BULL. AUSTRAL. MATH. SOC. MOS 0635
VOL. 1 (1969), 319-324.

# Sasaki projections on orthocomplemented posets

**P. D. Finch**

It is proved that an orthocomplemented poset $P$ is an orthomodular lattice if and only if it admits a suitable defined set of order preserving maps. These maps are called projections. They are, in fact, just the projections of the Baer *-semigroup associated with the orthomodular lattice.

## 1. Introduction

Foulis [1] has shown that orthomodular lattices arise whenever one has a Baer *-semigroup and conversely, that every orthomodular lattice can be coordinatised by a Baer *-semigroup. In [2] Foulis showed further that a necessary and sufficient condition that an involution poset be an orthomodular lattice is that its associated involution semigroup be a Baer *-semigroup. This result is of interest in the axiomatic development of non-relativistic quantum mechanics as in, for example, Mackey [3], where one wishes to determine conditions under which a quantum logic which has the structure of an orthomodular poset is, in fact, an orthomodular lattice.

A Baer *-semigroup is an involution semigroup in which the right annihilator of each element is a principal right ideal generated by a projection. If $L$ is an orthomodular lattice, and $S(L)$ is its associated Baer *-semigroup, then the projections in $S(L)$ are just the Sasaki projections introduced by Sasaki in [4]; these are the maps $\pi_x : L \to L$ defined for each $x$ in $L$ by the equation

(1) $$y\pi_x = (y \vee x^{\perp}) \wedge x , \qquad y \in L .$$

Received 5 May 1969.



https://doi.org/10.1017/S0004972700042192 Published online by Cambridge University Press

It is not difficult to show that for any $x$ and any $y$ in $L$ one has

(2) $$(y^{\perp}\pi_x)^{\perp}\pi_x = x \wedge y \ .$$

This equation suggests that if one could define an analogue of the Sasaki projections in an orthocomplemented poset, without the use of lattice operations, then one could establish the lattice structure directly by verifying that $(y^{\perp}\pi_x)^{\perp}\pi_x$ is the g.l.b. of $x$ and $y$. This idea motivates the discussion of the next section. The advantage of the procedure developed there is that one does not work with the whole Baer $*$-semigroup but only with its projections. It then becomes apparent that the property which ensures a lattice structure is not primarily the characteristic Baer property, namely that the right annihilator of each element in the semigroup is a principal right ideal generated by a projection, but simply the existence, on the orthocomplemented poset, of suitably defined projections. In all fairness it should be stated that this result is not very far below the surface in Foulis [1] and [2].

## 2. Sasaki sets of projections

Let $P$ be an orthocomplemented poset with least element $0$, greatest element $1$ and orthocomplementation $x \to x^{\perp}$. A set $S$ of maps $P \to P$ will be said to be a Sasaki set of projections on $P$ when

(i) for each $\pi$ in $S$, $x \leqq y$ implies $x\pi \leqq y\pi$,

(ii) if $\pi$, $\pi'$ are in $S$ then $1\pi \leqq 1\pi'$ implies $\pi'\pi = \pi$,

(iii) for each $\pi$ in $S$, $(x\pi)^{\perp}\pi \leqq x^{\perp}$ for all $x$ in $P$.

Note that (iii) is just the Foulis condition that $\pi$ be a self-adjoint element of the associated involution semigroup. We remark that a Sasaki set of projections on $P$ always exists; indeed define maps $\pi_0$, $\pi_1 : P \to P$ by writing $x\pi_0 = 0$ and $x\pi_1 = x$ for each $x$ in $P$, then the set $\{\pi_0, \pi_1\}$ is a Sasaki set of projections on $P$.

The following lemma lists some immediate consequences of the definition of a Sasaki set of projections. The results stated in it are used in the proof of the theorem below.

https://doi.org/10.1017/S0004972700042192 Published online by Cambridge University Press

**LEMMA.** *Let* $S$ *be a* Sasaki *set of projections on* $P$ *, then for* $\pi$ , $\pi'$ *in* $S$

*(a)* $\pi^2 = \pi$ ,

*(b)* $1\pi = 1\pi'$ *implies* $\pi = \pi'$ ,

*(c)* $1\pi \leqq 1\pi'$ *implies* $\pi\pi' = \pi = \pi'\pi$ ,

*(d)* $1\pi \leqq 1\pi'$ *implies* $(1\pi)\pi' = 1\pi$ ,

*(e)* $x\pi = 0$ *if and only if* $x \leqq (1\pi)^{\perp}$ .

Proof. *(a)* and *(b)* are immediate consequences of *(c)* and (ii). To prove *(c)* we use (ii), make repeated use of (iii) and follow through the following sequence of implications.

$$\begin{aligned} 1\pi \leqq 1\pi' &\Rightarrow \pi'\pi = \pi \\ &\Rightarrow (x\pi\pi')^{\perp}\pi = (x\pi\pi')^{\perp}\pi'\pi \leqq (x\pi)^{\perp}\pi \leqq x^{\perp} \\ &\Rightarrow x \leqq \left\{(x\pi\pi')^{\perp}\pi\right\}^{\perp} \\ &\Rightarrow x\pi \leqq x\pi\pi' . \end{aligned}$$

On the other hand

$$\begin{aligned} 1\pi \leqq 1\pi' &\Rightarrow \pi'\pi = \pi \\ &\Rightarrow (x\pi)^{\perp}\pi\pi' = (x\pi'\pi)^{\perp}\pi\pi' \leqq (x\pi')^{\perp}\pi' \leqq x^{\perp} \\ &\Rightarrow \left\{(x\pi)^{\perp}\pi\right\}^{\perp}\pi\pi' \leqq x\pi \\ &\Rightarrow x\pi\pi' \leqq \left\{(x\pi)^{\perp}\pi\right\}^{\perp}\pi\pi' \leqq x\pi . \end{aligned}$$

This proves *(c)*, and *(d)* is now obvious. Finally to prove *(e)* we use (i) and (iii), thus

$$x\pi = 0 \Rightarrow 1\pi \leqq x^{\perp} \Rightarrow x \leqq (1\pi)^{\perp}$$

and

$$x \leqq (1\pi)^{\perp} \Rightarrow x\pi \leqq (1\pi)^{\perp}\pi \leqq 0 .$$

We say that a Sasaki set $S$ of projections on $P$ is full when to each $x$ in $P$ there corresponds at least one (and hence, because of *(b)*, exactly one) element $\pi_x$ in $S$ with $1\pi_x = x$ . We prove now the

**THEOREM.** *An orthocomplemented poset is an orthomodular lattice if (and*

https://doi.org/10.1017/S0004972700042192 Published online by Cambridge University Press

*only if) it admits a full* Sasaki *set of projections.*

Proof. Suppose that $S$ is a full Sasaki set of projections on the orthocomplemented poset $P$. Let $x$ and $y$ be in $P$. Then

$$(y^{\perp}\pi_x)^{\perp}\pi_x \leqq 1\pi_x = x$$

and, by (iii),

$$(y^{\perp}\pi_x)^{\perp}\pi_x \leqq y \ .$$

Thus $(y^{\perp}\pi_x)^{\perp}\pi_x$ is a lower bound to $x$ and $y$; we show that it is their greatest lower bound. To do this let $t$ in $P$ be any lower bound of $x$ and $y$. Since $t \leqq x$ we have $\pi_x\pi_t = \pi_t$ and so, using $y^{\perp} \leqq t^{\perp} = (1\pi_t)^{\perp}$ and *(e)* above, we obtain

$$(y^{\perp}\pi_x)\pi_t = y^{\perp}\pi_t = 0 \ .$$

Using *(e)* again we deduce that $t \leqq (y^{\perp}\pi_x)^{\perp}$. Since $1\pi_t = t \leqq x = 1\pi_x$ we have $t\pi_x = t$ by *(d)*, and so

$$t = t\pi_x \leqq (y^{\perp}\pi_x)^{\perp}\pi_x \ .$$

This establishes that $x$ and $y$ do have a greatest lower bound in $P$ and, in fact, that

$$x \wedge y = (y^{\perp}\pi_x)^{\perp}\pi_x \ .$$

Since $x \vee y = (x^{\perp} \wedge y^{\perp})^{\perp}$ it is now immediate that $P$ is a lattice and it is trivial that it is a lattice orthocomplemented by $x \to x^{\perp}$. To establish that $P$ is orthomodular we prove that

$$x \wedge y = 0 \quad \text{and} \quad x \geqq y^{\perp} \Rightarrow x = y^{\perp}$$

This implication is one of the several well-known characterisations of orthomodularity. To establish it we use *(d)* to observe that

https://doi.org/10.1017/S0004972700042192 Published online by Cambridge University Press

$$x \geqq y^\perp \Rightarrow y^\perp\pi_x = y^\perp ,$$
$$\Rightarrow (y^\perp\pi_x)^\perp\pi_x = y\pi_x ,$$
$$\Rightarrow x \wedge y = y\pi_x .$$

Thus

$$x \wedge y = 0 \ \ \& \ \ x \geqq y^\perp \Rightarrow y\pi_x = 0$$
$$\Rightarrow x \leqq y^\perp$$

and this gives the desired conclusion.

Conversely if $L$ is an orthomodular lattice and, for each $x$ in $L$, $\pi_x$ is the Sasaki projection on $L$ given by equation (1), then $\{\pi_x : x \in L\}$ is a full Sasaki set of projections on $L$.

REMARK. The use of the term Sasaki set of projections is justified by the fact that under the hypothesis of the theorem one has equation (1) for each $x$ in $P$. To prove this we observe firstly that, under the hypothesis of the theorem,

(3) $$(x \vee y)\pi = x\pi \vee y\pi$$

for each $\pi$ in $S$ and any $x$, $y$ in $P$. To see this we remark that since $\pi$ is order preserving it is only necessary to show that $(x \vee y)\pi \leqq x\pi \vee y\pi$. To do so write $z = x\pi \vee y\pi$. Then $z^\perp \leqq (x\pi)^\perp$ and so $z^\perp\pi \leqq (x\pi)^\perp\pi \leqq x^\perp$. Thus $x \leqq (z^\perp\pi)^\perp$ and similarly $y \leqq (z^\perp\pi)^\perp$. It follows that

$$(x \vee y)\pi \leqq (z^\perp\pi)^\perp\pi \leqq z = x\pi \vee y\pi .$$

Using (3) we show easily that (1) holds. Thus

$$\begin{aligned}(y \vee x^\perp) \wedge x &= \left\{(y \vee x^\perp)^\perp\pi_x\right\}^\perp\pi_x \\ &= \left\{(y^\perp \wedge x)\pi_x\right\}^\perp\pi_x \\ &= (y^\perp \wedge x)^\perp\pi_x \\ &= (y \vee x^\perp)\pi_x \\ &= y\pi_x ,\end{aligned}$$

https://doi.org/10.1017/S0004972700042192 Published online by Cambridge University Press

since $x^{\perp}\pi_{x} = 0$ .

## References

[1] David J. Foulis, "Baer *-semigroups", *Proc. Amer. Math. Soc.* 11 (1960), 648-654.

[2] David J. Foulis, "A note on orthomodular lattices", *Portugal. Math.* 21 (1962), 65-72.

[3] George W. Mackey, *Mathematical foundations of quantum mechanics*, (W.J. Benjamin, Inc., New York, 1963).

[4] Usa Sasaki, "On orthocomplemented lattices satisfying the exchange axiom", *J. Sci. Hiroshima Univ., Ser. A* 17 (1954), 293-302.

Monash University,
Clayton, Victoria.

https://doi.org/10.1017/S0004972700042192 Published online by Cambridge University Press