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Almost sure invariance principle for some maps of an interval

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Abstract. We prove an almost sure invariance principle and a central limit theorem for the process $(F\circ f^n)_{n\geq 0}$, where f is a map of an interval with a non-positive Schwarzian derivative whose trajectories of critical points stay far from the critical points, and F is a measurable function with bounded p-variation ($p\geq 1$).

The almost sure invariance principle implies the Log-log laws, integral tests and a distributional type of invariance principle for the process $(F\circ f^n)_{n\geq 0}$.

0. *Introduction*

Let f be a map of an interval I and F a measurable function on I. Under some special assumptions (like expansiveness) on f, several authors have proved the existence of a probabilistic measure μ on I, f-invariant and absolutely continuous with respect to the Lebesgue measure.

In 1979 S. Wong [**9**] considered the process $(F\circ f^n)_{n\geq 0}$ and proved a central limit theorem for f piecewise C^2 and expanding in the case of a weak mixing μ. His method can be used to obtain an analogous theorem for maps of an interval which we consider below.

In 1980 G. Keller [**4**] proved a central limit theorem for expanding f using the ideas of M.I. Gordin [**2**].

In 1982 F. Hofbauer and G. Keller [**3**] proved for expanding maps f not only a central limit theorem but also an almost sure invariance principle, from which follow Log–log laws, integral tests and a distributional type of invariance principle.

In our paper all these theorems are proved for maps of an interval with a non-positive Schwarzian derivative considered by M. Misiurewicz [**5**], W. Szlenk [**8**] and B. Szewc [**7**]. A full list of assumptions and the formulation of the main theorems are given in § 1. § 2 contains basic technical estimations of derivatives. In § 3 we prove an exponential rate of decreasing of the diameter in the sense of Lebesgue measure of the partition $\bigvee_0^{n-1} f^{-i}\mathscr{A}$, where $\mathscr{A}$ is the partition into the intervals of monotonicity of f.

An exponential rate of decreasing of the diameter of the partition $\bigvee_0^{n-1} f^{-i}\mathscr{A}$ in the sense of an f-invariant probabilistic measure is proved in § 4. § 5 contains a proof of an exponential rate of decreasing of mixing coefficient. The proof of the main results formulated in § 1 is given in § 6.

1. *Assumptions and main results*

Let I be a closed interval, $A(g)\subset I$ its finite subset containing the ends of I, and $g: I\to I$ a continuous map, monotonic on the connected components of the set $I\setminus A(g)$. We assume that $g|_{I\setminus A(g)}$ satisfies the following conditions [5]:

(i) g is of class C^3;

(ii) $g\neq 0$;

(iii) $Sg\leq 0$, $(Sg=(g'''/g')-\frac{3}{2}(g''/g')^2)$;

(iv) if $g^p(x)=x$, then $|(g^p)'(x)|>1$;

(v) there exists a neighbourhood U of the set $A(g)$ such that for all $a\in A$, $n\geq 0$, $g^n(a)\in A(g)$ or $g^m(a)\notin U$ for all $m\geq n$;

(vi) for all $a\in A(g)$ there exists a neighbourhood U_a of a and constants α, ω, $\delta>0$, $u\geq 0$ such that

(a) $\alpha|x-a|^u\leq|f'(x)|\leq\omega|x-a|^u$,

(b) $|f''(x)|\leq\delta|x-a|^{u-1}$

for all $x\in U_a$.

[5, th. 6.2] ensures the existence of a probabilistic measure μ, g^k-invariant for some $k\geq 1$, absolutely continuous with respect to the Lebesgue measure λ and such that the system (g^k, μ) is exact. The map g^k also satisfies the conditions (i)-(vi), where $A(g^k)=\bigcup_{i=0}^{k-1} g^{-i}(A(g))$ plays the role of $A(g)$. In the following we will write f instead of g^k and A instead of $A(g^k)$.

Let $F:[0,1]\to\mathbb{R}$ be a function with bounded p-variation $p\geq 1$, i.e. $V^pF=\sup\sum_{i=1}^n|F(c_i)-F(c_{i-1})|^p<+\infty$, where the supremum is taken over all finite subsets $\{c_0,\ldots,c_n\}$, $c_0<c_1<\cdots<c_n$, of the interval I. We assume that $\int_I F\,d\mu=0$.

We will treat $(F\circ f^n)_{n\geq 0}$ as a stochastic process on the probabilistic space $([0,1], \mathscr{B}, \mu)$ ($\mathscr{B}$ is the Borel σ-field). Let $S_t=\sum_{m\leq t}F\circ f^m$.

The main result of this paper is the following:

THEOREM 1. *The series* $\sigma^2=\int F^2\,d\mu+2\sum_{i=1}^\infty F\cdot(F\circ f^i)\,d\mu$ *is absolutely convergent,* $\int(S_n)^2\,d\mu=n\sigma^2+O(1)$ *(for* $n\in\mathbb{N}$*) and if* $\sigma^2\neq 0$*, then*

(i) $\sup_{r\in\mathbb{R}}\mu\{S_n/\sigma\sqrt{n}\leq r\}-(1/\sqrt{2\pi})\int_{-\infty}^r e^{-x^2/2}\,dx|=O(n^{-\nu})$ *for some* $\nu>0$;

(ii) *without changing its distribution we can redefine the process* $(S_t)_{t\geq 0}$ *on a richer probabilistic space together with a standard Brownian process* $(B_t)_{t\geq 0}$ *such that almost surely*

$$|\sigma^{-1}S_t-B_t|=O(t^{1/2-\lambda})$$

for some $\lambda>0$.

Log-log laws, integral tests and a weak type of invariance principle follow from this theorem (see [3]). F. Hofbauer and G. Keller [3] proved the above theorem in the case of expanding f.

2. *Basic technical lemmas*

Let $B:=\overline{\bigcup_{n=1}^\infty f^n(A)}$ and V_0 be an open neighbourhood of B. Let $U:=\bigcup_{a\in A}U_a$. We can assume that the sets U_a, $a\in A$, are disjoint and:

$$f(U)\subset V_0; \tag{1}$$

$$(f^{-1}(A)\setminus A)\cap U=\varnothing. \tag{2}$$

It follows from (v) that $B\setminus A\subset I\setminus U$, $f(B\setminus A)\subset B\setminus A$ and $B\setminus A$ is closed. So, there exists a neighbourhood V_1 of the set $B\setminus A$ such that

$$f(V_1)\cap U=\emptyset. \tag{3}$$

Put $V:=V_0\cap(V_1\cup U)$.

LEMMA 1. *For every $\gamma>1$ there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ and all $x\in I$ if $f^n(x)\notin V$, then $|(f^n)'(x)|\geq\gamma$.*

Proof. For all $x\in I$ let $\Delta_n(x)$ be the connected component of the set $I\setminus\bigcup_{k=0}^{n-1}f^{-k}(A)$ which contains x. By [**5**, th. 4.6]

$$\lim_{n\to\infty}(\sup_{y\in I}\lambda(\Delta_n(y)))=0.$$

The proof follows by using the argument of W. Szlenk [**8**, prop. 4]. □

We remark that by integrating the inequality from condition (vi) we obtain

$$\alpha|x-a|^{u+1}/(u+1)\leq|f(x)-f(a)|\leq\omega|x-a|^{u+1}/(u+1) \tag{4}$$

for all $x\in U_a$, $a\in A$.

LEMMA 2. *Let $f^i(a)\in A$, $f^i(x)\in U_{f^i(a)}$ for $i=0,1,\ldots,k-1$, $f^{j+k}(x)\notin U$ for $j=0,\ldots,l-1$, $f^{l+k}(x)\notin V$, k, $l\geq1$. There exists a constant $L>0$ depending only on the size of U and V such that*

$$|(f^{k+1})'(x)|\geq\frac{L}{|x-a|}.$$

Proof. Let α_i, ω_i, u_i, δ_i be the constants from condition (vi) corresponding to $f^i(a)$. Assume that $f^k(a)\in A$. Then

$$|f^k(a)-f^k(x)|\geq\operatorname{dist}(A,I\setminus U)=:K_U,$$

and by (4)

$$|f^{k-1}(x)-f^{k-1}(a)|\geq\frac{(u_{k-1}+1)K_U}{\omega_{k-1}|f^{k-1}(x)-f^{k-1}(a)|^{u_{k-1}}}.$$

Hence, using (vi(a)) we obtain

$$|f^{k-1}(x)-f^{k-1}(a)|\geq\frac{(u_{k-1}+1)\alpha_{k-1}}{\omega_{k-1}}\frac{K_U}{|f'(f^{k-1}(x))|}.$$

Repeating the above argument k times we arrive at the inequality

$$|x-a|\geq\prod_{i=0}^{k-1}\frac{(u_i+1)\alpha_i}{\omega_i}\frac{K_U}{\prod_{i=0}^{k-1}f'(f^i(x))|}.$$

By [**5**, th. 1.3] there exists a $\nu>1$ and $m\geq1$ such that

$$\text{if } f^j(x)\notin U \text{ for } j=0,\ldots,m-1 \text{ then } |(f^m)'(x)|\geq\nu. \tag{5}$$

Therefore there exists a constant $c_0>0$ depending only on the size of U such that

$$|(f^{k+l})'(x)|\geq\frac{K_Uc_0}{|x-a|}\prod_{i=0}^{k-1}\frac{(u_i+1)\alpha_i}{\omega_i}. \tag{6}$$

Assume now that $f^k(a)\notin A$. By (v), $f^{j+k}(a)\notin U$ for all $j\geq0$. By assumption

$f^{l+k}(x) \notin V$, so

$$|f^{l+k}(x) - f^{l+k}(a)| \geq \text{dist}\,(B, I \setminus V) =: K_V.$$

Therefore for some $\theta \in (f^k(x), f^k(a))$ we have

$$|(f^l)'(f^k(x))| \geq \frac{|(f^l)'(f^k(x))|}{|(f^l)'(\theta)|} \frac{K_V}{|f^k(x) - f^k(a)|}.$$

If $[f^{j+k}(a), f^{j+k}(x)] \subset I \setminus U$ for $j = 0, \ldots, l-1$ then by (5) the same argument as in the case of an expanding f proves that there exists a constant $K > 0$ depending only on the size of the neighbourhood U such that

$$\frac{|(f^l)'(f^k(x))|}{|(f^l)'(\theta)|} \leq K.$$

So we have

$$|f^k(x) - f^k(a)| \geq \frac{K \cdot K_V}{|(f^l)'(f^k(x))|}. \tag{7}$$

But if $[f^j(f^k(a)), f^j(f^k(x))] \cap U \neq \emptyset$ for some j, $0 \leq j \leq l-1$, then

$$U_{a_0} \subset [f^j(f^k(a)), f^j(f^k(x))]$$

for some $a_0 \in A$ ($f^{j+k}(a) \notin U$ for $j \geq 0$ in virtue of (v)). If j is the smallest such iteration, then

$$|f^k(x) - f^k(a)| \geq \frac{K \cdot K_U}{|(f^j)'(f^k(x))|}. \tag{7'}$$

Now using (7) or (7′) (depending on the situation) we proceed as in the case $f^k(a) \in A$. The role of the constant K_U is now played by

$$\frac{K \cdot K_V}{|(f^l)'(f^k(x))|} \quad \text{or} \quad \frac{K \cdot K_U}{|(f^j)'(f^k(x))|}.$$

As a result we obtain an estimation similar to (6), but the constant $K_U \cdot c_0$ is now replaced by $K \cdot K_V$ or $K \cdot K_U \cdot c_0$. The set A being finite, there are finitely many expressions

$$\prod_{i=0}^{k-1} \frac{(u_i + 1)\alpha_i}{\omega_i}.$$

Multiplying the smallest of them by the smallest of the numbers $K_U \cdot c_0$, $K \cdot K_V$, $K \cdot K_U \cdot c_0$ we obtain L. □

From lemma 2 and (vi(b)) we derive immediately:

COROLLARY 1. *There exist constants E, $b > 0$ depending only on the size of U and V such that if the assumptions of lemma 2 are satisfied then*

(a) $|f''(x)|/|f'(x)||(f^{k+l})'(x)| \leq K$,

(b) $|(f^{k+l})'(x)| \geq b$.

Notice that in view of (iv) we can choose U_a such that if $a \in A$ is periodic with period p, then for all $x \in U_a$

$$|(f^p)'(x)| \geq \beta > 1. \tag{8}$$

LEMMA 3. *If the assumptions of lemma 2 are fulfilled then there exists a constant $H_0 > 0$ depending only on the size of U and V such that*

$$S_{k+l}(x) := \sum_{i=0}^{k+l-1} \frac{|f''(f^i(x))|}{|f'(f^i(x))||(f^{k+l-i})'(f^i(x))|} \le H_0.$$

Proof. Suppose the assumptions of lemma 2 are satisfied. If $a \in A$ is periodic there is nothing to do in view of (i), (8) and (5). If a is not periodic then $k \le k_0$ for some $k_0 \in \mathbb{N}$, because A is finite. In view of corollary 1 the sum of the first k terms of the sum $S_{k+l}(x)$ does not exceed $k_0 \cdot E$. By (5) and (i) the sum of remaining terms does not exceed (up to a constant) the sum of a geometric series with the quotient $1/\nu$. □

Notice that we can make the sets U_a so small that

$$f(U_a) \cap \Bigg(\bigcup_{\substack{b \ne f(a) \\ b \in A}} U_b \Bigg) = \varnothing \qquad \text{for all } a \in A. \tag{9}$$

LEMMA 4. *There exists $H > 0$ depending only on the size of U and V such that if $f^n(y) \notin V$, then*

$$\left| \left(\frac{1}{(f^n)'(y)} \right)' \right| \le S_n(y) \le H.$$

Proof. Suppose $f^n(y) \notin V$. On account of (1), (3) and (9) we can divide the sequence $y, f(y), \ldots, f^n(y)$ (except for the first $q_0 \ge 0$ terms not belonging to U) into blocks satisfying the assumptions of lemma 2.

Let $\gamma > 1$ and $n_0 \in \mathbb{N}$ be as in lemma 1. Suppose that we have $m \cdot n_0 + r$ ($m, r \in \mathbb{N} \cup \{0\}$, $r < n_0$) blocks in the sequence $y, f(y), \ldots, f^n(y)$. In view of (5) there exists $H_1 > 0$ depending only on the size of U such that $S_{q_0}(y) \le H_1$. Hence, the desired estimate is obtained by using lemma 1, corollary 1(b) and lemma 3. □

LEMMA 5. *There exist $c > 0$, $\theta > 1$ depending only on the size of U and V such that if the assumptions of lemma 2 are satisfied then $|(f^{k+l})'(x)| \ge c \cdot \theta^{k+l}$.*

Proof. If a is periodic there is nothing to do by (8) and (5).

Suppose a is not periodic. Let α_i, ω_i, u_i be the constants from condition (vi) corresponding to $f^i(a)$. Using the inequalities (4) k times we arrive at

$$|f^k(x) - f^k(a)| \ge \frac{\alpha_{k-1}}{u_{k-1}+1} \left(\frac{\alpha_{k-2}}{u_{k-2}+1} \right)^{(u_{k-1}+1)} \cdots \left(\frac{\alpha}{u+1} \right)^{(u_{k-1}+1)\cdots(u_1+1)} |x-a|^{(u_{k-1}+1)\cdots(u+1)}. \tag{10}$$

But $k \le k_0$ (cf. the proof of lemma 3), so there exist $c_3 > 0$, $q > 0$ depending only on the size of U such that

$$|f^k(x) - f^k(a)| \ge c_3 |x-a|^q. \tag{11}$$

Let j be the greatest integer such that $[f^{i+k}(x), f^{i+k}(a)] \cap U = \varnothing$ for $i = 0, \ldots, j-1$. By (5) there exists a constant $c_4 > 0$ depending only on the size of U such that

$$|I| \ge f^{j+k}(x) - f^{j+k}(a)| \ge c_4 \nu^{j/m} |f^k(x) - f^k(a)|.$$

The last inequalities together with (11) give

$$|x-a| \le \frac{\sqrt[q]{|I|}}{c_3\sqrt[q]{c_4}} \frac{1}{(\sqrt[q]{\nu})^j} \tag{12}$$

The same argument as in the proof of lemma 2 shows that

$$|(f^{k+j})'(x)| \ge L_1/|x-a| \tag{13}$$

where $L_1 > 0$ depends only on U and V.

By (12) and (13) there exists $c_5 > 0$ depending only on U and V such that

$$|(f^{k+j})'(x)| \ge c_5(\sqrt[q]{\nu})^{j+k}.$$

Now the desired estimate follows from the last inequality and (5). □

LEMMA 6. *There exist constants $G > 0$, $\nu > 1$, depending only on U and V such that if $f^n(y) \notin V$, then $|(f^n)'(y)| \ge G \cdot \nu^n$.*

Proof. If $f^n(y) \notin V$, then the sequence $y, f(y), \ldots, f^n(y)$ (except the first $q_0 \ge 0$ terms not belonging to U) can be divided into k blocks each of which satisfies the assumptions of lemma 2. Hence by lemma 5

$$|(f^n)'(y)| \ge b_0 c^k \theta^n, \tag{14}$$

where $b_0 > 0$ depends only on U.

If $c \ge 1$, we obtain the desired inequality. Suppose that $c < 1$. Let $n_1 \in \mathbb{N}$ be such that $c \cdot \theta^{n_1/2} > 1$. If $k < n/n_1$ then by (14),

$$\begin{aligned} |(f^n)'(y)| &\ge b_0 c^{n/n_1}(\theta^{n_1})^{n/n_1} \\ &= b_0[(c \cdot \theta^{n_1/2})^{1/n_1}(\theta^{n_1/2})^{1/n_1}]^n \\ &> b_0(\sqrt{\theta})^n. \end{aligned}$$

If $k > n/n_1$, we have $k = ln_0 + r$, $0 \le r < n_0$, where n_0 is as in lemma 1, and

$$|(f^n)'(y)| \ge b_0 \gamma^l c^r \ge b_0 c^{n_0} \gamma^l. \tag{15}$$

By assumption $k > n/n_1$, so $l = (k-r)/n_0 > (n/n_1 - r)/n_0 > (n/n_1 n_0) - 1$. From (15) and the above estimate we obtain

$$|(f^n)'(y)| \ge \frac{b_0 c^{n_0}}{\gamma}(\gamma^{1/n_0 n_1})^n. \qquad \square$$

3. *Estimation of the diameter of $\bigvee_0^{n-1} f^{-i}\mathcal{A}$ in the sense of Lebesgue measure*

Let $\mathcal{A}$ be the partition of I given by the points of A. We will show that the number $\max_{\Delta \in \bigvee_0^{n-1} f^{-i}\mathcal{A}} \lambda(\Delta)$ decreases exponentially with respect to n. We will call this number the diameter of $\bigvee_0^{n-1} f^{-i}\mathcal{A}$ in the sense of Lebesgue measure.

LEMMA 7. *Let $\Delta = (v, w) \in \bigvee_0^{n-1} f^{-i}\mathcal{A}$, $\Delta \notin \bigvee_0^{n} f^{-i}\mathcal{A}$ and l be the smallest iteration such that $f^l(w) \in A$. There exists $z \in (v, w)$ such that*

(a) $|(f^n)'(z)| \ge D\nu^n$;

(b) *for some $n_z \ge n - l$ the sequence $f^l(z), f^{l+1}(z), \ldots, f^{l+n_z}(z)$ satisfies the assumptions of lemma 2 and*

$$|f^{j+l}(z) - f^{j+l}(w)| < K_V \qquad \text{for } j = 0, \ldots, n_z - 1,$$

where the constant $D>0$ depends only on U and V, ν is as in lemma 6 and K_V as in the proof of lemma 2.

Proof. We can assume that n is the biggest natural number such that $\Delta=(v,w)\in \bigvee_0^{n-1} f^{-1}\mathcal{A}$ (such an n exists because of the lack of homtervals). Hence

$$f^n(\Delta)\cap A\neq\emptyset. \tag{16}$$

Let l be the smallest iteration such that $f^l(w)=b\in A$. Write $x=f^l(v)$.

Suppose that $x\in U_b$. The ends of the interval $f^n(\Delta)$ belong to B, so in view of (16) there exists $y\in[x,b)$ (or $(b,x]$) such that the sequence $y, f(y),\dots,f^{n-l}(y)$ satisfies the assumptions of lemma 2 (with b instead of a) and

$$|f^j(y)-f^j(b)|<K_V \qquad \text{for } j=0,\dots,n-l-1. \tag{17}$$

Therefore by lemma 5

$$|(f^{n-l})'(y)|\geq c\theta^{n-l}. \tag{18}$$

Let $z\in(v,w)$ be such that $f^l(z)=y$. It is easy to show by using (2), (9) and the monotonicity of f on the components of $I\setminus A$, that $f^{l-1}(z)\notin U$. Hence by (3) and the definition of V we have $f^{l-1}(z)\notin V$ and we can use lemma 6 to estimate $|(f^l)'(z)|$ by ν^l up to a constant depending only on U and V. This estimate together with (18) gives

$$|(f^n)'(z)|\geq R_1\nu^n, \tag{19}$$

where $R_1>0$ is a constant depending only on U and V.

Let us now suppose $x\notin U_b$ and let $y\in(x,b)$ be the centre of the interval $U_b\cap(x,b)$ (or $U_b\cap(b,x)$). Denote by n_b the smallest iteration such that the sequence $y,f(y),\dots,f^{n_b}(y)$ satisfies the assumptions of lemma 2 and $|f^j(y)-f^j(b)|<K_V$ for $j=0,\dots,n_b-1$.

If $n_b<n-l$, then there exists $y'\in U_b\cap(x,b)$ (or $U_b\cap(b,x)$) such that the sequence $y',f(y'),\dots,f^{n-l}(y')$ satisfies the assumptions of lemma 2 and

$$|f^j(y')-f^j(b)|<K_V \qquad \text{for } j=0,\dots,n-l-1.$$

Let $z'\in(v,w)$ be such that $f^l(z')=y'$. In exactly the same way as for z, we prove that the estimate (19) remains true for z'.

If $n_b\geq n-l$ then

$$|(f^{n_b})'(y)|=|(f^{n_b-n-l})'(f^{n-l}(y))|\,|(f^{n-l})'(y)|,$$

where the first factor on the right side of the above equality is bounded from above by a constant K_0 depending on U because the set of numbers n_b, $b\in A$, is finite. This argument together with lemma 5 gives

$$|(f^{n-l})'(y)|\geq\frac{c}{K_0}\,\theta^{n-l}.$$

If $z\in(v,w)$ is such that $f^l(z)=y$, then as above we obtain the estimate analogous to (19):

$$|(f^n)'(z)|\geq\frac{R_1}{K_0}\,\nu^n. \qquad \square$$

Remark. We can formulate and prove in a similar way a lemma analogous to lemma 7 for the left end point of Δ.

THEOREM 2. *There exists $d>0$ such that for $n\in\mathbb{N}$, $\max_{\Delta\in\bigvee_0^{n-1}f^{-i}\mathcal{A}}\lambda(\Delta)\le d/\nu^n$.*

Proof. We can assume than n is the biggest number such that $\Delta\in\bigvee_0^{n-1}f^{-i}\mathcal{A}$. Let $\Delta=(v,w)$, z and n_z be as in lemma 7. Write $y=f^l(z)$, $b=f^l(w)$. For all $m\ge n_z$ we define $I_m=\{x\in(y,b): |f^j(x)-f^j(b)|<K_V$ for $j=0,\dots,m-1$ and $|f^m(x)-f^m(b)|\ge K_V\}$. It is easy to see that I_m is an interval and $f^m|_{I_m}$ is monotonic. Hence, by using lemma 5, we obtain

$$\lambda(I_m)\le\lambda(I)/c\theta^m.$$

Observe that by lemma 7(b) $(y,b)=\bigcup_{m=n_z}^{\infty}I_m$. Therefore

$$\lambda((y,b))\le\sum_{m=n_z}^{\infty}\lambda(I_m)\le\frac{\theta\lambda(I)}{c(\theta-1)}\frac{1}{\theta^{n-l}} \tag{20}$$

Now, using the standard argument we prove that for all $t\in(z,w)$ $|(f^l)'(t)|$ can be estimated from below by ν^l, up to a constant depending only on U and V. From this estimate and (20) we obtain

$$\lambda((z,w))\le R_2(1/\nu^n), \tag{21}$$

where $R_1>0$ depends only on U and V. Now using the version of lemma 7 for the left end point of Δ we find a point $u\in\Delta$ such that

$$\lambda((v,u))\le R_2(1/\nu^n), \tag{22}$$

and by lemma 7(a) and its version for the left end point of (u,z) we get, if we observe that $|f'|$ has no positive local minima:

$$\lambda((u,z))\le\frac{\lambda(I)}{D}\frac{1}{\nu^n}; \tag{23}$$

from (21), (22) and (23) we obtain the desired estimate for $\lambda(\Delta)$. □

Definition 1. If m is a measure on I then we define the *diameter* of a partition $\mathscr{C}$ as $\max_{C\in\mathscr{C}}m(C)$.

Theorem 2 says that the diameter of $\bigvee_0^{n-1}f^{-i}\mathcal{A}$ in the sense of λ decreases exponentially with n.

4. *Estimation of the diameter of $\bigvee_0^{n-1}f^{-i}\mathcal{A}$ in the sense of the invariant measure*

The exponential rate of decrease of the diameters of $\bigvee_0^{n-1}f^{-i}\mathcal{A}$ in the sense of the invariant measure μ follows from theorem 2 and the following:

THEOREM 3. *If $\lambda(G)$ is sufficiently small then $\mu(G)\le R\sqrt[s]{\lambda(G)}$ for $R>0$, $s>1$ depending only on U and V.*

Proof. Let $m_0\ge1$ be such that f^{m_0} satisfies the conditions (i)–(vi) and (vii), (viii) [**5**]:

(vii) $|(f^{m_0})'|>1$ on $U(f^{m_0})$ $(U(f^{m_0})=\bigcup_{i=0}^{m_0-1}f^{-i}(U))$;

(viii) If $a\in A(f^{m_0})$ is a periodic point for f^{m_0}, then it is a fixed point for f^{m_0}.

Such an m_0 exists – see lemma 3.1 of [**5**].

Let $E_n=\{x\in I: f^{km_0}(x)\notin U(f^{m_0}),\ k=0,\dots,n-1\}$. E_n is a neighbourhood of $B\setminus A(f^{m_0})$ and it is proved in [**5**] (the proof of lemma 3.7) that there are constants

$d_1>0$, ξ_1, $\xi_2\in(0,1)$ such that for every $n\in\mathbb{N}$

$$\sup_{k\in\mathbb{N}}\int_{E_n} f_*^{km_0}(1)\,d\lambda\le\sum_{l=n}^{\infty}(l+1)d_1\xi_1^l+\lambda(I)\xi_2^n. \tag{24}$$

Let a be an end point of a component of E_n and $b\in B$ a point from this component. For some $k\le n-1$ we have $f^{km_0}(a)\in U(f^{m_0})$ and

$$|f^{km_0}(a)-f^{km_0}(b)|\ge\operatorname{dist}(B\setminus A(f^{m_0}),U)=:h_B.$$

Hence

$$|a-b|\ge\frac{h_B}{F_0^k}\ge\frac{h_B}{F_0^n},$$

where $F_0=\sup_{x\in I}|(f^{m_0})'(x)|$. Since the component of E_n and the point b are arbitrary, we have shown

$$\operatorname{dist}(B\setminus A(f^{m_0}),I\setminus E_n)\ge\frac{h_B}{F_0^n}. \tag{25}$$

Let $A_1(f^{m_0})$ denote the set of periodic points of the set $A(f^{m_0})$. For $a\in A_1(f^{m_0})$ we define

$$V_n(a)=\{x\in I: f^{m_0 i}(x)\in U_a\quad\text{for } i=0,\ldots,n-1\},$$

as in the proof of lemma 3.6 of [5]. It is shown in this proof that there exist $d_2>0$, ξ_3, $\xi_4\in(0,1)$ such that

$$\sup_{k\in\mathbb{N}}\int_{V_n(a)} f_*^{km_0}(1)\,d\lambda\le d_2\sum_{l=n}^{\infty}\xi_3^l+\lambda(I)\xi_4^n \tag{26}$$

Write $V_n=\bigcup_{a\in A_1(f^{m_0})}V_n(a)$, $h_A=\operatorname{dist}(A_1(f^{m_0}),I\setminus U(f^{m_0}))$. As in (25) we prove that

$$\operatorname{dist}(A_1(f^{m_0}),I\setminus V_n)\ge h_A/F_0^n. \tag{27}$$

Denote by $A_2(f^{m_0})$ the set of non-periodic points of $A(f^{m_0})$. The set $A_2(f^{m_0})$ is finite so it is easy to construct a neighbourhood G_n of $A_2(f^{m_0})$ being a finite sum of intervals such that

$$\operatorname{dist}(A_2(f^{m_0}),I\setminus G_n)=h_A/F_0^n. \tag{28}$$

From [5, (3.4)] there are $d_3>0$, $\xi_5\in(0,1)$ such that

$$\sup_{k\in\mathbb{N}}\int_{G_n} f_*^{km_0}(1)\,d\lambda\le d_3\left(\frac{h_A}{F_0^n}\right)^{1-\xi_5}. \tag{29}$$

Put $W_n=E_n\cup V_n\cup G_n$. If n is sufficiently large then the sets E_n, V_n and G_n are disjoint and by (25), (27) and (28) we have for $h\le h_A$, h_B

$$\operatorname{dist}(B,I\setminus W_n)\ge h/F_0^n. \tag{30}$$

The estimates (29), (26) and (24) imply the existence of $P_0>0$ and $\xi\in(0,1)$ such that

$$\sup_{k\le 0}\int_{W_n} f_*^{km_0}(1)\,d\lambda\le P_0\xi^n. \tag{31}$$

Fix $\varepsilon>0$. If we take $n=[\log_{F_0}(\varepsilon/3P_0)/\log_{F_0}(\xi)]+1$, then

$$P_0\xi^n\le\varepsilon/3 \tag{32}$$

and

$$\frac{h}{F_0^n} \geq \frac{h}{F_0} \cdot \left(\frac{\varepsilon}{3P_0}\right)^{-1/\log_{F_0}(\xi)}. \tag{33}$$

Now we can follow the proof of [**5**, prop. 3.8] with W_n instead of W. [**5**, 3.10] is satisfied because of (31) and (32). Following the proof we find a δ which satisfies

$$\delta \geq d_4 \varepsilon \operatorname{dist}(B, I \setminus W_n), \tag{34}$$

where d_4 depends only on U and is such that

$$\text{if } \lambda(G) < \delta, \quad \text{then } \int_G f_*^n(1)\, d\lambda \leq \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{35}$$

But in view of (30), (33) and (34)

$$\delta \geq d_5 \varepsilon^{1+\sigma} \tag{36}$$

for some $d_5 > 0$, $\sigma > 0$ which do not depend on δ and ε. Now the theorem follows from (35), (36) and [**5**, th. 6.2(e)]. □

5. *Estimation of the rate of decrease of the mixing coefficient*

Definition 2. For two partitions $\mathscr{C}$, $\tilde{\mathscr{C}}$ of I we define

$$D(\mathscr{C}, \tilde{\mathscr{C}}) = \sum_{C \in \mathscr{C}} \sum_{\tilde{C} \in \tilde{\mathscr{C}}} |\mu(C \cap \tilde{C}) - \mu(C)\mu(\tilde{C})|.$$

We will show that $D(\bigvee_0^{n-1} f^{-i}\mathscr{A}, \bigvee_{n+l}^{n+l+k-1} f^{-i}\mathscr{A})$ decreases exponentially with respect to l uniformly in n and k.

Let $F = \sup_{x \in I} |f'(x)|$ and take $p_0 \in \mathbb{N}$ such that $\sqrt[p_0]{F} < \nu$, where ν is as in theorem 2.

Definition 3. Fix $M \in \mathbb{N}$. For every $m \in \mathbb{N} \cup \{0\}$ we define:

(a) $\mathscr{D}_{m,M}^0$ to be the set of all atoms of $\bigvee_0^{m+[M/p_0]-1} f^{-i}\mathscr{A}$ whose end points and their $m-1$ consecutive images do not belong to A;

(b) $\mathscr{D}_{m,M}$ to be the set of all atoms of $\bigvee_0^{m+M-1} f^{-i}\mathscr{A}$ which are subsets of the atoms belonging to $\mathscr{D}_{m,M}^0$.

Write $h_n(y) = (1/(f^n)'(y))$ for $n \in \mathbb{N}$, $y \in I$.

LEMMA 8. *The function h_n is monotonic on the elements of the partition $\bigvee_0^{n-1} f^{-i}\mathscr{A}$.*

Proof. In view of (iii) $1/\sqrt{|(f^n)'|}$ is convex on the elements of $\bigvee_0^{n-1} f^{-i}\mathscr{A}$ (see [**5**, (iii), (iii')]). The square of a non-negative convex function is a convex function, hence $1/|(f^n)'|$ is convex on the elements of $\bigvee_0^{n-1} f^{-i}\mathscr{A}$. This proves the assertion of the lemma. □

LEMMA 9. *Let $\Delta \in \mathscr{D}_{m,M}^0$. For every $y \in \Delta$, $|h_m(y)| \leq H(\sqrt[p_0]{F})^M$, where H is the constant of lemma* 4.

Proof. Fix $\Delta \in \mathscr{D}_{m,M}^0$. By the definition of $\mathscr{D}_{m,M}^0$ h_m is finite on all $\bar{\Delta}$. In view of lemma 8 $h_m|_{\bar{\Delta}}$ assumes its greatest value at one of the end points of Δ, denote it by z. Now it suffices to estimate $S_m(z)$, because $|h_m(z)| \leq S_m(z)$.

Let r be the smallest positive integer such that $f^{m+r}(z) \in A$ (it is clear that $r < [M/p_0]$). By (2), (3) and the definition of V, $f^{m+r-1}(z) \notin V$. Therefore by lemma 4

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$S_{m+r-1}(z) \le H$. On the other hand

$$S_{m+r-1}(z) = S_m(z)\frac{1}{|(f^{r-1})'(f^m(z))|} + S_{r-1}(f^m(z))$$
$$\ge S_m(z)\frac{1}{F^{M/p_0}}$$

Therefore $S_m(z) \le H(\sqrt[p_0]{F})^M$. □

LEMMA 10. *Let $\Delta \in \mathcal{D}_{m,M}$, y', $y'' \in \Delta$. If M is sufficiently large, then*

$$\left|\frac{(f^m)'(y'')}{(f^m)'(y')} - 1\right| \le 2H(\sqrt[p_0]{F})^M |f^m(y') - f^m(y'')|.$$

Proof. Let $\Delta \in \mathcal{D}_{m,M}$, y', $y'' \in \Delta$. By definition Δ is contained in an atom of $\mathcal{D}^0_{m,M}$, so we can use lemma 9 to obtain

$$\left|\frac{1}{(f^m)'(y')} - \frac{1}{(f^m)'(y'')}\right| \le H(\sqrt[p_0]{F})^M |y' - y''|.$$

The function $|(f^m)'||_{[y',y'']}$ assumes its minimum at one of the ends points of $[y', y'']$ (by (iii) and the definition of $\mathcal{D}_{m,M}$), say at y''. Hence

$$\left|\frac{(f^m)'(y'')}{(f^m)'(y')} - 1\right| \le H(\sqrt[p_0]{F})^M |f^m(y') - f^m(y'')|.$$

If $|(f^m)'|$ has its minimum at y' we obtain just an analogous estimation of $|[(f^m)'(y')/(f^m)'(y'')] - 1|$. In this case, if M is sufficiently large, then the right side of this inequality does not exceed $\frac{1}{2}$ (because of theorem 2 and the choice of p_0 such that $\sqrt[p_0]{F} < \nu$). Now it is easy to obtain the desired inequality. □

Let m_0 be such that f^{m_0} satisfies (i)-(vi) and (vii), (viii). Denote by $A'(f^{m_0})$ the set of the points from $A(f^{m_0})$ and the ends points of the components of $U(f^{m_0})$. Let $\mathcal{A}'$ be a partition defined by the points of $A'(f^{m_0})$. Let W_n, for sufficiently large $n \in \mathbb{N}$, be the neighbourhood of B defined in the proof of theorem 3. By the definition of W_n:

(37) W_n is a sum of some atoms belonging to $\bigvee_0^{nm_0-1} f^{-i}(\mathcal{A}')$ and of G_n (which is a finite sum of intervals).

Definition 4. Fix a large $M \in \mathbb{N}$. For every $m \in \mathbb{N} \cup \{0\}$ we define a family $\mathcal{A}_{m,M}$ of good atoms of $\bigvee_0^{m+M-1} f^{-i}(\mathcal{A}')$ as a family of all $\Delta \in \bigvee_0^{m+M-1} f^{-i}\mathcal{A}'$ such that

(a) $\Delta \subset \operatorname{supp} \mu$;

(b) Δ is contained in an atom belonging to $\mathcal{D}_{m,M}$;

(c) the end points of Δ and their $m-1$ consecutive images do not belong to $A'(f^{m_0})$;

(d) $(\Delta \cup f^m(\Delta)) \cap W_{[M/2p_0m_0]} = \varnothing$.

Let φ be the density of μ with respect to the Lebesgue measure λ. B. Szewc proved in [7] that if f satisfies (i)-(vi), then $\varphi \in C^2$ and for $j = 0, 1, 2$, $x \in I \setminus B$

$$|\varphi^{(j)}(x)| \le \sum_{n=0}^{\infty} \sum_{\substack{a\in A\\ f(a)\notin A}} \frac{1}{|(f^n)'(f(a))|^{\eta(a)}} \frac{1}{|x - f^{n+1}(a)|^{\xi(a)+j}} + 1, \tag{38}$$

where $\xi(a)=u_a/(u_a+1)$ (u_a is the constant from condition (vi) corresponding to the point a), $\eta(a)=1-\xi(a)$.

LEMMA 11. *There exist $L_0>0$, $\xi_0\in(0,1)$ such that if $x\in I\setminus B$, then for $j=0,1,2$*

$$|\varphi^{(j)}(x)|\le L_0/|x-b|^{\xi_0+j},$$

where b is the point of B nearest to x.

Proof. It is a simple consequence of (38) and (5). □

LEMMA 12. *There exist $Z>0$, $\zeta\in(0,1)$ such that if $\Delta\in\mathscr{A}_{m,M}$, $m\in\mathbb{N}\cup\{0\}$, $x,y\in\Delta$, then*

$$\left|\frac{\varphi(x)}{\varphi(y)}-1\right|\le Z\zeta^M.$$

Proof. Fix $\Delta\in\mathscr{A}_{m,M}$. By definition Δ is contained in the complement of a neighbourhood of singularities of φ and in supp φ. Hence $\varphi_{\bar\Delta}$ is positive, convex ([**5**]), and

$$\sup_{x\in\Delta}|\varphi'(x)|=|\varphi'(v)|,$$

where v is one of the end points of Δ. So we have

$$|\varphi(x)-\varphi(y)|\le|\varphi'(v)||x-y|\qquad\text{for } x,y\in\Delta,$$

and by definition 4 and theorem 2,

$$|\varphi(x)-\varphi(y)|\le d|\varphi'(v)|/\nu^M. \tag{39}$$

In view of lemma 11

$$|\varphi'(v)|\le\frac{L_0}{|v-b|^{\xi_0+1}},\qquad b\in B. \tag{40}$$

By definition 4, $\Delta\cap W_{[M/2p_0m_0]}=\varnothing$, so using (30) we obtain

$$|v-b|\ge\operatorname{dist}(I\setminus W_{[M/2p_0m_0]},B)\ge\frac{h}{F^{M/2p_0}}.$$

From the last estimate, (40) and (39) it follows that

$$|\varphi(x)-\varphi(y)|\le\frac{dL_0}{h^{\xi_0+1}}\left(\frac{\sqrt[p_0]{F}}{\nu}\right)^M,$$

which ends the proof because φ is separated from zero by a positive constant on supp φ. □

The next lemma is analogous to [**3**, lemma 12].

LEMMA 13. *There exist $S>0$, $\Omega>1$ such that if $M\in\mathbb{N}$ is sufficiently large, $m\in\mathbb{N}\cup\{0\}$, then*:

(a) *for every $\Delta\in\mathscr{A}_{m,M}$, $f^m(\Delta)\in\mathscr{A}_{0,M}$*;

(b) $\mu(\bigcup\mathscr{A}_{m,M})>1-(S/\Omega^M)$;

(c) *if $\tilde\Delta$ is a measurable subset of $\Delta\in\mathscr{A}_{m,M}$ then*

$$\left|\frac{\mu(f^m(\tilde\Delta))}{\mu(f^m(\Delta))}-\frac{\mu(\tilde\Delta)}{\mu(\Delta)}\right|<\frac{S}{\Omega^M}\frac{\mu(\tilde\Delta)}{\mu(\Delta)}.$$

Proof. (a) is a consequence of definition 4(c). For a proof of (b) let us notice that for $\Delta\in\bigvee_0^{m+M-1}f^{-i}\mathscr{A}'$, $\Delta\notin\mathscr{A}_{m,M}$ if and only if it satisfies one of the following four conditions:

(1) $\Delta \cap (I \setminus \operatorname{supp} \mu) \neq \emptyset$;

(2) Δ is not contained in an atom in $\mathcal{D}_{m,M}$;

(3) there exists k, $0 \leq k < m$ such that
$$\overline{f^k(\Delta)} \cap A'(f^{m_0}) \neq \emptyset;$$

(4) $(\Delta \cup f^m(\Delta)) \cap W_{[M/2p_0m_0]} \neq \emptyset$.

There are only two atoms in $\bigvee_0^{m+M-1} f^{-i}\mathcal{A}'$ with a non-zero μ-measure which satisfy the first condition. This follows from the fact that supp μ is an interval (remark 5.6 and theorem 6.2(b) of [**5**]). Hence a suitable estimate of the sum of atoms satisfying condition (1) is an immediate consequence of theorems 2 and 3.

A suitable estimate of the sum of atoms satisfying condition (2) or (3) can be obtained by using theorems 2, 3 and a standard argument (see the proof in [**1**]).

A suitable estimate of the sum of atoms satisfying condition (4) follows from (37), (31) and theorems 2 and 3.

For a proof of (c) take a measurable subset $\tilde{\Delta}$ of an atom $\Delta \in \mathcal{A}_{m,M}$ and choose a point $x_0 \in \tilde{\Delta}$. We have
$$\frac{\mu(f^m(\tilde{\Delta}))}{\mu(f^m(\Delta))} = \frac{\displaystyle\int_{\tilde{\Delta}} \frac{\varphi(f^m(x))}{\varphi(f^m(x_0))} \frac{|(f^m)'(x)|}{|(f^m)'(x_0)|} \frac{\varphi(x_0)}{\varphi(x)}\, d\mu(x)}{\displaystyle\int_{\Delta} \frac{\varphi(f^m(x))}{\varphi(f^m(x_0))} \frac{|(f^m)'(x)|}{(f^m)'(x_0)|} \frac{\varphi(x_0)}{\varphi(x)}\, d\mu(x)}$$

Now we obtain the desired estimate by using lemmas 12 and 10. If the constants obtained in the proofs of (b) and (c) are different, we may change them to obtain the estimates with the same constants, which we will call $S > 0$, $\Omega > 1$. □

Now we will quote some results of B. Szewc ([7]). For $\varepsilon \in [0, \min_{a \in A}(1-\xi(a))/2) \cup \mathbb{N}$, let
$$\varphi_\varepsilon(x) = \left[\sum_{n=0}^{\infty} \sum_{a \in \tilde{A}} |(f^n)'(f(a))|^{-\eta(a)} \frac{1}{|x - f^{n+1}(a)|^{\varepsilon + \xi(a)}}\right] + 1.$$

Let $\mathcal{J}$ denote the set of all components of $I \setminus B$. B. Szewc defines for ρ: $I \setminus B \to \mathbb{R}$
$$\|\rho\|_\varepsilon = \sup_{J \in \mathcal{J}} \sup_J \frac{|\rho|}{\varphi_\varepsilon}.$$

If additionally ρ is Lipschitz on compact subsets of $I \setminus B$, then Szewc also defines
$$|\rho|_{(0)+1} = \sup_{J \in \mathcal{J}} \operatorname*{essup}_J \frac{|\rho|}{\varphi_1}$$

and
$$\|\rho\|_{(0)+1} = \max\{\|\rho\|_\varepsilon, |\rho|_{(0)+1}\}.$$

He considers the space $C_{f,\varepsilon}^{(0)+1}$ of all functions ρ which are Lipschitz on compact subsets of $I \setminus B$ and such that $\|\rho\|_{(0)+1} < +\infty$.

Let ϕ be the operator on $L^1(\lambda)$ given by the formula
$$\phi(\psi) = f_*(\psi) - \left(\int \psi \, d\lambda\right) \cdot \phi \qquad \text{for } \psi \in L^1(\lambda);$$

(recall that f_* is the Perron–Frobenius operator for the Lebesgue measure λ).

B. Szewc proved that $C_{f,\varepsilon}^{(0)+1}$ is a Banach space ([7, remark 4.3(b)]) invariant with respect to f_* and ϕ, and there exist constants $\Gamma>0$, $\rho\in(0,1)$ such that for $\psi\in C_{f,\varepsilon}^{(0)+1}$

$$\|\phi^n(\psi)\|_{L^1}\le\Gamma\cdot\rho^n\|\psi\|_{(0)+1}. \tag{41}$$

LEMMA 14. *If* $\Delta\in\bigvee_0^{M-1}f^{-i}\mathcal{A}$, *then* $\phi^M(\varphi\cdot\chi_\Delta)\in C_{f,\varepsilon}^{(0)+1}$ *and* $\|\phi^M(\varphi\cdot\chi_\Delta)\|_{(0)+1}\le P$, *where* P *is a constant which does not depend on* Δ *nor on* M.

Proof. The proof of this lemma follows easily from the considerations conducted in the proof of proposition 5.4 in [7]. □

LEMMA 15. *There exists* $\Gamma_1>0$ *such that for all* M, n, $l\in\mathbb{N}$, $l\ge M$,

$$D\left(\bigvee_0^{M-1}f^{-i}\mathcal{A},\ \bigvee_{M+l}^{M+l+n-1}f^{-i}\mathcal{A}\right)\le\Gamma_1\left(\frac{N}{\rho}\right)^M\cdot\rho^l,$$

where N *denotes the number of elements of* $\mathcal{A}$ *and* ρ *is the constant from* (41).

Proof. Using an argument analogous to that in the proof of lemma 13 in [3] we show that

$$D\left(\bigvee_0^{M-1}f^{-i}\mathcal{A},\ \bigvee_{M+l}^{M+l+n-1}f^{-i}\mathcal{A}\right)\le 2\cdot\sum_{\Delta\in\bigvee_0^{M-1}f^{-i}\mathcal{A}}\|f_*^l(\varphi\cdot\chi_\Delta)-\mu(\Delta)\varphi\|_{L^1}.$$

Observe that $f_*^l(\varphi\cdot\chi_\Delta)-\mu(\Delta)\varphi=\phi^l(\varphi\cdot\chi_\Delta)$. In view of (41) and lemma 14 we have $\|\phi^l(\varphi\cdot\chi_\Delta)\|_{L^1}\le\Gamma\cdot\rho^{l-M}P$. □

LEMMA 16. *If* $l\ge 3M$ *then*

$$D\left(\bigvee_0^{M-1}f^{-i}\mathcal{A},\ \bigvee_{M+l}^{M+l+n-1}f^{-i}\mathcal{A}\right)\le\Gamma_1\left(\frac{N^2}{\rho^3}\right)^M\rho^l+M\frac{\Gamma_2}{(\sqrt[s]{\nu})^M},$$

where $\Gamma_2>0$ *does not depend on* M *nor on* n.

Proof. By theorems 2 and 3 the atoms of $\bigvee_0^{M-1}f^{-i}\mathcal{A}$ are contained in the atoms of $\mathcal{A}'$, up to atoms whose sum has measure not greater than $2\operatorname{card}\mathcal{A}'R\sqrt[s]{d}/(\sqrt[s]{\nu})^M$. Hence we have

$$\begin{aligned}
&D\left(\bigvee_0^{M-1}f^{-i}\mathcal{A}',\ \bigvee_{M+l}^{M+l+n-1}f^{-i}\mathcal{A}\right)\\
&\le D\left(\bigvee_0^{M-1}f^{-i}\left(\mathcal{A}'\vee\bigvee_0^{M-1}f^{-j}\mathcal{A}\right),\ \bigvee_{M+l}^{M+l+n-1}f^{-i}\mathcal{A}\right)\\
&\le D\left(\bigvee_0^{M-1}f^{-i}\left(\bigvee_0^{M-1}f^{-j}\mathcal{A}\right),\ \bigvee_{M+l}^{M+l+n-1}f^{-i}\mathcal{A}\right)+2M\,2\operatorname{card}\mathcal{A}'\frac{R\sqrt[s]{d}}{(\sqrt[s]{\nu})^M},
\end{aligned}$$

but $\bigvee_0^{M-1}f^{-i}(\bigvee_0^{M-1}f^{-j}\mathcal{A})=\bigvee_0^{2M-2}f^{-i}\mathcal{A}$, and we can use lemma 15 to estimate $D(\bigvee_0^{2M-2}f^{-i}\mathcal{A},\bigvee_{M+l}^{M+l+n-1}f^{-i}(\mathcal{A})$. □

THEOREM 4. *There exist constants* $P>0$, $p\in(0,1)$ *such that for arbitrary* n, k, $l\in\mathbb{N}$ *we have*

$$D\left(\bigvee_0^{n-1}f^{-i}\mathcal{A},\ \bigvee_{n+l}^{n+l+k-1}f^{-i}\mathcal{A}\right)\le P\cdot p^l.$$

Proof. We proceed in the same way as F. Hofbauer and G. Keller in the proof of [3, theorem 4].

Fix δ, $0<\delta<\frac{1}{4}$ such that $\rho(N^2/\rho^3)^\delta<1$. It suffices to prove the theorem for sufficiently large l and $n\geq[\delta l]+1$.

Put $M:=[\delta l]+1$, $m:=n-M$. If l is sufficiently large, then M is so large that we can use all the lemmas proved above. In view of lemma 13(b)

$$D\left(\bigvee_0^{n-1} f^{-i}\mathscr{A}, \bigvee_{n+l}^{n+l+k-1} f^{-i}\mathscr{A}\right)\leq D\left(\bigvee_0^{n-1} f^{-i}\mathscr{A}', \bigvee_{n+l}^{n+l+k-1} f^{-i}\mathscr{A}\right)$$
$$\leq D\left(\mathscr{A}_{m,M}, \bigvee_{n+l}^{n+l+k-1} f^{-i}\mathscr{A}\right)+2S/\Omega^M. \qquad (42)$$

Put $\mathscr{G}:=\bigvee_{n+l}^{n+l+k-1} f^{-i}\mathscr{A}$. Performing estimations similar to those of the proof of theorem 4 of [3] and using lemma 13(a), (c) we prove that

$$D(\mathscr{A}_{m,M}, \mathscr{G})\leq (S/\Omega^M)+D\left(\bigvee_0^{M-1} f^{-i}\mathscr{A}', \bigvee_{M+l}^{M+l+k-1} f^{-i}\mathscr{A}\right).$$

Notice, that $\delta<\frac{1}{4}$, so $3M=3[\delta l]+1\leq l$ and we can use lemma 16. We obtain

$$D(\mathscr{A}_{m,M}, \mathscr{G})\leq\frac{S}{\Omega^M}+\Gamma_1\left(\frac{N^2}{\rho^3}\right)^M\rho^l+M\frac{\Gamma_2}{(\sqrt[s]{\nu})^M}.$$

But $M=[\delta l]+1$, $(N^2/\rho^3)^\delta\,\rho<1$, so the last inequality together with (42) gives an estimate for $n\geq[\delta l]+1$ and n sufficiently large. □

6. *Concluding remarks*; *proof of theorem* 1

In § 1 we have formulated the main result of this paper, theorem 1, which implies other important theorems like Log-log laws, integral tests, etc.... Now we are in the position to prove theorem 1:

Let F and $(S_t)_{t\geq0}$ be as in theorem 1. To prove theorem 1 (ii) it suffices to know that $F\circ f^i$, $i=0,1,\ldots$, can be treated as a functional of some process $(\zeta_n)_{n\in\mathbb{N}}$ and to check that the assumptions of theorem 7.1 of [6] are satisfied in this case.

Define, for $n\in\mathbb{N}$, a random variable ξ_n on $([0,1]), \mathscr{B}, \mu)$ by the formula

$$\xi_n(x)=j \qquad \text{if } f^n(x)\in[a_j, a_{j+1}),$$

where a_i, $i=0,\ldots,s_0$ are the points of A.

In view of theorem 2 the process $(F\circ f^i)_{i=0,1,\ldots}$ can be treated as a functional of 'label-process' $(\zeta_n)_{n=0,1,\ldots}$. Now by using theorems 2, 3 and 4, we check as in the proof of theorem 4 in [3] that the assumptions of theorem 7.1 of [6] are satisfied. □

In § 1 we have assumed the continuity of f. This assumption is not necessary and was made only for simplicity. It suffices to assume the continuity of f on components of the set $I\setminus A$.

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