. However, we have computed some exact solutions, which have group theoretic connections.

5. Conclusion

Starting with the nonlinear diffusion equation with source term (equation (1)), we have, using the isovector method, obtained an isogroup classification in the form of a table (see Table 1) for the various choices of the functions $D_1(T)$ and $D_2(T)$. Through these we have given symmetry reductions of the more general possible (solutions of the system (23)–(26)) case, in which $D_1(T)$ and $D_2(T)$ takes power-law forms. For this case, we have given an exact solution, which is new. Further study of (2), a particular form of (1), *via* the group analysis of the travelling wave reduction of it leads to many insights into this problem. That is, we have obtained and recovered all the available solutions that have been obtained by different workers (Hereman and Takaoka [10] and Wang [17]) through different techniques with integrability conditions and group theoretic explanations for the Fisher, Newell–Whitehead and Fitzhugh–Nagumo equations in addition to the solution for the general case (see equation (56)). Furthermore, results obtained *via* group theoretic analysis have been confirmed *via* Painlevé analysis (see Section 4). A special mention in this study is the integrability conditions obtained for the equations of Fisher, Newell–Whitehead and Fitzhugh–Nagumo and the existence of two parameter groups, namely, X_1, X_2 with the property $[X_1, X_2] = $ constant X_1 and the relation between them as a one-one correspondence. Finally, we have presented another exact solution to the Fisher equation which is not derivable by group analysis but by an *ad hoc* method. This solution is new and complex-valued (it may become real under certain circumstances).

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons. Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).

[2] W. F. Ames and C. Rogers, *Nonlinear Boundary value Problems in Science and Engineering* (Academic Press, New York, 1989).

[3] D. J. Arrigo, J. M. Hill and P. Broadbridge, "Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source", *IMA J. Appl. Math.* **52** (1994), 1–24.

[4] O. P. Bhutani and K. Vijayakumar, "On the isogroups of the generalised diffusion equation", *Int. J. Eng. Sci.* **28** (1990) 375–387.

[5] Bluman and G. W. Kumei, *Symmetries and Differential Equations* (Springer, New York, 1989).

[6] F. Cariello, and M. Tabor, "Painlevé expansions for nonintegrable evolution equations", *Physica D* **39** (1989) 77–94.

[7] V. A. Dorodnitsyn, "On invariant solutions of the equation of non-linear heat conduction with a source", *USSR Comp. Math. Phys.* **22** (1982) 115–122.

[8] D. G. B. Edelen, *Applied Exterior Calculus* (Willey-Interscience, New York, 1985).

[9] D. G. B. Edelen and J. Wang, *Transformation Methods for Nonlinear Partial Differential Equations* (World Scientific, Singapore, 1992).

[10] W. Hereman and M. Takaoka, "Solitary wave solutions of nonlinear evolution and wave equations using a direct method and MACSYMA", *J. Phys. A: Math Gen.* **23** (1990) 4805–4822.

[11] J. M. Hill, *Differential Equations and Group Methods for Scientists and Engineers* (CRC Press, London, 1992).

[12] M. C. Nucci and P. A. Clarkson, "The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh–Nagumo equation", *Phys. Lett. A* **164** (1992) 49–56.

[13] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer (GTM(106)), Berlin, 1986).

[14] Ovsyannikov, *Group Analysis of Differential Equations* (Trans. W. F. Ames) (Academic Press, New York, 1982).

[15] A. Ramani, B. Grammaticos and T. Bountis, "The Painleve property and singularity analysis of integrable and non-integrable systems", *Physics Reports* **180** (1989) 159–245.

[16] P. L. Sachdeva, *Nonlinear Diffusive Waves* (Cambridge University Press, Cambridge, 1987).

[17] X. Y. Wang "Exact and explicit solitary wave solutions for the generalised Fisher equation", *Phys. Lett. A* **131** (1988) 277–279.

[18] H. Wilhelmson, "Explosive instabilities of reaction-diffusion equations", *Physical Review A* **36** (1987) 965–966.