BULL. AUSTRAL. MATH. SOC. 30C45, 30C55
VOL. 24 (1981), 381-388.

A RADIUS OF CONVEXITY PROBLEM

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The authors determine the sharp radius of convexity for functions analytic and starlike in the unit disc having power series representation of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$ where a_{n+1} is fixed. The estimate obtained is an improvement over the corresponding fixed second coefficient result. It is expected that this approach will lead to sharpening and improvement of a number of earlier known results.

1. Introduction

Let $w = f(z)$ be regular and univalent in the unit disc $\Delta \equiv \{z : |z| < 1\}$ and be normalized by the conditions $f(0) = 0$, $f'(0) = 1$. The power series representation for such a function is

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k . \tag{1.1}$$

Gronwall in [1] showed that there is a positive number r_0 such that $w = f(z)$ maps $|z| \le r_0$ onto a convex region. In fact $r_0 \ge 2 - \sqrt{3}$ for all functions $w = f(z)$ regular and univalent in Δ. Thus, with every such function, one can associate a radius of convexity r_0 which is the largest number such that $w = f(z)$ maps $|z| \le r_0$ onto a convex region

Received 26 June 1981. The research by the first author was supported by a research grant of Central Research Committee of University of Khartoum.

and $|z| \le r$ with $r > r_0$ need not be mapped onto a convex region.

If $f(z)$, given by (1.1), satisfies the condition

$$\text{(1.2)} \qquad \operatorname{Re}\{z(f'(z)/f(z))\} > 0$$

for all $z \in \Delta$, then it is well known (see, for example, [5], [6]) that (1.2) is both necessary and sufficient for $f(z)$ to be univalent and star-like with respect to the origin in Δ. The radius of convexity for the class of starlike functions is known to be $2 - \sqrt{3}$ (see [8]). The radius of convexity for various other subclasses of univalent functions have also been obtained (see, for example, [3], [8], and so on).

Recently, considerable attention has been paid to study various aspects of univalent and regular function $f(z)$ with power series representation (1.1) whose second coefficient a_2 is fixed throughout (see, for example, [2], [4], [9], and so on). This has, in particular, led to sharpening of radius of convexity estimates which in turn has given rise to refining a number of results. For example, Tepper [10] showed that the radius of convexity r_0 for starlike functions with a_2 fixed is given by

$$\text{(1.3)} \qquad r_0 \ge \left(a+\sqrt{(a^2+32)}-\sqrt{\left[2a^2+2a\sqrt{(a^2+32)}+16\right]}\right)/4 \ge 2 - \sqrt{3} .$$

Here $|a_2| = a$. The estimate (1.3) led Tepper [10] to an improvement of an old estimate in support of Schild's conjecture [7].

The aim of the present paper is to consider functions $f(z)$ regular in Δ and having power series representation

$$\text{(1.4)} \qquad f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots$$

where we take a_{n+1} to be fixed throughout. We then obtain sharp radius of convexity estimates for $f(z)$ when it is starlike, that is, when it satisfies conditions of the form (1.2). Here it may be pointed out that it is perhaps for the first time that this problem is being solved by taking the general $(n+1)$th coefficient fixed. This yields not only fixed second coefficients results for $n = 1$ but it is expected that this approach will also give rise to refinement and improvement of various known results.

https://doi.org/10.1017/S0004972700004925 Published online by Cambridge University Press

2. Preliminary lemmas

We need the following lemmas.

LEMMA 1. *Let* $\omega(z) = c_n z^n + c_{n+1} z^{n+1} + \dots$ *be regular in* Δ *and* $|\omega(z)| < 1$. *Then* $|c_n| \le 1$ *and*

$$(2.1) \qquad |\omega(z)| \le r^n\big((r+|c_n|)/(1+|c_n|r)\big) , \quad |z| = r .$$

Proof. Let $\omega_1(z) \equiv \omega(z)/z^n = c_n + \dots$. Then $\omega_1(z)$ is regular and satisfies $|\omega_1(z)| \le 1$ with $|c_n| \le 1$. Define

$$(2.2) \qquad \omega_2(z) = \big(\omega_1(z)-c_n\big)/\big(1-\bar{c}_n\omega_1(z)\big) .$$

Then $\omega_2(0) = 0$ and $|\omega_2(z)| \le 1$. Thus, by Schwarz's lemma [5],

$$|\omega_2(z)| \le |z| .$$

But, from (2.2),

$$\omega_1(z) = \big(\omega_2(z)+c_n\big)/\big(1+\bar{c}_n\omega_2(z)\big) .$$

Therefore

$$\begin{aligned} |\omega(z)/z^n| = |\omega_1(z)| &= \big|\big(\omega_2(z)+c_n\big)/\big(1+\bar{c}_n\omega_2(z)\big)\big| \\ &\le \big(|\omega_2(z)|+|c_n|\big)/\big(1+|c_n||\omega_2(z)|\big) \\ &\le \big(r+|c_n|\big)/\big(1+|c_n|r\big) . \end{aligned}$$

Now the result immediately follows.

LEMMA 2. *If* $p(z) = 1 + b_n z^n + \sum_{k=n+1}^{\infty} b_k z^k$ *is regular and has* $\operatorname{Re} p(z) > 0$ *for* $z \in \Delta$, *then, for* $|z| = r$,

$$(2.3) \qquad \operatorname{Re} p(z) \ge \big[2+b_n r-b_n r^n-2r^{n+1}\big]/\big[2+b_n r+b_n r^n+2r^{n+1}\big]$$

$$(2.4) \qquad |p(z)| \le \big[2+b_n r+b_n r^n+2r^{n+1}\big]/\big[2+b_n r-b_n r^n-2r^{n+1}\big]$$

where $0 \le b_n \le 2$. *Both the estimates are sharp.*

Proof. Since $\operatorname{Re} p(z) > 0$ and $p(0) = 1$, the function $p(z)$ is

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subordinate to the function $(1+z)/(1-z)$. Therefore there exists a function $h(z)$ which is regular for $z \in \Delta$ with $h(0) = 0$ and $|h(z)| < 1$ such that

$$(2.5) \qquad p(z) = \big(1+h(z)\big)/\big(1-h(z)\big) = 1 + b_n z^n + \sum_{k=n+1}^{\infty} b_k z^k .$$

A direct computation gives $h(z) = \big(b_n/2\big)z^n + \ldots$. Since $h(z)$ satisfies the conditions of Lemma 1, applying it we have

$$(2.6) \qquad |h(z)| \le r^n\big(\big(r+\tfrac{1}{2}b_n\big)/\big(1+\tfrac{1}{2}b_n r\big)\big) = r^n\big(\big(2r+b_n\big)/\big(2+b_n r\big)\big) .$$

Another direct computation shows that

$$(2.7) \qquad \operatorname{Re}\, p(z) = \big(1-|h(z)|^2\big)/\big(|1-h(z)|^2\big) \ge \big(1-|h(z)|\big)/\big(1+|h(z)|\big) .$$

Since the right hand side of (2.7) is monotone decreasing with respect to $|h(z)|$, applying (2.6) to (2.7), we get (2.3) immediately.

To prove (2.4) consider the function

$$q(z) = \big(p(-z)\big)^{-1} .$$

Since $q(z)$ satisfies the hypothesis of Lemma 2 we have

$$1/|p(z)| = |q(-z)| \ge \operatorname{Re}\, q(z) \ge \left(2+b_n r-b_n r^n-2r^{n+1}\right)/\left(2+b_n r+b_n r^n+2r^{n+1}\right)$$

which gives (2.4).

It is easy to see that the estimate in (2.3) is sharp for the function $p(z) = \left(2+b_n z-b_n z^n-2z^{n+1}\right)/\left(2+b_n z+b_n z^n+2z^{n+1}\right)$ while the estimate in (2.4) is sharp for the function $p(z) = \left(2+b_n z+b_n z^n+2z^{n+1}\right)/\left(2+b_n z-b_n z^n-2z^{n+1}\right)$ for each b_n , $0 \le b_n \le 2$.

LEMMA 3. *If* $p(z) = 1 + b_n z^n + \sum_{k=n+1}^{\infty} b_k z^k$ *is regular and has* $\operatorname{Re}\, p(z) > 0$ *for* $z \in \Delta$ *, then, for* $|z| = r$ *,*

$$(2.8) \qquad |z\big(p'(z)/p(z)\big)| \le 2r^n \cdot \left\{\left(2nb_n+\left(4n+4+(n-1)b_n^2\right)r+2nb_n r^2\right)/\left(\big(2+b_n r\big)^2-r^{2n}\big(2r+b_n\big)^2\right)\right\}$$

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where $b_n \geq 0$.

Proof. As in Lemma 2, there exists a function $h(z)$ regular in Δ with $|h(z)| < 1$ such that

$$p(z) = \big(1+h(z)\big)/\big(1-h(z)\big) .$$

Furthermore, since $h(0) = 0$, $|h(z)| < 1$ and $h(z)$ has the power series expansion $h(z) = \big(b_n/2\big)z^n + \ldots$, there exists a function $\phi(z)$ regular in Δ with $|\phi(z)| < 1$ such that $h(z) = z^n\phi(z)$. Using this, we obtain

$$p(z) = \big(1+z^n\phi(z)\big)/\big(1-z^n\phi(z)\big) . \tag{2.9}$$

Taking the logarithmic derivatives on both sides of equation (2.9), we have

$$p'(z)/p(z) = \big(2\{nz^{n-1}\phi(z)+z^n\phi'(z)\}\big)/\big(1-z^{2n}\{\phi(z)\}^2\big) .$$

Thus

$$|p'(z)/p(z)| \leq 2\big(\big(nr^{n-1}|\phi(z)|+r^n|\phi'(z)|\big)/\big(1-r^{2n}|\phi(z)|^2\big)\big) . \tag{2.10}$$

But it is well known [5] that

$$|\phi'(z)| \leq \big(1-|\phi(z)|^2\big)/\big(1-|z|^2\big) . \tag{2.11}$$

Since the right hand side of (2.10) is monotone increasing with respect to $|\phi'(z)|$, by substituting (2.11) into (2.10), we obtain

$$\begin{aligned} &|p'(z)/p(z)| \\ &\quad \leq \big(2/\big(1-r^2\big)\big)\big[\big(nr^{n-1}\big(1-r^2\big)|\phi(z)|+r^n\big(1-|\phi(z)|^2\big)\big)/\big(1-r^{2n}|\phi(z)|^2\big)\big] . \end{aligned} \tag{2.12}$$

It is easy to check that the expression in the square bracket in (2.12) is monotone increasing with respect to $|\phi(z)|$. Furthermore,

$$|\phi(z)| = |h(z)/z^n| \leq \big(2r+b_n\big)/\big(2+b_nr\big) .$$

Hence

$$\begin{aligned} &|p'(z)/p(z)| \\ &\quad \leq 2\left(\left(n\big(1-r^2\big)r^{n-1}\left(\frac{2r+b_n}{2+b_nr}\right)+r^n\left\{1-\left(\frac{2r+b_n}{2+b_nr}\right)^2\right\}\right)\bigg/\left(\big(1-r^2\big)\left\{1-r^{2n}\left(\frac{2r+b_n}{2+b_nr}\right)^2\right\}\right)\right) \\ &\quad = 2r^{n-1}\left(\left(2nb_n+\left(4+4n+(n-1)b_n^2\right)r+2nb_nr^2\right)\bigg/\left(\big(2+b_nr\big)^2-r^{2n}\big(2r+b_n\big)^2\right)\right) \end{aligned}$$

which completes the proof.

3. Radius of convexity estimates

We now obtain an exact estimate for radius of convexity for starlike functions whose power series expansion is of the form

$$(3.1) \qquad f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots .$$

There is no loss of generality in assuming that $a_{n+1} \geq 0$ in (3.1). If this is not the case, then consider the function $w = e^{i\alpha/n}f\left(ze^{-i\alpha/n}\right)$ where $\arg a_{n+1} = \alpha$. We now prove the following theorem.

THEOREM. *Let* $f(z) = z + a_{n+1}z^{n+1} + \sum_{k=n+2}^{\infty} a_kz^k$ *be regular and starlike with respect to origin in* Δ *; then* f *is convex in* $|z| < r_0$ *, where* r_0 *is the smallest positive root of the equation*

$$(3.2) \quad 4r^{2n+2} + 4na_{n+1}r\left(1+r^{2n}\right) + n^2a_{n+1}^2r^2\left(1+r^{2n-2}\right) - 4n(n+1)a_{n+1}r^n\left(1+r^2\right) - 2\left[8+4n+n^3a_{n+1}^2\right]r^{n+1} + 4 = 0 .$$

The estimate is sharp for each a_{n+1} *,* $0 \leq a_{n+1} \leq 2/n$ *.*

Proof. If

$$p(z) = z\left(f'(z)/f(z)\right) = 1 + na_{n+1}z^n + \dots ,$$

then, since f satisfies (1.2), Lemma 2 gives

$$(3.3) \quad \operatorname{Re} p(z) \geq \left[2+na_{n+1}r-na_{n+1}r^n-2r^{n+1}\right] / \left[2+na_{n+1}r+na_{n+1}r^n+2r^{n+1}\right] .$$

Direct computation gives

$$(3.4) \qquad 1 + z\left(f''(z)/f'(z)\right) = p(z) + z\left(p'(z)/p(z)\right) .$$

Therefore, applying (3.3) and (2.8) with $b_n = na_{n+1}$ to (3.4), we obtain

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(3.5)

$$\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\}$$

$$\geq \operatorname{Re}\,p(z) - \left|z\frac{p'(z)}{p(z)}\right|$$

$$\geq \frac{2+n\delta r-n\delta r^{n}-2r^{n+1}}{2+n\delta r+n\delta r^{n}+2r^{n+1}} - 2r^{n}\frac{2n^{2}\delta+\left(4n+4+(n-1)n^{2}\delta^{2}\right)r+2n^{2}\delta r^{2}}{(2+n\delta r)^{2}-r^{2n}(2r+n\delta)^{2}}$$

$$= \frac{4r^{2n+2}+4n\delta r\left(1+r^{2n}\right)+n^{2}\delta^{2}r^{2}\left(1+r^{2n-2}\right)-4n(n+1)\delta r^{n}\left(1+r^{2}\right)-2\left(8+4n+n^{3}\delta^{2}\right)r^{n+1}+4}{(2+n\delta r)^{2}-r^{2n}(2r+n\delta)^{2}},$$

where $\delta = a_{n+1}$. Since a univalent function $f(z)$ maps $|z| \leq r$ onto a convex region if and only if $\operatorname{Re}\left(1+z\left(f''(z)/f'(z)\right)\right) > 0$ for $|z| \leq r$ [5], [6], it follows from (3.5) that $f(z)$ is convex in $|z| < r_0$ where r_0 is the smallest positive root of equation (3.2). This completes the proof.

The function $f(z)$ given by

$$z\left(f'(z)/f(z)\right) = \left(2+na_{n+1}z-na_{n+1}z^{n}-2z^{n+1}\right)/\left(2+na_{n+1}z+na_{n+1}z^{n}+2z^{n+1}\right)$$

shows that the estimate (3.2) is sharp for each a_{n+1}, $0 \leq a_{n+1} \leq 2/n$.

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