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A NOTE ON PROXIMINALITY IN $C(S\times T)$ WITH THE L_1-NORM

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1. Introduction

Let S and T be compact Hausdorff spaces and G and H finite-dimensional subspaces of $C(S)$ and $C(T)$ respectively. Suppose μ and ν are regular Borel measures on S and T respectively such that $\mu(S)=\nu(T)=1$. The product measure $\mu\times\nu$ will be denoted by σ. Set $U=G\otimes C(T)$, $V=C(S)\otimes H$ and $W=U+V$. If G and H possess continuous proximity maps, then U and V are proximinal subspaces of $C(S\times T)$ when this linear space is equipped with the L_1-norm, [**4**, Lemma 2]. That is, every $z\in C(S\times T)$ possesses at least one best approximation from U and from V. A metric selection $A_U:C(S\times T)\to U$ is a mapping which associates each $z\in C(S\times T)$ with one of its best approximations in U. The metric selection A_V is similarly defined. In [**4**] the behaviour of the Diliberto–Straus algorithm was investigated. For a given $z\in C(S\times T)$ this algorithm generates a sequence $\{z_n\}$ by taking $z_1=z$ and setting

$$\begin{array}{l} z_{2n}=z_{2n-1}-A_U z_{2n-1} \\ z_{2n+1}=z_{2n}-A_V z_{2n} \end{array},\quad n=1,2,3,\ldots.$$

Under suitable hypothesis on G,H,A_U,A_V and z it was established in [**4**] that $\|z_n\|\to\operatorname{dist}(z,W)$. An essential ingredient of the proof was an application of the Ascoli Theorem to the sequence $\{z_n\}$. It is clear from the construction of this sequence that it is bounded in the L_1-norm. However, the Ascoli Theorem requires boundedness in the supremum norm. This is not an obvious result and was only established implicitly by the discussion in [**4**]. In this note we establish a result which is sufficient to guarantee this boundedness explicitly. It also has applications to questions about proximinality in $C(S\times T)$ with the L_1-norm. These results parallel those of Respess and Cheney [**6**] in the same linear space with the usual supremum norm.

2. Notation and the basic result

We shall retain much of the notation of [**4**]. In particular unadorned norm symbols will always denote the L_1-norm. Select bi-orthonormal bases $\{g_i,\phi_i\}_1^n$ for $\{h_i,\psi_i\}_1^m$ for G and H respectively, where each is equipped with the L_1-norm. Assume that there are (supremum norm) continuous proximity maps $A_G:C(S)\to G$ and $A_H:C(T)\to H$. Then these can be extended by Lemma 2 of [**4**] to proximity maps $A_U:C(S\times T)\to U$ and $A_V:C(S\times T)\to V$ by taking $(A_Uz)(s,t)=(A_Gz^t)(s)$ and $(A_Vz)(s,t)=(A_Hz_s)(t)$. Here z^t,z_s are the sections of z defined by $z^t(s)=z(s,t)$ and $z_s(t)=z(s,t)$. We shall need to assume that $A_U(z+u)=A_Uz+u$ for all $z\in C(S\times T)$ and $u\in U$ with a similar assumption for A_V. These requirements are met if A_G and A_H satisfy the corresponding properties (i.e. for A_G, for example, we have $A_G(x+g)=A_Gx+g$ for all $x\in C(S)$ and $g\in G$).

Definition 2.1.

(i) A sequence $\{z_k\}$ in $C(S\times T)$ is said to be admissible if there is some $z\in C(S\times T)$ such that $z-z_k\in W$ for all $k\in\mathbb{N}$, and $\|z_k\|\leqq\|z\|$ for all k.

(ii) Let $\{z_k\}$ be a sequence in $C(S\times T)$. Then its derived sequence $\{z_k'\}$ is defined by $z_k'=(I-A_V)(I-A_U)z_k$, $k\in\mathbb{N}$.

Lemma 2.2. *There exists a constant c such that each element $w\in W$ has a representation $w=u+v$, with $u\in G\otimes C(T)$, $v\in C(S)\otimes H$ and $\|u\|+\|v\|\leqq c\|w\|$.*

Proof. Take $w\in W$. Then w is in the larger space $G\otimes L_1(T)+L_1(S)\otimes H$ which is closed in $L_1(S\times T)$, [3, Lemma 4.2]. Hence there is a constant c such that each element $w\in W$ can be written $w=u+v$ where $u\in G\otimes L_1(T)$, $v\in L_1(S)\otimes H$ and $\|u\|+\|v\|\leqq c\|w\|$. Now we claim $u\in U$, $v\in V$. It will suffice to show $u\in G\otimes C(T)$. Write $u=\sum g_iy_i$ and $v=\sum x_ih_i$ with $x_i\in L_1(S)$ and $y_i\in L_1(T)$. Then

$$\|x_i\|=\int|x_i(s)|\,ds=\int|\psi_i(v_s)|\,ds\leqq\int\|v_s\|\,ds=\|v\|.$$

Also,

$$\|v^t-v^\tau\|=\|\sum x_i[h_i(t)-h_i(\tau)]\|$$
$$\leqq\sum\|x_i\|\,|h_i(t)-h_i(\tau)|\leqq\|v\|\sum|h_i(t)-h_i(\tau)|.$$

Now

$$|y_i(t)-y_i(\tau)|=|\phi_i(u^t-u^\tau)|$$
$$=|\phi_i(w^t-v^t-w+v^\tau)|$$
$$\leqq\|w^t-w^\tau\|+\|v^t-v^\tau\|$$
$$\leqq\|w^t-w^\tau\|_\infty+\|v\|\sum|h_i(t)-h_i(\tau)|.$$

This inequality demonstrates the continuity of y_i and hence of u. □

Lemma 2.3. *Suppose $z_1\in C(S\times T)$ and $z_1=z-u-v$ for some $u\in U$, $v\in V$, where $\|u\|+\|v\|\leqq c\|z\|$. If $A_Vz_1=0$ then $\|v\|_\infty\leqq cM\|z\|_\infty$ while if $A_Uz_1=0$ then $\|u\|_\infty\leqq cM\|z\|_\infty$. Here $M=4\sum\|h_i\|_\infty\sum\|g_i\|_\infty$.*

Proof. We consider the case $A_U z_1=0$, the other case being similar. Set $u=\sum g_i y_i$ and $v=\sum x_i h_i$. Then

$$\|x_i\|=\int|x_i(s)|\,ds=\int|\psi_i(v_s)|\,ds\leqq\int\|v_s\|\,ds=\|v\|\leqq c\|z\|\leqq c\|z\|_\infty.$$

Also

$$\|v^t\|=\|\sum h_i(t)x_i\|\leqq\sum|h_i(t)|\,\|x_i\|\leqq c\|z\|_\infty\sum\|h_i\|_\infty.$$

Now $0=A_U z_1=A_U(z-v)-u$, and so $u=A_U(z-v)$ and $u^t=A_G(z^t-v^t)$. Thus

$$\begin{aligned}|y_i(t)|=|\phi_i(u^t)|\leqq\|u^t\|\leqq 2\|z^t-v^t\|\leqq 2\|z^t\|+2\|v^t\|\\ \leqq 2\|z\|_\infty+2c\|z\|_\infty\sum\|h_i\|_\infty.\end{aligned}$$

This gives

$$\begin{aligned}\|u\|_\infty&\leqq\sum\|g_i\|_\infty\|y_i\|_\infty\leqq(2\|z\|_\infty+2c\|z\|_\infty\sum\|h_i\|_\infty)\sum\|g_i\|_\infty\\ &\leqq 4c\|z\|_\infty\sum\|h_i\|_\infty\sum\|g_i\|_\infty=cM\|z\|_\infty.\end{aligned}$$

□

Theorem 2.4. *Let $\{z_k\}$ be an admissible sequence in $C(S\times T)$ and let $\{z_k'\}$ be its derived sequence. Then there is a constant M such that $\|z_k'\|_\infty\leqq M$ for all $k\in\mathbb{N}$.*

Proof. Since $\{z_k\}$ is admissible we may write $z-z_k=w$ where $w\in W$ and by 2.2 we may set $w=u+v$ where

$$\|u\|+\|v\|\leqq c\|w\|\leqq c\|z-z_k\|\leqq 2c\|z\|\leqq 2c\|z\|_\infty.$$

Now set $u^*=A_U(z-v)$ and $v^*=A_V(z-u^*)$. Then

$$z_k-A_U z_k=z_k-A_U(z-u-v)=z-u-v+u-A_U(z-v)=z-u^*-v.$$

Also

$$z_k'=z-u^*-v-A_V(z-u^*-v)=z-u^*-v+v-A_V(z-u^*)=z-u^*-v^*.$$

Now

$$\|u^*\|\leqq 2\|z-v\|\leqq 2(1+c)\|z\|\leqq 4c\|z\|$$

and,

$$\|v^*\|\leqq 2\|z-u^*\|\leqq 2(1+4c)\|z\|,$$

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so that,

$$\|u^*\|+\|v^*\|\leqq 14c\|z\|.$$

Since $A_V z_k'=0$ we can apply 2.3 to get $\|v^*\|_\infty\leqq 14cM\|z\|_\infty$. Similarly

$$\|u^*\|+\|u\|\leqq 6c\|z\|$$

and $A_U(z-u^*-v)=0$. Thus 2.3 again gives $\|u^*\|_\infty\leqq 6cM\|z\|_\infty$. Finally

$$\|z_k'\|_\infty\leqq\|z\|_\infty+\|u^*\|_\infty+\|v^*\|_\infty\leqq 21cM\|z\|_\infty. \qquad \square$$

If we now return to the algorithm we see that

$$z_{2n+1}=z_{2n}-A_V z_{2n}=z_{2n-1}-A_U z_{2n-1}-A_V(z_{2n-1}-A_U z_{2n-1})=(I-A_V)(I-A_U)z_{2n-1}.$$

This shows that the odd iterates in the algorithm are members of the sequence derived from the odd iterates (except, of course for z_1). Similarly the even iterates are derived from the previous ones by

$$z_{2n+2}=(1-A_U)(I-A_V)z_{2n}.$$

Thus 2.4 and its symmetric counterpart give:

Corollary 2.5. *The iterates in the L_1-version of the Diliberto–Straus algorithm in $C(S\times T)$ are bounded in the supremum norm.*

We now consider the question of proximinality and obtain the analogue of the "Sitting Duck Theorem" of Respess and Cheney [6].

3. Proximinality in $C(S\times T)$ with the L_1-norm

We shall continue with the notation established in Section 2 with two additional hypotheses. Firstly, we shall assume that G has a Lipschitz continuous proximity map. Thus we shall assume that there is a constant λ such that $\|A_G x_1-A_G x_2\|_\infty\leqq\lambda\|x_1-x_2\|_\infty$ for all x_1,x_2 in $C(S)$. Secondly, we shall assume H possesses a continuous proximity map with respect to the supremum norm. This has the force of making the mapping from U to V defined by $u\to A_V(z-u)$ continuous for fixed z in $C(S\times T)$; see Lemma 3 of [4] for details.

Lemma 3.1. *Let $\{z_k\}$ be an admissible sequence in $C(S\times T)$. Let $\{z_k'\}$ be its derived sequence and let $\{z_k''\}$ be the derived sequence of $\{z_k'\}$. Then $\{z_k''\}$ is a bounded, equicontinuous sequence in $C(S\times T)$.*

Proof. The sequence $\{z_k'\}$ is certainly admissible and so 2.4 shows that both $\{z_k'\}$ and

$\{z_k''\}$ are bounded. From the proof of 2.4 we may write

$$z_k' = z - u_k - v_k \quad \text{where} \quad \|u_k\|_\infty \leqq M \quad \text{and} \quad \|v_k\|_\infty \leqq M$$

$$z_k'' = z - a_k - b_k \quad \text{where} \quad \|a_k\|_\infty \leqq M, \|b_k\|_\infty \leqq M$$

and

$$a_k = A_U(z - v_k),\ b_k = A_V(z - a_k).$$

Now,

$$\begin{aligned} |a_k(s_1,t_1) - a_k(s,t)| &\leqq |a_k(s_1,t_1) - a_k(s_1,t)| + |a_k(s_1,t) - a_k(s,t)| \\ &\leqq \|a_k^{t_1} - a_k^t\|_\infty + \|(a_k)_{s_1} - (a_k)_s\|_\infty \\ &= \|A_G(z-v_k)^{t_1} - A_G(z-v_k)^t\|_\infty + \|(a_k)_{s_1} - (a_k)_s\|_\infty \\ &\leqq \lambda\|z^{t_1} - z^t\|_\infty + \lambda\|v_k^{t_1} - v_k^t\|_\infty + \|(a_k)_{s_1} - (a_k)_s\|_\infty. \end{aligned}$$

Now set $v_k = \sum x_i h_i$ and $a_k = \sum g_i y_i$. Then

$$|x_i(s)| = |\psi_i(v_s)| \leqq \|v_s\| \leqq \|v\|_\infty \leqq M$$

and so

$$\|v_k^{t_1} - v_k^t\|_\infty \leqq \sum \|x_i\|_\infty |h_i(t_1) - h_i(t)| \leqq M \sum |h_i(t_1) - h_i(t)|.$$

In a similar manner,

$$\|(a_k)_{s_1} - (a_k)_s\|_\infty \leqq \sum \|y_i\|_\infty |g_i(s_1) - g_i(s)| \leqq M \sum |g_i(s_1) - g_i(s)|.$$

Hence

$$|a_k(s_1,t_1) - a_k(s,t)| \leqq \lambda\|z^{t_1} - z^t\|_\infty + \lambda M\left(\sum |h_i(t_1) - h_i(t)| + \sum |g_i(s_1) - g_i(s)|\right).$$

Now the sections $\{z^t : t \in T\}$ form an equicontinuous family in $C(S)$ and so given ε we may force the three terms on the right to be each at most $\varepsilon/3$ by taking (s_1, t_1) sufficiently close to (s,t). Hence the $\{a_k\}$ form an equicontinuous family in $C(S\times T)$. Thus $\{z - a_k\}$ is also an equicontinuous family and so, since A_V is a continuous mapping, $\{b_k\}$ forms an equicontinuous family. Finally, these combine to give $\{z_k''\}$ equicontinuous. □

One of the results from [6] which is always needed in this type of problem is that the subspace W is closed. This only rests on the finite-dimensionality of G and H.

Theorem 3.2. *Let G be a finite dimensional subspace of C(S) having a Lipschitz continuous proximity map. Let H be a finite dimensional subspace of C(T) having a continuous proximity map. Then* $C(S)\otimes H+G\otimes C(T)$ *is proximinal in* $C(S\times T)$ *when the* L_1*-norm is employed.*

Proof. Fix $z\in C(S\times T)$ and pick a minimising sequence $\{w_k\}$ in $W=C(S)\otimes H+G\otimes C(T)$ such that $\|z-w_k\|\downarrow \operatorname{dist}(z,W)$. Set $z_k=z-w_k$. Then $\{z_k\}$ is an admissible sequence. Let its derived sequence be $\{z_k'\}$ and the derived sequence of $\{z_k'\}$ be $\{z_k''\}$. Then we have $\|z_k''\|\leqq\|z_k'\|\leqq\|z_k\|$, and so $\{z_k''\}$ is also a minimising sequence. It is also bounded and equicontinuous and so, by the Ascoli theorem, has a cluster point. This cluster point is in W, since W is closed, and is a best approximation to z from W. □

It is easy to provide examples of the above result. If G is a one-dimensional subspace generated by a function g which is bounded away from zero then the proximity map A_G is Lipschitz continuous. If H is a Chebyshev subspace of $C(T)$ with respect to the L_1-norm (i.e. each element y possesses a unique best L_1-approximation in H) then an old result (see, for example, Holmes [1]) guarantees that A_H is continuous.

Corollary 3.3. *Let G be a one-dimensional subspace of C(S) generated by a function which is bounded away from zero. Let H be a finite dimensional Chebyshev subspace of C(T) with respect to the* L_1*-norm. Then* $C(S)\otimes H+G\otimes C(T)$ *is proximinal in* $C(S\times T)$ *under the* L_1*-norm.*

An example of subspaces G and H which satisfy the conditions of 3.3 are π_0 and π_n respectively, where π_n denotes the subspace of $C(S)$ or $C(T)$ consisting of polynomials of degree at most n. That A_G is Lipschitz may be found in **[5]** while the Chebyshev property of H is an old result of Jackson [2].

Corollary 3.4. *The subspace* $C(S)\otimes\pi_n+\pi_0\otimes C(T)$ *is proximinal in* $C(S\times T)$ *under the* L_1*-norm.*

This result includes a result from **[5]** which stated that $C(S)+C(T)$, which is of course shorthand for $C(S)\otimes\pi_0+\pi_0\otimes C(T)$, is proximinal in $C(S\times T)$ when the L_1-norm is used.

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