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UNBOUNDED NEGATIVE DEFINITE FUNCTIONS

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0. Introduction. Negative definite functions (all definitions are given in § 1 below) on a locally compact, σ-compact group G have been used in several different contexts recently [**2**, **5**, **7**, **11**]. In this paper we show how such functions relate to other properties such a group may have. Here are six properties which G might have. They are grouped into three pairs with one property of each pair involving negative definite functions. We show that the paired properties are equivalent and, where possible, give counter-examples to other equivalences. We assume throughout that G is not compact.

(1A) G does not have property T.

(1B) There is a continuous, negative definite function on G which is unbounded.

(2A) G has the (weak and/or strong) dual R-L property.

(2B) For every closed, non-compact set $Q \subset G$ there is a continuous, negative definite function on G which is unbounded on Q.

(3A) $C_0(G)$ has an approximate unit consisting of positive definite functions.

(3B) There is a continuous, negative definite function ψ on G such that for any $M > O$ there is a compact set $K \subset G$ with $|\psi(x)| > M$ for all $x \in G\backslash K$.

Note that (3B) $\Rightarrow$ (2B) $\Rightarrow$ (1B) in an obvious way, but the relationships among (1A), (2A) and (3A) are not so apparent at first. We shall prove the equivalences (1A) $\Leftrightarrow$ (1B), (2A) $\Leftrightarrow$ (2B) and (3A) $\Leftrightarrow$ (3B) as well as several other results related to these properties. As applications we give simpler proofs for (and/or stronger versions of) some results of Connes and Kazhdan. An interesting open problem is the possible equivalence between (2B) and (3B).

This paper was inspired in part by conversations with Alain Connes and Uffe Haagerup. We thank the referee for pointing out [**12**].

1. Notation and preliminaries. We shall assume familiarity with basic information about C^*-algebras as found in the first few chapters of [**4**, **9** or **10**]. Recall that the positive linear functionals of norm 1 on a C^*-algebra A are called *states*, and $S(A)$ denotes the positive linear

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functionals on A of norm less than or equal to 1. $S(A)$ is a convex set which is compact in the weak* topology determined by A. The second dual space A^{**} of A is a von Neumann algebra [**4**, **9**, **10**] containing A in its canonical embedding. The multiplier algebra $M(A)$ is defined to be the idealizer of A in A^{**}. Elements of $M(A)$ are weak* continuous as functions on the set $S_1(A)$ of all states of A. A projection p in A^{**} is called *open* [**9**, Section 3.11] if there is a net $\{a_\alpha\}$ of positive operators in A which is increasing and satisfies $\lim a_\alpha = p$ in the weak* topology of A^{**}. If $A_0 = \{a \in A: pa = ap = a\}$, then A_0 is a C^*-algebra with dual space A_0^* canonically isometrically isomorphic to $pA^*p = \{pfp: f \in A^*\}$, where $pfp(a) = f(pap)$ for all $a \in A^{**}$ (the Sakai product [**10**]). In particular we get that the extreme points of $\{f \in S(A): f(p) = 1\}$ are extreme points of $S(A)$ (i.e., pure states or 0) and that every state g of A satisfying $g(p) = 1$ can be approximated by convex combinations of such extreme points in the weak* topology.

Now let G be a locally compact, σ-compact group. We shall assume familiarity with the last few chapters of [**4**] which deal with such groups. We let $P(G)_1$ denote the set of continuous positive definite functions f on G satisfying $f(e) = 1$, where e is the identity of G. The group C^*-algebra is denoted by $C^*(G)$ as in [**4**], and $P(G)_1$ is exactly the set of states of $C^*(G)$. Further, there is a natural embedding of G into $M(C^*(G))$ so that an element x of G defines a weak* continuous function on $P(G)_1$ by $x(f) = f(x)$, where $f(x)$ has the same value when f is considered as a function on the group or as a functional on $M(C^*(G))$ in which G is embedded. Further, the weak* topology on $P(G)_1$ is equivalent to the topology of uniform convergence on compact subsets of G, and we shall use this fact several times in our proofs. One particular element of $P(G)_1$ is of special interest; set $f_0(x) = 1$ for all $x \in G$. Let $z_0 \in C^*(G)^{**}$ be the one dimensional central projection such that $z_0 f_0 = f_0$ (see [**1**]). The extreme points of $P(G)_1$ are called *pure positive definite functions*.

We say that G has the *weak* (resp. *strong*) *dual R-L* property [**1**] if for every $\epsilon > 0$ and for every weak* neighborhood V of f_0 in $P(G)_1$ (resp. every open projection $p \geqq z_0$ in $C^*(G)^{**}$) there is a compact neighborhood K of e in G such that for every $x \in G\backslash K$ there exists a pure, continuous positive definite function f in V (resp. with $f(p) = 1$) such that $\mathrm{Re}(f(x)) \leqq \epsilon$. In [**1**] the weak and strong dual R-L properties are shown to be equivalent.

Let $C_0(G)$ denote the continuous functions on G which vanish at infinity. The sup norm on $C_0(G)$ is denoted by $\|\cdot\|_\infty$.

If G is a discrete group (as in Theorem 9), we say G is an ICC group [**10**] if $\{xyx^{-1}: x \in G\}$ is infinite for all $y \in G\backslash\{e\}$. Following Effros, we say G is *inner amenable* [**8**] if there is a mean on $l^\infty(G)$ which is invariant under inner automorphisms of G. Paschke [**8**] introduces the representation of G as unitary operators on $L^2(G)$ by: if $x \in G$ and η is in $L^2(G)$, then

$$U_x(\eta)(y) = \eta(x^{-1}yx).$$

He lets δ be the characteristic function of $\{e\}$ and P_δ the projection of $L^2(G)$ onto the subspace spanned by δ. Paschke shows that the group G is inner amenable if and only if P_δ is not in the C^*-algebra generated by $\{U_x: x \in G\}$.

A complex function ψ on G is called *negative definite* if, for all n-tuples $x_1, \ldots, x_n$ in G, the $n \times n$ matrix with entries

$$(\psi(x_i) + \overline{\psi(x_j)} - \psi(x_j^{-1}x_i))$$

is positive semi-definite. These functions are discussed in great detail in Chapter II, § 7-9 of [**2**]. While [**2**] assumes that G is abelian, the results we need all hold in the general case. Further information on negative definite functions is given in [**7**]. For this paper we need only a few facts. It follows from Section 7.14 of [**2**] that if ψ is a continuous negative definite function on G, then so is

$$\psi_0 = \text{Re } [(\psi - \psi(e))^{1/2}].$$

The function ψ_0 is not only continuous and negative definite, but it is bounded if and only if ψ is bounded, $\psi_0(e) = 0$ and ψ_0 takes only non-negative real values. If φ is a positive definite function on G, then $\varphi(e) - \varphi = \psi$ is a negative definite function. If $\{\psi_n\}_{n=1}^{\infty}$ are continuous, negative definite functions on G and if $\sum_{n=1}^{\infty} \psi_n$ converges uniformly on compact sets, then the sum is a continuous, negative definite function.

Kazhdan [**6**] introduces the notion that G has property T if the trivial representation is isolated in the dual space of G. A full discussion of the dual space of G and its topologies is given in [**4**] and [**3**].

2. The results. In this section we prove the equivalences (1A) $\Leftrightarrow$ (1B), (2A) $\Leftrightarrow$ (2B) and (3A) $\Leftrightarrow$ (3B) described in § 0 as well as some related results. Before we do so, however, let us look at some examples. There are a number of groups which have property T [**6**] and hence (Theorem 3) have no unbounded negative definite functions at all; $SL(3, \mathbf{R})$ is an example. Given such a group, we can cross it with the integers $\mathbf{Z}$ and get a group which does not have property T (see Proposition 8). However, $SL(3, \mathbf{R}) \times \mathbf{Z}$ does not satisfy (2B) since the restriction of any continuous, negative definite function to the non-compact set $SL(3, \mathbf{R}) \times \{0\}$ must be bounded.

Any amenable group satisfies (3A) since the Fourier algebra, $A(G)$, contains an approximate unit of positive definite functions in this case. Any free group satisfies (3B) (see [**7**]) so there are non-amenable groups in this category as well.

LEMMA 1. *If A is a C^*-algebra, p is an open projection in A^{**} and $\{f_\alpha\}$*

is a net of states of A with $\lim f_\alpha = f$ *weak* and* $f(p) = 1$, *then* $f_\alpha(p) \to 1$, $\|pf_\alpha p - f_\alpha\| \to 0$ *and* $pf_\alpha p \to f$ *weak*.*

Proof. Since p is lower semi-continuous on $S_1(A)$ [**9**, Section 3.11], $f(p) = p(f) = 1$ and $p(f_\alpha) = f_\alpha(p) \leqq 1$, it follows that $f_\alpha(p) \to 1$. For any $b \in A$ with $\|b\| \leqq 1$,

$$\begin{aligned}&|pf_\alpha p(b) - f_\alpha(b)|\\ &= |f_\alpha((1-p)bp) + f_\alpha(pb(1-p)) + f_\alpha((1-p)b(1-p))|\\ &\leqq 3|f_\alpha(1-p)|^{1/2},\end{aligned}$$

so

$$\|pf_\alpha p - f_\alpha\| \to 0.$$

This implies $pf_\alpha p \to f$ weak* as well.

LEMMA 2. *The following are equivalent.*

(a) *G has property T.*

(b) *z_0 lies in $C^*(G)$, not merely in $C^*(G)^{**}$.*

(c) *For any net $\{g_\alpha\} \subset P(G)_1$ with $g_\alpha \to f_0$ weak* we have*

$$\|g_\alpha - f_0\| \to 0.$$

(d) *For any net $\{g_\alpha\} \subset P(G)_1$ with $g_\alpha \to f_0$ weak* we have*

$$\|g_\alpha - f_0\|_\infty \to 0.$$

Proof. If G has property T, then by [**4**, Section 3.2], $C^*(G) = I \oplus J$, where I and J are closed, 2-sided ideals and $C^*(G) \to C^*(G)/I \cong J$ extends the trivial representation of G. Thus J has a unit p which is clearly the central projection z_0 supporting the trivial representation of G. Thus (b) follows.

Assuming (b), the projection z_0 is in $C^*(G)$ (and hence is open); so in the notation of (c), $g_\alpha(z_0) \to f_0(z_0) = 1$. Thus

$$z_0 g_\alpha \to z_0 f_0 = f_0$$

in norm and by Lemma 1,

$$\|z_0 g_\alpha - g_\alpha\| \to 0,$$

hence (c) follows.

That (c) $\Rightarrow$ (d) is trivial.

Assuming (a) is false and using the notation of Lemma 1 of [**3**], we get a net $\{g_\alpha\}$ of normalized positive definite functions from representations orthogonal to the trivial representations such that $g_\alpha \to f_0$ uniformly on compact sets. However, using Lemma 2 of [**3**], if $\|g_\alpha - f_0\|_\infty \to 0$, then the representation from which g_α is taken must eventually contain the trivial representation, a contradiction.

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THEOREM 3. (1A) ⇔ (1B).

Proof. First suppose that G has an unbounded, continuous, negative definite function ψ. We can assume $\psi(e) = 0$ and $\psi(x) \geqq 0$ for all $x \in G$. By Schoenberg's Theorem [**2**, Theorem 7.8] the functions $g_t = e^{-t\psi}$ for $t \geqq 0$ are continuous, positive definite and satisfy $g_t(e) = 1$. Clearly

$$\lim_{t\to 0} g_t(x) = 1$$

uniformly on any compact subset K of G since ψ is bounded on K, so $\lim_{t\to 0} g_t = f_0$ in the weak topology of $B(G)$. If G had property T, then by Lemma 2 we must have

$$\lim_{t\to 0} g_t(x) = 1$$

uniformly on G. However, for any $t > 0$ there is an $x_t \in G$ such that $\psi(x_t) > t^{-1}$ since ψ is unbounded. Thus

$$g_t(x_t) = e^{-t\psi(x_t)} < e^{-1} < 1$$

for every $t > 0$, so uniform convergence fails. Thus G cannot have property T.

Now suppose G does not have property T. By Lemma 2 let $\{g_\alpha\}$ be a net of continuous, positive definite functions on G with $g_\alpha(e) = 1$, $g_\alpha(x) \to 1$ uniformly on compact subsets of G but $g_\alpha(x) \not\to 1$ uniformly on all of G. By passing to a subnet we can find $\epsilon > 0$ such that for any α there is an $x_\alpha \in G$ with $\mathrm{Re}(g_\alpha(x_\alpha)) < (1 - \epsilon)$. Since G is σ-compact, there is an increasing sequence $K_1 \subset K_2 \ldots$ of compact neighborhoods of e with $G = \cup_{k=1}^{\infty} K_n$. By a simple induction we can get a sequence $\{\varphi_n\} \subset \{g_\alpha\}$ and $\{x_n\} \subset G$ such that $\mathrm{Re}(\varphi_n(x_n)) < 1 - \epsilon$ and $\mathrm{Re}(1 - \varphi_n(x)) < 4^{-n}$ for all $x \in K_n$ and for all n. Set

$$\psi(x) = \sum_{k=1}^{\infty} 2^k \mathrm{Re}\,(1 - \varphi_k(x)),$$

and note that the series converges uniformly on each compact neighborhood K_n (and hence uniformly on any compact subset K of G since $K \subset K_n$ for large n) because

$$|2^k \mathrm{Re}(1 - \varphi_k(x))| < 2^{-k}$$

for $k > n$ and all $x \in K$. Thus ψ is a continuous, negative definite function which is unbounded since

$$\psi(x_n) \geqq 2^n \mathrm{Re}(1 - \varphi_n(x_n)) > 2^n \epsilon.$$

COROLLARY 4 ([**6**, **12**]). *If G has property* T, *then G is compactly generated.*

Proof. Since G is σ-compact, there is an increasing sequence $U_1 \subset U_2 \subset \ldots$ of open neighborhoods of e with compact closures such that

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$G = \bigcup_{n=1}^{\infty} U_n$. Let G_n be the subgroup generated by U_n. The characteristic function χ_n of G_n is continuous (since G_n is open) and positive definite. However, $\chi_n(x) \to 1$ for all $x \in G$ but convergence is not uniform unless $G_n = G$ for all $n \geqq n_0$. Thus G is compactly generated by Lemma 2(d).

PROPOSITION 5. *If a subset K of G satisfies $K^m = G$ for some $m > 0$ and no continuous negative definite function on G is unbounded on K, then G has property* T.

Proof. If ψ were an unbounded, continuous negative definite function on G then so is

$$\theta = [\mathrm{Re}(\psi - \psi(e))^{1/2}]^{1/2}$$

by [**2**, Proposition 7.15] and

$$0 \leqq \theta(xy) \leqq \theta(x) + \theta(y)$$

for all $x, y \in G$. By induction we get

$$\theta(x) \leqq m \sup \{\theta(y) : y \in K\}$$

for all $x \in G$ since $K^m = G$. This contradicts the unboundedness of θ.

COROLLARY 6 (see [**3**, Theorem 4]). *If G contains closed subgroups $G_1, \ldots, G_n$, each having property* T *and satisfying $K^m = G$ for $K = \bigcup_{i=1}^{n} G_i$ and some integer m, then G has property* T.

The next result was first proved in [**12**, p. 25].

COROLLARY 7. *If $\Gamma \subset G$ is a closed subgroup with property* T *such that G/Γ, the topological quotient space, is compact, then G has property* T.

Proof. Let $\pi: G \to G/\Gamma$ be the quotient map. Then π is an open map so $\{\pi(U) : U$ is an open neighborhood of e in G with compact closure $\bar{U}\}$ is an open covering of G/Γ, so there is a finite subcover $\pi(U_1), \ldots, \pi(U_n)$. Set

$$K = \Gamma \cup \bigcup_{i=1}^{n} \bar{U}_i.$$

Clearly $K^2 = G$. If ψ is a continuous, negative definite function on G, then ψ is bounded on Γ (since Γ has property T) and ψ is bounded on $\bigcup_{i=1}^{n} \bar{U}_i$ by compactness. Thus ψ is bounded on K, hence bounded on G by Proposition 5. Theorem 3 now implies that G has property T.

Remark. Calvin Moore has mentioned to us a counter-example to a converse of Corollary 7, namely $G = SL(3, \mathbf{R})$, $\Gamma =$ all matrices in $SL(3, \mathbf{R})$ which have only zeros above the main diagonal. Then G has property T and G/Γ is compact, yet Γ does not have property T. This is also mentioned in [**12**]. Also see [**12**, p. 26] for the converse of [**3**, Theorem 3].

PROPOSITION 8 (see [**6**]). *If G_1 and G_2 are locally compact, σ-compact groups, then $G_1 \times G_2$ has property* T *if and only if both G_1 and G_2 have property* T.

Proof. If G_1 and G_2 have property T, so does $G_1 \times G_2$ by Corollary 6. For the converse, if (say) G_1 does not have property T, we can apply (d) of Lemma 2 to get a net $\{(g_\alpha)_1\}$ of continuous positive definite functions on G_1 which converges to 1 uniformly on compacta but not uniformly on G_1. Defining g_α on $G_1 \times G_2$ by

$$g_\alpha(x, y) = (g_\alpha)_1(x)$$

we get a similar sequence for $G_1 \times G_2$, contradicting Lemma 2.

THEOREM 9. (2A) $\Leftrightarrow$ (2B).

Proof. Assume (2A). Let $Q \subset G$ be non-compact and let $G = \bigcup_{n=1}^{\infty} K_n$ be an increasing union of compact neighborhoods of e in G (since G is σ-compact). By (2A) there exist pure positive definite functions $\{f_n\}$ on G such that

$$\mathrm{Re}(1 - f_n(x)) < 4^{-n} \quad \text{for all } x \in K_n$$

and there exist $x_n \in Q$ with

$$\mathrm{Re}(f_n(x_n)) < \tfrac{1}{2}.$$

Set $\psi_n = \mathrm{Re}(1 - f_n)$. Each ψ_n is negative definite and $\sum_{n=1}^{\infty} \psi_n(x)2^n$ converges uniformly on each K_n, so $\psi = \sum_{n=1}^{\infty} \psi_n$ is continuous, negative definite and unbounded on Q since

$$\psi(x_n) \geqq \psi_n(x_n)2^n \geqq 2^{n-1}.$$

Now assume (2B) and suppose that G does not have the (strong) dual R-L property. Thus there is an $\epsilon > 0$, an open projection $p \geqq z_0$ and a sequence $\{x_n\} \subset G$ with $x_n \in G\backslash K_n$ and $\mathrm{Re}(f(x_n)) > \epsilon$ for every pure positive definite function f with $f(p) = 1$. Since p is open, the discussion in § 1 implies that for any state g with $g(p) = 1$ we have $\mathrm{Re}(g(x_n)) \geqq \epsilon$ for every $n = 1, 2, \ldots$. Let $Q = \{x_n\}_{n=1}^{\infty}$, a non-compact set, and let ψ be a continuous negative definite function on G which is unbounded on Q. We can assume ψ is real valued with $\psi(e) = 0$. For each n choose $t_n > 0$ such that $g_n = e^{-t_n\psi}$ satisfies

$$|g_n(x) - 1| < 2^{-n} \quad \text{for all } x \in K_n.$$

Since $g_n \rightarrow f_0$ weak*, we get $pg_np \rightarrow f_0$ weak* by Lemma 1. Since

$$(pg_np/\|pg_np\|) = h_n$$

is a state with $h_n(p) = 1$, our assumption gives

$$\mathrm{Re}(h_n(x_k)) \geqq \epsilon \quad \text{for every } k \text{ and } n.$$

However, since (using a subsequence if necessary) $\psi(x_k) \to \infty$ as $k \to \infty$, for fixed n, Lemma 1 implies that $h_n(x^k) \to 0$, as $k \to \infty$, a contradiction.

THEOREM 10. (3A) $\Rightarrow$ (3B).

Proof. Assume (3B) and we can assume ψ is real-valued with $\psi(e) = 0$. Then

$$\lim_{t\to 0} e^{-t\psi} = 1$$

uniformly on compact sets while

$$\lim_{x\to\infty} e^{-t\psi(x)} = 0$$

for each t. Thus (3A) holds.

Assume (3A) and let $\{g_\alpha\}$ be a net of positive definite functions in $C_0(G)$ with $g_\alpha(x) \to 1$ uniformly on compact sets. We can assume $g_\alpha(e) = 1$ for all α. Write

$$G = \bigcup_{i=1}^{\infty} K_i$$

where $K_1 \subset K_2 \subset \ldots$ are compact neighborhoods of e in G. Set $K_0 = \{e\}$. Suppose we have chosen $\{g_{\alpha_i}\}_{i=1}^{n}$ and integers $0 = j_0, j_1, \ldots, j_n$ such that for all $i = 1, \ldots, n$;

(a) $|g_{\alpha_i}(x) - 1| < 4^{-i}$ for all $x \in K_{j_{i-1}}$ and

(b) $|g_{\alpha_i}(x)| < 4^{-i}$ for all $x \notin K_{j_i}$.

We can easily chose $g_{\alpha_{n+1}}$ so that

$$|g_{\alpha_{n+1}}(x) - 1| < 4^{-n} \quad \text{for all } x \in K_{j_n}$$

since $g_\alpha \to 1$ uniformly on K_{j_n}; and, since $g_{\alpha_{n+1}}$ vanishes at infinity, we can choose j_{n+1} such that

$$|g_{\alpha_{n+1}}(x)| < 4^{-n} \quad \text{for all } x \notin K_{j_{n+1}}.$$

This proves the induction step. Now set

$$\psi_n = \mathrm{Re}\,(1 - g_{\alpha_n}) \quad \text{and} \quad \psi = \sum_{n=1}^{\infty} \psi_n 2^n.$$

Clearly the series converges uniformly on each K_n, so ψ is negative definite and continuous. Further, for $x \in G\backslash K_{j_n}$

$$\psi(x) > 2^n\psi_n(x) = \mathrm{Re}(1 - g_{\alpha_n}(x))2^n > 2^n(1 - 4^{-n}) > 2^{n-1} \to \infty \quad \text{as } n \to \infty.$$

Thus (3B) holds.

Remark. These results leave open the question, "Does (2B) imply (3B)?"

Remark. If G_1 and G_2 are non-trivial discrete groups, their free product $G_1{*}G_2$ always satisfies (1B) and hence cannot have property T (see [**7**]). In fact, it is shown in [7] that if each free factor satisfies (2B) (resp. (3B)), then the free product satisfies (2B) (resp. (3B)).

The last theorem relates inner amenability to property T. The proof given here was discovered during a discussion with Alain Connes who had originally found a different proof.

THEOREM 11. *A countably infinite discrete* ICC *group with property* T *is not inner amenable.*

Proof. Using the notation of [**8**] we need only show that $P_\delta \in C^*(U_G)$. If φ: $C^*(G) \to C^*(U_G)$ is the representation determined by $\varphi(x) = U_x$ for all $x \in G$, then we claim $P_\delta = \varphi(z_0)$.

Clearly $\varphi(z_0)\delta = \delta$ since $\varphi(x)\delta = \delta$ for all $x \in G$. Suppose $\varphi(z_0)\eta = \eta$ for some unit vector $\eta \in L^2(G)$ which is perpendicular to δ. Then

$$\varphi(x)\eta = \varphi(x)\varphi(z_0)\eta = \varphi(xz_0)\eta = \varphi(z_0)\eta = \eta,$$

so η will be invariant under $C^*(U_G)$. Write

$$\eta = \sum_{x\in G}\lambda_x x \quad \text{with} \quad \sum_{x\in G} |\lambda_x|^2 < \infty$$

and assume $\lambda_{x_0} \neq 0$ for some $x_0 \neq e$. Then for each $y \in G$,

$$\eta = U_y\eta = \sum_{x\in G} \lambda_x yxy^{-1} = \sum_{x\in G}\lambda_x x.$$

Thus $\lambda_{y^{-1}x_0y} = \lambda_{x_0}$ for each $y \in G$. Since

$$\lambda_{x_0} \neq 0 \quad \text{and} \quad \sum_{x\in G} |\lambda_x|^2 < \infty,$$

this contradicts the assumption that G is an ICC group, i.e., that $\{yx_0y^{-1} = y \in G\}$ is infinite. Thus $\varphi(z_0) = P_\delta$, so G is not inner amenable.

REFERENCES

1. C. A. Akemann and M. E. Walter, *The Riemann-Lebesgue property for arbitrary locally compact groups*, Duke Math. J. *43* (1976), 225–236.

2. C. Berg and G. Forst, *Potential theory on locally compact abelian groups*, Ergebnine der Mathematik unde ihrer Grenzgebiete *87* (Springer-Verlag, Berlin, 1975).

3. C. Delaroche and A. Kirillov, *Sur les relations entre l'espace dual d'un groupe et la structure de ses sous-groupes fermés*, Seminaire Bourbaki no. *343* (1967/68).

4. J. Dixmier, *C*-algebras* (North-Holland, 1977).

5. U. Haagerup, *An example of a non-nuclear C*-algebra, which has the metric approximation property* (with addendum), Invent. Math. *50* (1978–79) 279–293.

6. D. A. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, Functional Analysis and its Applications *1* (1967), 63–65.

7. T-Y Lee, Ph.D. Thesis, University of California at Santa Barbara (1979).
8. W. Paschke, *Inner amenability and conjugation operators*, Proc. Amer. Math. Soc. *71* (1978), 117–118.
9. G. K. Pedersen, *C*-algebras and their automorphism groups* (Academic Press, 1979).
10. S. Sakai, *C*-algebras and W*-algebras* (Springer-Verlag, Berlin, 1971).
11. M. E. Walter, *Differentiation on the dual of a group, An introduction*, submitted.
12. S. P. Wang, *On isolated points in the dual spaces of locally compact groups*, Math. Ann. *218* (1975), 19–34.

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