QUASI-STATIC EVOLUTION OF A THREE-DIMENSIONAL FORCE-FREE MAGNETIC FIELD

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1 - INTRODUCTION

In this Communication, we consider in the half-space $\{z > 0\}$ a fully 3-D force-free magnetic field $\underset{\sim}{B}$ embedded in a perfectly conducting plasma and its quasi-static evolution driven by motions imposed to the feet of its lines on the boundary $\{z = 0\}$. Assuming that the field lines of $\underset{\sim}{B}$ have a simple topology - i.e. that it is possible to choose a (non-unique) set of nested magnetic surfaces which are either "arcade-like" or "tube-like" - we first establish a global representation of $\underset{\sim}{B}$ in terms of two Euler potentials u and v (with v multivalued) and derive new formulae giving in particular the relative helicity and the energy of $\underset{\sim}{B}$ as functions of the values of u and v (and of their derivatives) on $\{z = 0\}$. We thus establish analytically some general qualitative features of the behaviour of $\underset{\sim}{B}$. We show in particular that an indefinite shearing of the feet of the lines leads to an illimited inflation of the magnetic surfaces and then eventually to their opening, the currents concentrating in infinitesimally thin current sheets.

2 - REPRESENTATION OF THE FIELD

Let us consider in the half-space $\Omega = \{z > 0\}$ a magnetic field which satisfies the following assumptions:

i) $\underset{\sim}{B} \neq 0$ in Ω (no neutral points);

ii) the parts $\partial\Omega^{+/-}$ of the boundary $\partial\Omega = \{z = 0\}$ on which $B_z > 0$ / < 0, respectively, are separated by a simple curve L (neutral line) which is either closed (Fig. 1a) or open (Fig. 2a);

iii) all the field lines of $\underset{\sim}{B}$ cut twice $\partial\Omega$ (there are no open lines); there are no lines meeting $\partial\Omega$ tangentially along the neutral line.

With these assumptions, the lines of $\underset{\sim}{B}$ establish a one-to-one correspondance (magnetic mapping) between $\partial\Omega^+$ and $\partial\Omega^-$. Then let us choose in $\partial\Omega^+$ an arbitrary covering set of nested regular curves C_u^+ (when L is closed, we assume, without loss of generality, that $\partial\Omega^+$ is inside that curve), parametrized by the magnetic flux

$$u = \int_{\Sigma_u^+} B_z \, d\sigma \qquad (1)$$

where Σ_u^+ is the area enclosed inside C_u^+ (then $L = C_\Phi^+$, where $\Phi = \int_{\partial\Omega^+} B_z \, d\sigma$).

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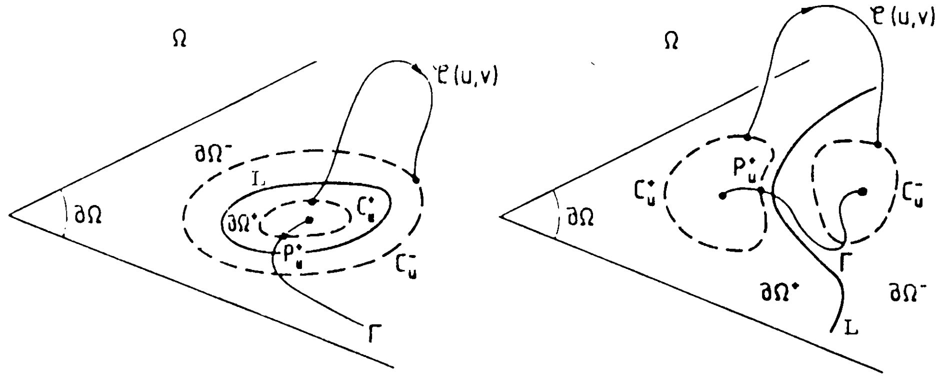

(a) (b)

Figure 1 (see the text)

Now we choose on $\partial\Omega^+$ an arc Γ^+ cutting once and only once (at P_u^+) each curve C_u^+ and set (∇_t = component of ∇ parallel to $\partial\Omega$)

$$v(u,s) = \int_{o\,C_u^+}^{s} \frac{B_z}{|\nabla_t u|}\, ds' \Big/ \int_{o\,C_u^+}^{\ell_u^+} \frac{B_z}{|\nabla_t u|}\, ds' \tag{2}$$

where s ($0 \leqslant s < \ell_u^+$) is the arc length - counted from P_u^+ - along the counter-clockwise oriented C_u^+ and ℓ_u^+ = length C_u^+. Then, if we define u and v at each point $\underset{\sim}{r}$ of Ω by seting: $u(\underset{\sim}{r}) = u(\underset{\sim}{r}_0)$ and $v(\underset{\sim}{r}) = v(\underset{\sim}{r}_0)$, where $\underset{\sim}{r}_0$ is that point of $\partial\Omega^+$ where the line of $\underset{\sim}{B}$ passing though $\underset{\sim}{r}$ originates, it may be shown that the following Euler representation of $\underset{\sim}{B}$ holds: $\underset{\sim}{B} = \nabla u \times \nabla v$ (3)

The following points are worth noticing:

i) the set of curves C_u^+ naturally defines in Ω a set of nested magnetic surfaces Σ_u (on which u = const.) which are arcade-like (Fig. 1a) or tube-like (Fig. 1b); however, the C_u^+ are quite arbitrary, and then there are not intrinsically defined magnetic surfaces in our type of configuration (in contrast with the case of general toroidal configurations) ;

ii) our representation of $\underset{\sim}{B}$ is global; v, however, is discontinuous accross the surface S formed by the lines originating from Γ^+ ($[v]_s = 1$); equivalently, we may consider v as being multivalued ;

iii) the relative helicity of $\underset{\sim}{B}$ (Berger and Field, 1984), which is an important topological quantity, may be expressed as

$$H = \int_{\partial\Omega^-} u\, u_0\, (\nabla_t v \times \nabla_t v_0)\, .\, \hat{z}\, d\sigma \tag{4}$$

where we have set $\underset{\sim}{B}_0 = \nabla u_0 \times \nabla v_0$ ($\underset{\sim}{B}_0$ is defined by $\nabla \times \underset{\sim}{B}_0 = 0$ in Ω and $B_{0z} = B_z$ on $\partial\Omega$) and choosen for that potential field, also assumed to satisfy our topological assumptions, the same set of curves $\{C_u^+\}$ as for $\underset{\sim}{B}$.

3 - EQUATIONS DESCRIBING THE QUASI-STATIC EVOLUTION OF A FORCE-FREE FIELD

Let us now assume that Ω is filled up with a perfectly conducting plasma and that the field $\underset{\sim}{B}$ evolves quasi-statically through a sequence of force-free configurations as a consequence of a velocity field $\underset{\sim}{v}$ (with $v_z = 0$) imposed on the boundary $\partial\Omega$. Then:

i) at each time t, the potentials u and v satisfy (Barnes and Sturrock, 1972)

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$$\nabla \cdot \{\nabla u \times (\nabla u \times \nabla v)\} = \nabla \cdot \{\nabla v \times (\nabla u \times \nabla v)\} = 0 \tag{5}$$

ii) the boundary conditions $(u,v)(x,y,o,t) = (U,V)(x,y,t)$ for (5) are obtained by solving

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \underset{\sim}{v} \cdot \nabla_t U = \frac{dV}{dt} = \frac{\partial V}{\partial t} + \underset{\sim}{v} \cdot \nabla_t V = 0 \tag{6}$$

$$(U,V)(x,y,o) = (U_0, V_0)(x,y) \tag{7}$$

where (U_0, V_0) are the boundary values of the potentials of the initial field (e.g. a "good" potential field) ;

iii) we complete the boundary conditions by the "asymptotic" condition

$$\int_\Omega B^2 d\underset{\sim}{r} = \int_\Omega |\nabla u \times \nabla v|^2 = \int_0^{\psi} du \int_{C_u} v \, \underset{\sim}{B}_t \cdot d\underset{\sim}{s} < \infty \tag{8}$$

where we have introduced for the energy of a force-free field a new formula in which C_u is a closed oriented curve constituted of C_u^+, C_u^- and that part Γ_u of $\Gamma^+ \cup \Gamma^-$ joining them (C_u^- and Γ^- corresponding to C_u^+ and Γ^+, respectively, by the magnetic mapping).

4 - METHODS OF SOLUTION

Two methods have been used to try to prove that the shearing problem stated in § 3 has a solution. However, no definite results have yet been obtained.

4.1 - Variational method

One tries to minimize the functional

$$C[u,v] = \int_\Omega |\nabla u \times \nabla v|^2 \, d\underset{\sim}{r} \tag{9}$$

- which admits (5) as its Euler-Lagrange equations - over the set of functions (u,v) satisfying the right boundary conditions. A possible way to effect the minimization is as follows. One first fixes a set of magnetic surfaces Σ_u meeting the given curves $C_u^{\pm}$ and one minimizes with respect to v. This first step amounts to solve the linear equation

$$\nabla_t \cdot \{|\nabla u| \nabla_t v\} = 0 \tag{10}$$

on each Σ_u. One then gets a unique solution for $v[u]$, which determines the optimum shape for the field lines on Σ_u. Reporting this $v[u]$ into (9), one obtains a functional of u alone, which is still under study.

4.2 - Perturbation method

One starts from a known equilibrium (u,v) (e.g. a potential field). Thus one changes slightly the boundary conditions $[(U,V) \to (U + \epsilon U_1, V + \epsilon V_1)]$ and looks for a new solution of the form $(u+\epsilon u_1, v+\epsilon v_1)$. Then (u_1, v_1) must be solution of

$$L(\underset{\sim}{r})(u_1, v_1) = \epsilon M(\underset{\sim}{r}, u_1, v_1) \tag{11}$$