ELEMENTARY EQUIVALENCE AND THE COMMUTATOR SUBGROUP

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(Received 23 September, 1980)

If G and H are elementarily equivalent groups (that is, no elementary statement of group theory distinguishes between G and H) then the definable subgroups of G are elementarily equivalent to the corresponding ones of H. But the commutator subgroup G' of G, consisting of all products of commutators $[a, b] = a^{-1}b^{-1}ab$ of elements a and b of G, may not be definable. Must G' and H' be elementarily equivalent?

A first thought might be that if G' is not definable, even allowing parameters, then it will follow that there is $H \equiv G$ with $H' \not\equiv G'$. However, this naive idea fails.

EXAMPLE. Let G be the direct product of a countably-infinite free-nilpotent-of-class-2-and-exponent-3 group with a countably-infinite group of exponent 3. If $H \equiv G$, then H' and G' are infinite abelian groups of exponent 3 and hence $H' \equiv G'$. But G' is not definable (by an obvious modification of the lemma below).

Despite this example, we will show the following theorem.

THEOREM. *There exists groups G and H such that $G \equiv H$ (in fact $G \prec H$) but $G' \not\equiv H'$.*

If $G \equiv H$, then no quantifier-free sentence will distinguish between G' and H', and the same is true for an $\forall$-sentence (and hence an $\exists$-sentence). About the simplest sentence to consider next would be something of the form $\forall x\, \exists y(x = y^2)$, so our construction begins with a group F of nilpotent class 2 and reduces modulo a normal subgroup that makes each commutator a square, but there is no uniform bound on the length of the commutator word being squared. An ultrapower of the resulting group G produces H with some non-squares in its commutator subgroup. Now for the details. We will say that G is a *nil*-2 *group* if it is a group satisfying $\forall x\, \forall y\, \forall z[x, y, z] = 1$, where $[x, y, z] = [[x, y], z]$.

LEMMA. *There is a nil-2 group G with the following pair of properties:*

(a) *every element of G' is the square of an element of G',*

(b) *for every $n < \omega$, there exist g_n and h_n in G such that $[g_n, h_n]$ is not the square of any product of at most n commutators.*

Proof of the lemma. Our group G will be F/K where F is the free nil-2 group on $\{a_n : n < \omega\}$ and K will be a central subgroup generated by relations R_n $(n < \omega)$. For any word w in F', let $l(w)$ be the minimum number of commutators required to witness that w is in F' and note that since F is free nil-2 on $\{a_n : n < \omega\}$ then $[a, bc] = [a, b][a, c]$ and F' is

[1] The authors would like to thank Bowling Green State University, Talisker Malt, and the Central States Universities, Inc., for making these results possible. Research was supported in part by a number of granting agencies.

Glasgow Math. J. **23** (1982) 115–117.

a free abelian group with basis $\{[a_i, a_j]: i<j<\omega\}$ (see [1]). Hence for each $m<\omega$ there exists w in F' (for example $w=[a_0, a_1][a_2, a_3]\ldots[a_{2m}, a_{2m+1}]$) such that $l(w^t)\geq m$ for all integers $t\neq 0$. For, suppose $([a_0, a_1]\ldots[a_{2m}, a_{2m+1}])^t$ were the product of fewer than m commutators for some $t\neq 0$. As F' is free abelian on $\{[a_p, a_q]: p<q<\omega\}$, we have for some integers $x(i, j)$ and $y(i, j)$

$$\prod_{0\leq p\leq m}[a_{2p}, a_{2p+1}]^t = \prod_{0\leq j<m-1}\left[\prod_{i\leq 2m+1} a_i^{x(i,j)}, \prod_{i\leq 2m+1} a_i^{y(i,j)}\right]$$

$$= \prod_{0\leq p<q\leq 2m+1}[a_p, a_q]^{v(p,q)},$$

where

$$v(p, q)=\sum_{0\leq j<m-1}\{x(p, j)y(q, j)-x(q, j)y(p, j)\}.$$

Hence the number

$$v(p, q)=\sum_{0\leq j<m-1}\{x(p, j)y(q, j)-x(q, j)y(p, j)\}$$

equals t if p is even and $p+1=q$ and is 0 otherwise. If we let

$$A=\begin{bmatrix} x(0,0) & -y(0,0) & \ldots & x(0,m-2) & -y(0,m-2)\\ \cdot & \cdot & \cdots & \cdot & \cdot \\ x(2m+1,0) & -y(2m+1,0) & \ldots & x(2m+1,m-2) & -y(2m+1,m-2)\end{bmatrix}_{(2m+2)\times(2m-2)}$$

and

$$B=\begin{bmatrix} y(1,0) & y(0,0) & \ldots & y(2m+1,0) & y(2m,0)\\ x(1,0) & x(0,0) & \ldots & x(2m+1,0) & x(2m,0)\\ \cdot & \cdot & \cdots & \cdot & \cdot \\ y(1,m-2) & y(0,m-2) & \ldots & y(2m+1,m-2) & y(2m,m-2)\\ x(1,m-2) & x(0,m-2) & \ldots & x(2m+1,m-2) & x(2m,m-2)\end{bmatrix}_{(2m-2)\times(2m+2)}$$

then $A\cdot(1/t)B=J_{2m+2}$, where J_{2m+2} is a diagonal matrix with pp entry $(-1)^p$. Hence

$$\text{column rank }(A)=\text{row rank }(A)=2m+2.$$

Since A has only $2m-2$ columns, we have the desired contradiction. (This simple argument is due to Pat Rogers—see the proof of Theorem 3.10 in [2].)

We are now ready to define the relations R_n, and words w_n, inductively. To ease our notation, let i and j denote the standard enumerations of ω such that $i(n)<j(n)\leq n+1$ for all $n<\omega$ and $\{(i(n), j(n)): n<\omega\}=\{(r, s): r<s<\omega\}$. Define $w_0=[a_2, a_3]$ and $R_0=[a_0, a_1]^{-1}w_0^2$; and for each $n<\omega$, pick w_n such that $l(w_n^t)>2+n+\sum_{m<n} l(R_m)$ for all integers $t\neq 0$—possible by the discussion above—and put $R_n=[a_{i(n)}, a_{j(n)}]^{-1}w_n^2$. As stated before, we let K be the subgroup generated by $\{R_n: n<\omega\}$ and put $G=F/K$.

Property (a) of the lemma now follows from the definition of G. (Since G is nil-2, it sufficed to arrange for the generators of G' to be squares.) To prove (b) choose $r<s$ such

that $[a_r, a_s]$ does not appear in R_m for $m<n$ and suppose $[a_r, a_s]$ were the square of some w, a product of n commutators (modulo K). Then in F, $[a_r, a_s]^{-1}w^2 = R_0^{k_0}R_1^{k_1}\dots R_t^{k_t}$ for some integers $k_0, k_1, \dots, k_t$, where $k_t \neq 0$. By the choice of r and s, $t \geq n$. Recall that $R_t = [a_{i(t)}, a_{j(t)}]^{-1}w_t^2$; so $l(w_t^{2k_t}) \leq 2+n+\sum_{m<t} l(R_m)$ which contradicts the choice of w_t.

Proof of the theorem. Let G be as in the lemma; if σ denotes $\forall x\, \exists y(x=y^2)$ then $G' \models \sigma$. To construct H (with $H' \models \neg\sigma$) either take a non-trivial ultrapower of G, or use a simple compactness argument on

$$T = \mathrm{Th}(G) \cup \{\text{“}[c, d] \text{ is not the square of the product of } n \text{ commutators”} : n<\omega\},$$

where c and d are constants added to the language of groups. T is consistent since G can be interpreted as satisfying any finite subset of T, and H can be any model of T.

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