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FAMILIES OF PARTIAL REPRESENTING SETS

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Abstract

Assume GCH. Let $\kappa, \lambda, \mu, \Sigma$ be cardinals, with κ infinite. Let $\mathcal{Q}$ be a family consisting of λ pairwise almost disjoint subsets of Σ each of size κ, whose union is Σ. In this note it is shown that for each μ with $1 \leqslant \mu \leqslant \min(\lambda, \Sigma)$, there is a "large" almost disjoint family $\mathcal{T}$ of μ-sized subsets of Σ, each member of $\mathcal{T}$ having non-empty intersection with at least μ members of the family $\mathcal{Q}$.

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1. Introduction

If λ and κ are cardinals, a (λ, κ) *family* is an indexed family $(S_i; i \in I)$ of sets where $|I| = \lambda$ and $|S_i| = \kappa$ for each i in I.

A family $\mathcal{X}$ of sets is said to be *almost disjoint* if $|X \cap X'| < \min(|X|, |X'|)$ for all pairs X, X' of elements of $\mathcal{X}$. The *degree of disjunction*, $\delta(\mathcal{X})$, of the family $\mathcal{X}$ is the least cardinal θ such that $|X \cap X'| < \theta$ for all pairs X, X' of elements of $\mathcal{X}$. A set T is called a *representing set* of $\mathcal{X}$ if $T \subseteq \bigcup \mathcal{X}$ and $T \cap X \neq \varnothing$ for each X in $\mathcal{X}$.

Suppose κ is an infinite cardinal and $\mathcal{Q}$ is an almost disjoint family of κ-sized sets. In Balanda [1] it was shown (assuming GCH) that $\mathcal{Q}$ need not possess an almost disjoint pair of representing sets if $|\mathcal{Q}| > \kappa$. Almost disjoint families of representing sets are studied further in Balanda [2]. This paper is concerned with families of sets, each of which is a representing set of some fixed sized subfamily of $\mathcal{Q}$.

The Generalized Continuum Hypothesis (GCH) is assumed throughout the general discussion.

Suppose κ is an infinite cardinal and $\mathcal{Q}$ is an almost disjoint (λ, κ) decomposition of the cardinal Σ. If $1 \leqslant \mu \leqslant \min(\lambda, \Sigma)$ then a μ-sized representing set of some μ-sized subfamily of $\mathcal{Q}$ is called a *μ-partial representing set* of $\mathcal{Q}$. We are interested in the 'maximum' cardinality of an almost disjoint family of μ-partial representing sets of $\mathcal{Q}$. The following definition is useful.

DEFINITION. Suppose θ, μ are cardinals with $1 \leqslant \theta \leqslant \mu \leqslant \min(\lambda, \Sigma)$. Let $RS_\theta(\mu, \mathcal{Q}) = \sup\{|\mathcal{T}|;\ \mathcal{T}$ is a family of μ-partial representing sets of $\mathcal{Q}$ and $\delta(\mathcal{T}) \leqslant \theta\}$. $RS_\mu(\mu, \mathcal{Q})$ is often written $RS(\mu, \mathcal{Q})$.

Our aim is to establish the following theorem.

THEOREM. (GCH). *Suppose $\mu, \lambda, \kappa, \Sigma$ are cardinals with κ infinite, $\kappa \leqslant \Sigma$ and $1 \leqslant \mu \leqslant \min(\lambda, \Sigma)$. Let $\mathcal{Q}$ be an almost disjoint (λ, κ) decomposition of Σ.*

(i) *If $\theta < \mu$ or if $\mu' \neq \Sigma'$, then $RS_\theta(\mu, \mathcal{Q}) = \Sigma$*

(ii) *If $\mu' = \Sigma'$ then $RS(\mu, \mathcal{Q}) = \Sigma^+$.*

Moreover, the supremum in the definition of $RS_\theta(\mu, \mathcal{Q})$ is a maximum and not a strict supremum.

This theorem is proved in Section 2 in a series of propositions. The cardinal $RS_\theta(\mu, \mathcal{Q})$ is 'as large as possible' in the following sense. Suppose $1 \leqslant \theta \leqslant \mu \leqslant \Sigma$ and Σ is infinite, and let

$$S_\theta(\mu, \Sigma) = \sup\left\{|\mathcal{X}|;\ \mathcal{X} \subseteq [\Sigma]^\mu \text{ and } \delta(\mathcal{X}) \leqslant \theta\right\}.$$

It follows from Baumgartner [3] that $RS_\theta(\mu, \mathcal{Q}) = S_\theta(\mu, \Sigma)$ always, and hence that $RS_\theta(\mu, \mathcal{Q})$ is as large as possible.

Our set notation is standard. An ordinal is identified with the set of its predecessors and cardinals are identified with initial ordinals. We use $\alpha, \beta, \gamma, \delta, \ldots$ to denote ordinals and $\lambda, \kappa, \Sigma, \mu, \theta, \ldots$ to denote cardinals. The cardinal κ will always be infinite. If λ and κ are cardinals, a (λ, κ) family is an indexed family $(S_i, i \in I)$ of sets where $|I| = \lambda$ and $|S_i| = \kappa$ for each i in I. The symbol $[S]^\mu$ denotes $\{S';\ S' \subseteq S \text{ and } |S'| = \mu\}$. The *cofinality* λ' of a non-zero cardinal λ is the least cardinal μ such that λ can be expressed as the sum of μ cardinals all less than λ. We say λ is *regular* if $\lambda' = \lambda$; otherwise λ is *singular* in which case $\lambda' < \lambda$. A *λ-sequence* is a sequence $\langle \lambda_\sigma;\ \sigma < \lambda' \rangle$ of cardinals all less than λ such that $\lambda = \Sigma(\lambda_\sigma;\ \sigma < \lambda')$. If λ is singular then strictly increasing λ-sequences exist. An *η-transversal* of a family $\mathcal{X}$ is a subset T of $\bigcup \mathcal{X}$ such that $1 \leqslant |T \cap X| < \eta$ for each X in $\mathcal{X}$. A 2-transversal is called a *transversal*. If $\mathcal{X} = (X_i: i \in I)$ and $I' \subset I$, then $\mathcal{X}[I']$ denotes $(X_i;\ i \in I')$. The family $\mathcal{X} = (X_i;\ i \in I)$ is said to be a *$\Delta(\mu)$ family* if $|I| = \mu$ and there is a set K such that $X_i \cap X_j = K$ for all pairs

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$\{i, j\}$ in $[I]^2$. Such a system is called a *delta family*. The symbol

$$(\lambda, \kappa) \to \Delta(\mu) \text{ means: Every } (\lambda, \kappa) \text{ family contains a } \Delta(\mu) \text{ subfamily.}$$

Delta families were studied in Erdos and Rado [4] and we refer the reader to this paper for details when needed. We refer the reader to Williams [5] for any further set theoretical background.

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2. Proof of Theorem

Throughout this section λ, κ, Σ, μ and θ denote non-zero cardinals such that κ is infinite, $\kappa \leqslant \Sigma$ and $\theta \leqslant \mu \leqslant \min(\lambda, \Sigma)$. Note that although it follows that Σ is infinite; neither λ, μ nor θ need be.

The first two results are concerned with the cardinalities of maximal families of μ-partial representing sets. Note that every almost disjoint $(< \kappa', \kappa)$ family possesses a transversal.

LEMMA 1. (GCH). *Suppose $\kappa < \Sigma$ and $\mu < \kappa'$. Let $\mathcal{A}$ be an almost disjoint (λ, κ) decomposition of Σ and suppose $\mathfrak{T}$ is a family of μ-partial representing sets of $\mathcal{A}$ such that $\delta(\mathfrak{T}) \leqslant \theta$ and $|\mathfrak{T}| < \Sigma$. Then $\mathfrak{T}$ is not maximal with respect to $\delta(\mathfrak{T}) \leqslant \theta$.*

PROOF. Write $\mathcal{A} = (A_\alpha; \alpha < \lambda)$. Since $\bigcup \mathcal{A} = \Sigma$ and $\Sigma > \kappa$, it follows that $\lambda \geqslant \Sigma$. The conditions on the cardinals imply that $\mu < \Sigma$. Hence $|\bigcup \mathfrak{T}| < \Sigma$ and $|\Sigma - \bigcup \mathfrak{T}| = \Sigma$.

To show that $\mathfrak{T}$ is not maximal we construct a μ-sized subset X of λ and a μ-sized transversal T of $\mathcal{A}[X]$ such that $T \cap \bigcup \mathfrak{T} = \varnothing$. The construction of X and T depends on whether Σ is regular or not.

Case 1. Σ regular. Let $M = \{\alpha < \lambda;\ A_\alpha - \bigcup \mathfrak{T} \neq \varnothing\}$. Since $\Sigma - \bigcup \mathfrak{T} \subseteq \bigcup \{A_\alpha; \alpha \in M\}$ it follows that $|M| \geqslant \Sigma$. We may assume, without loss of generality, that if $\{\alpha, \beta\} \in [M]^2$ then $A_\alpha - \bigcup \mathfrak{T} \neq A_\beta - \bigcup \mathfrak{T}$. For each α in M we have $|A_\alpha - \bigcup \mathfrak{T}| \leqslant \kappa$ and we partition the ordinals α in M according to $|A_\alpha - \bigcup \mathfrak{T}|$. Since Σ is regular there is a set M' in $[M]^\Sigma$ and a cardinal ρ with $1 \leqslant \rho \leqslant \kappa$ such that $|A_\alpha - \bigcup \mathfrak{T}| = \rho$ for all α in M'. If $\rho = \kappa$ choose X from $[M']^\mu$. Then $(A_\alpha - \bigcup \mathfrak{T};\ \alpha \in X)$ is an almost disjoint (μ, κ) family and choose T to be a μ-sized transversal of this family. The set T is a μ-sized transversal of $\mathcal{A}[X]$ and $T \cap \bigcup \mathfrak{T} = \varnothing$. If $\rho < \kappa$ then $\rho^+ < \Sigma$ and $(\Sigma, \rho) \to \Delta(\mu)$, noting that $\mu < \Sigma$. (See

Erdös and Rado [4].) Thus there is a set X in $[M']^\mu$ such that $(A_\alpha - \bigcup\mathfrak{T};\ \alpha \in X)$ is a $\Delta(\mu)$ system. Let T be a μ-sized transversal of this family. (This is possible because $(A_\alpha - \bigcup\mathfrak{T};\ \alpha \in X)$ is a $\Delta(\mu)$ family of pairwise distinct sets.) Then T is a μ-sized transversal of $\mathfrak{A}[X]$ and $T \cap \bigcup\mathfrak{T} = \varnothing$.

Case 2. Σ singular. In this case let $L = \{\alpha < \lambda;\ |A_\alpha - \bigcup\mathfrak{T}| < \kappa\}$. Then $(A_\alpha \cap \bigcup\mathfrak{T};\ \alpha \in L)$ is an almost disjoint $(|L|, \kappa)$ family of subsets of $\bigcup\mathfrak{T}$ and $|L| \leqslant |\bigcup\mathfrak{T}|^+ < \Sigma$ since $|\bigcup\mathfrak{T}| < \Sigma$ and Σ is a limit cardinal. Hence $|\lambda - L| = \lambda$, and we choose X from $[\lambda - L]^\mu$ and let T be a μ-sized transversal of the almost disjoint (μ, κ) family $(A_\alpha - \bigcup\mathfrak{T};\ \alpha \in X)$. Then T is a μ-sized transversal of $\mathfrak{A}[X]$ and $T \cap \bigcup\mathfrak{T} = \varnothing$.

This completes the proof of Lemma 1.

LEMMA 2. (GCH). *Suppose $\kappa < \Sigma$, $\mu < \kappa'$, μ is infinite and $\mu' = \Sigma'$. Let $\mathfrak{A}$ be an almost disjoint (λ, κ) decomposition of Σ and suppose $\mathfrak{T}$ is an almost disjoint family of μ-partial representing sets of $\mathfrak{A}$ with $|\mathfrak{T}| \leqslant \Sigma$. Then $\mathfrak{T}$ is not maximal with respect to almost disjointness.*

PROOF. Write $\mathfrak{A} = (A_\alpha;\ \alpha < \lambda)$ and let $\mathfrak{T} = (T_\beta;\ \beta < \Sigma)$ where repetitions occur if $|\mathfrak{T}| < \Sigma$. Note that the conditions on the cardinals imply that $\Sigma \leqslant \lambda$, $\mu < \Sigma$ and Σ is singular. Let $\langle \mu_\sigma;\ \sigma < \mu' \rangle$ be a μ-sequence and let $\langle \Sigma_\delta;\ \delta < \mu' \rangle$ be a strictly increasing Σ-sequence.

We construct sets X from $[\lambda]^\mu$ and T from $[\Sigma]^\mu$ such that T is a transversal of $\mathfrak{A}[X]$ and $|T \cap T_\beta| < \mu$ for each β less than Σ. This establishes that $\mathfrak{T}$ is not maximal.

Inductively, define a pairwise disjoint family $(X_\sigma;\ \sigma < \mu')$ of subsets of λ such that $|X_\sigma| = \mu_\sigma$ for each σ less than μ'. Suppose that $\sigma < \mu'$ and each member of the set $\mathfrak{X}_\sigma = \{X_\delta;\ \delta < \sigma\}$ has been defined. Let $S_\sigma = \bigcup\{T_\beta;\ \beta < \Sigma_\sigma\}$ and let $I_\sigma = \{\alpha < \lambda;\ |A_\alpha \cap S_\sigma| = \kappa\}$. Then the family $(A_\alpha \cap S_\sigma;\ \alpha \in I_\sigma)$ is an almost disjoint $(|I_\sigma|, \kappa)$ family of subsets of S_σ and $|I_\sigma| \leqslant |S_\sigma|^+$, where $|S_\sigma|^+ < \Sigma$ since $|S_\sigma| \leqslant \mu \cdot \Sigma_\sigma < \Sigma$ and Σ is a limit cardinal. Also, $|\bigcup\mathfrak{X}_\sigma| = \Sigma(\mu_\delta;\ \delta < \sigma) < \mu < \Sigma$. Hence $|\lambda - (I_\sigma \cup \bigcup\mathfrak{X}_\sigma)| = \lambda$ and we choose X_σ from $[\lambda - (I_\sigma \cup \bigcup\mathfrak{X}_\sigma)]^{\mu_\sigma}$. Note that if $\alpha \in X_\sigma$ then $|A_\alpha \cap S_\sigma| < \kappa$. Put $X = \bigcup\{X_\sigma;\ \sigma < \mu'\}$. For each α in X let $\sigma(\alpha)$ be the unique σ less than μ' such that $\alpha \in X_\sigma$ and set $S = \bigcup\{A_\alpha \cap S_{\sigma(\alpha)};\ \alpha \in X\}$. Since S is the union of μ sets each of power less than κ and $\mu < \kappa'$, it follows that $|S| < \kappa$. Hence $(A_\alpha - S;\ \alpha \in X)$ is an almost disjoint (μ, κ) family and we choose T to be a μ-sized transversal of this family. This defines X and T.

Since T is a transversal of $(A_\alpha - S;\ \alpha \in X)$ and $A_\alpha \cap T = (A_\alpha - S) \cap T$ for each α in X, it follows that T is a transversal of $\mathfrak{A}[X]$. To show that T is almost disjoint from each member of $\mathfrak{T}$ suppose that $\beta < \Sigma$ and let $\delta(\beta)$ be the least δ less than μ' such that $\beta < \Sigma_\delta$. If $\delta(\beta) \leqslant \sigma < \mu'$ then $(A_\alpha \cap T) \cap T_\beta = \varnothing$ for each α in X_σ. To prove this we argue by contradiction. Suppose that $\delta(\beta) \leqslant \sigma < \mu'$, $\alpha \in X_\sigma$ and $t \in (A_\alpha \cap T) \cap T_\beta$. Then $T_\beta \subseteq S_\sigma$ since $\beta < \Sigma_{\delta(\beta)} \leqslant \Sigma_\sigma$; and $t \in A_\alpha \cap S_\sigma = A_\alpha \cap S_{\sigma(\alpha)} \subseteq S$. On the other hand; $t \notin S$ since $t \in T$ and $T \cap S = \varnothing$; a contradiction. Therefore:

$$T \cap T_\beta = \bigcup \{(A_\alpha \cap T) \cap T_\beta; \alpha \in X\}$$
$$\subseteq \bigcup \{A_\alpha \cap T; \alpha \in \bigcup \{X_\sigma; \sigma < \delta(\beta)\}\},$$

and so

$$|T \cap T_\beta| < \left|\bigcup \{X_\sigma; \sigma < \delta(\beta)\}\right| < \mu,$$

since $|A_\alpha \cap T| = 1$ for each α in X, $|X_\sigma| < \mu$ for each σ less than $\delta(\beta)$ and $|\delta(\beta)| < \mu'$.

This completes the proof of Lemma 2.

The following two propositions deal with the case when $\Sigma > \kappa$ and $\mu \geqslant \kappa'$. Note that GCH is not required and the family $\mathcal{A}$ need not be almost disjoint.

PROPOSITION 3. *Suppose $\Sigma > \kappa$ and $\mu \geqslant \kappa'$. Let $\mathcal{A}$ be a (λ, κ) decomposition of Σ. There exists a pairwise disjoint (Σ, μ) decomposition $\mathcal{T}$ of Σ such that each member of $\mathcal{T}$ is a μ-transversal of some μ-sized subfamily of $\mathcal{A}$.*

PROOF. Write $\mathcal{A} = (A_\alpha; \alpha < \lambda)$. The conditions on the cardinals imply that $\lambda \geqslant \Sigma$. Let $\langle \mu_\sigma; \sigma < \mu' \rangle$ be a μ-sequence and let $(L_\sigma; \sigma < \mu')$ be a pairwise disjoint (μ', Σ) decomposition of Σ.

We inductively define families $(X_\alpha; \alpha < \Sigma)$ and $(T_\alpha; \alpha < \Sigma)$ of sets such that

(i) $X_\alpha \in [\lambda]^\mu$, $T_\alpha \in [\Sigma]^\mu$ and T_α is a μ-transversal of $\mathcal{A}[X_\alpha]$ for each α less than Σ,

(ii) $T_\beta \cap T_\alpha = \varnothing$ if $\beta < \alpha < \Sigma$, and

(iii) $|T_\alpha \cap L_\sigma| \leqslant \mu_\sigma$ if $\langle \alpha, \sigma \rangle \in \Sigma \times \mu'$.

Suppose that $\alpha < \Sigma$ and X_β, T_β have been defined for each β less than α. Let $\mathcal{X}_\alpha = \{X_\beta; \beta < \alpha\}$ and let $\mathcal{T}_\alpha = \{T_\beta; \beta < \alpha\}$. Note that, for each σ less than μ',

$$\left|L_\sigma \cap \bigcup \mathcal{T}_\alpha\right| = \left|\bigcup \{T_\beta \cap L_\sigma; \beta < \alpha\}\right| \leqslant \mu_\sigma \cdot |\alpha| < \Sigma,$$

and $\left|L_\sigma - \bigcup \mathcal{T}_\alpha\right| = \Sigma$. For each σ less than μ' let

$$I_\sigma = \left\{\alpha < \lambda; A_\alpha \cap \left(L_\sigma - \bigcup \mathcal{T}_\alpha\right) \neq \varnothing\right\}.$$

Since $\Sigma > \kappa$ it follows that $|I_\sigma| \geqslant \Sigma$ for each σ less than μ'.

To define X_α and T_α we inductively define two pairwise disjoint families $(Y_\sigma; \sigma < \mu')$ and $(S_\sigma; \sigma < \mu')$ such that $|Y_\sigma| = |S_\sigma| = \mu_\sigma$ for each σ less than μ'. Suppose that $\sigma < \mu'$ and Y_δ, S_δ have been defined for each δ less than σ. Let $\mathcal{Y}_\sigma = \{Y_\delta; \delta < \sigma\}$ and let $\mathcal{S}_\sigma = \{S_\delta; \delta < \sigma\}$. To define Y_σ and S_σ we inductively define sequences $\langle y^\sigma(\gamma); \gamma < \mu_\sigma \rangle$, $\langle s^\sigma(\gamma); \gamma < \mu_\sigma \rangle$ of pairwise distinct elements

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of λ, Σ respectively. Suppose that $\gamma < \mu_\sigma$ and $y^\sigma(\nu)$ $s^\sigma(\nu)$ have been defined for each ν less than γ; and let

$$L_\sigma(\gamma) = \left(L_\sigma - \bigcup \mathfrak{T}_\sigma\right) - \left(\bigcup \mathbb{S}_\sigma \cup \bigcup \mathcal{Q}\left[\bigcup \mathbb{S}_\sigma\right] \cup \{s^\sigma(\nu); \nu < \gamma\}\right).$$

Then $|L_\sigma(\gamma)| = \Sigma$ since

$$\left|\bigcup \mathbb{S}_\sigma\right| = \sum(\mu_\delta; \delta < \sigma) < \mu \leqslant \Sigma,$$

$$\left|\bigcup \mathcal{Q}\left[\bigcup \mathbb{S}_\sigma\right]\right| \leqslant \kappa \cdot \left|\bigcup \mathbb{S}_\sigma\right| < \Sigma,$$

$\left|L_\sigma - \bigcup \mathfrak{T}_\sigma\right| = \Sigma$ and $|\gamma| < \mu_\sigma < \Sigma$. Since $\Sigma > \kappa$ the set

$$I_\sigma(\gamma) = \{\alpha \in I_\sigma; A_\alpha \cap L_\sigma(\gamma) \neq \varnothing\}$$

has cardinality at least Σ. Also

$$\left|\bigcup \mathfrak{Y}_\sigma\right| = \sum(\mu_\delta; \delta < \sigma) < \mu \leqslant \Sigma.$$

Hence

$$\left|I_\sigma(\gamma) - \left(\bigcup \mathfrak{Y}_\sigma \cup \{y^\sigma(\nu); \nu < \gamma\}\right)\right| = |I_\sigma(\gamma)| \geqslant \Sigma$$

and we choose $y^\sigma(\gamma)$ from this set. (Hence, $y^\sigma(\gamma) \notin \bigcup \mathfrak{Y}_\sigma$ and $y^\sigma(\gamma) \neq y^\sigma(\nu)$ for any ν less than γ.) Since $y^\sigma(\gamma) \in I_\sigma(\gamma)$ it follows that $A_{y^\sigma(\gamma)} \cap L_\sigma(\gamma) \neq \varnothing$ and we choose $s^\sigma(\gamma)$ from this set. (Hence, $s^\sigma(\gamma) \notin \bigcup \mathbb{S}_\sigma$ and $s^\sigma(\gamma) \neq s^\sigma(\nu)$ for any ν less than γ.) This defines $y^\sigma(\gamma)$ and $s^\sigma(\gamma)$. Set $Y_\sigma = \{y^\sigma(\gamma); \gamma < \mu_\sigma\}$ and set $S_\sigma = \{s^\sigma(\gamma); \gamma < \mu_\sigma\}$. Put $X_\alpha = \bigcup\{Y_\sigma; \sigma < \mu'\}$ and put $T_\alpha = \bigcup\{S_\sigma; \sigma < \mu'\}$.

The sets X_α, T_α will do. Since $|Y_\sigma| = |S_\sigma| = \mu_\sigma$ for each σ less than μ' and the cardinals μ_σ sum to μ, it follows that $|X_\alpha| = |T_\alpha| = \mu$. We show that T_α is a μ-transversal of $\mathcal{Q}[X_\alpha]$. Now $X_\alpha = \{y^\sigma(\gamma); \sigma < \mu' \text{ and } \gamma < \mu_\sigma\}$ and $s^\sigma(\gamma) \in A_{y^\sigma(\gamma)}$ always. Hence $T_\alpha \subseteq \bigcup \mathcal{Q}[X_\alpha]$ and $T_\alpha \cap A_y \neq \varnothing$ for each y in X_α. Next, suppose that $\sigma < \mu'$ and $\gamma < \mu_\sigma$. If $\sigma < \delta < \mu'$ and $\varepsilon < \mu_\delta$ then $s^\delta(\varepsilon) \notin A_{y^\sigma(\gamma)}$ since $A_{y^\sigma(\gamma)} \subseteq \bigcup \mathcal{Q}[S_\delta]$, $s^\delta(\varepsilon) \in L_\delta(\varepsilon)$ and $L_\delta(\varepsilon) \cap \bigcup \mathcal{Q}[S_\delta] = \varnothing$. Hence

$$\left|T_\alpha \cap A_{y^\sigma(\gamma)}\right| \leqslant \left|\bigcup\{S_\delta; \delta \leqslant \sigma\}\right| = \sum(\mu_\delta; \delta \leqslant \sigma) < \mu,$$

and T_α is a μ-transversal of $\mathcal{Q}[X_\alpha]$ as claimed. If $\beta < \alpha$ then $T_\beta \cap T_\alpha = \varnothing$ since $T_\alpha \subseteq \Sigma - \bigcup \mathfrak{T}_\alpha$. Finally, if $\sigma < \mu'$ then $T_\alpha \cap L_\sigma = S_\sigma$ and $|T_\alpha \cap L_\sigma| \leqslant \mu_\sigma$ as required. This completes the construction of X_α and T_α.

The family $\mathfrak{T} = (T_\alpha; \alpha < \Sigma)$ is a pairwise disjoint family of μ-sized subsets of Σ and each member of $\mathfrak{T}$ is a μ-transversal of some μ-sized subfamily of $\mathcal{Q}$.

The next proposition is a modification of Proposition 3 and gives a related result in the case when $\mu' = \Sigma'$.

PROPOSITION 4. *Suppose $\Sigma > \kappa$, $\mu \geqslant \kappa'$ and $\mu' = \Sigma'$. Let $\mathcal{Q}$ be a (λ, κ) decomposition of Σ. There exists an almost disjoint (Σ^+, μ) decomposition $\mathfrak{T}$ of Σ such that each member of $\mathfrak{T}$ is a μ-transversal of some μ-sized subfamily of $\mathcal{Q}$.*

PROOF. The proof involves only minor modifications to the proof of Proposition 3 to deal with the inductive step when $|\mathfrak{T}_\alpha| = \Sigma$. We refer to the proof of Proposition 3 for details. Write $\mathcal{A} = (A_\alpha;\ \alpha < \lambda)$. Let $\langle \mu_\sigma;\ \sigma < \mu' \rangle$ be a μ-sequence and let $\langle \Sigma_\sigma;\ \sigma < \mu' \rangle$ be a Σ-sequence. Suppose $\mathfrak{M} = (M_\sigma;\ \sigma < \mu')$ is a pairwise disjoint decomposition of Σ such that $|M_\sigma| = \Sigma_\sigma$ for each σ less than μ'. Let $(L_\sigma;\ \sigma < \mu')$ be a pairwise disjoint (μ', Σ) decomposition of Σ. As in Proposition 3, we inductively construct families $(X_\alpha;\ \alpha < \Sigma^+)$ and $(T_\alpha;\ \alpha < \Sigma^+)$ such that

(i) $X_\alpha \in [\lambda]^\mu$, $T_\alpha \in [\Sigma]^\mu$ and T_α is a μ-transversal of $\mathcal{A}[X_\alpha]$ for each α less than Σ^+,

(ii) $|T_\beta \cap T_\alpha| < \mu$ if $\beta < \alpha < \Sigma^+$,

(iii) $|T_\alpha \cap L_\sigma| \leqslant \mu_\sigma$ if $\langle \alpha, \sigma \rangle \in \Sigma^+ \times \mu'$.

The families $(X_\alpha;\ \alpha < \Sigma)$ and $(T_\alpha;\ \alpha < \Sigma)$ were constructed in Proposition 3. Next, suppose that $\Sigma \leqslant \alpha < \Sigma^+$ and X_β, T_β have been defined for each β less than α. The families $\mathfrak{X}_\alpha$, $\mathfrak{T}_\alpha$ are as before and we re-index $\mathfrak{T}_\alpha$ by the ordinals ε less than Σ: write $\mathfrak{T}_\alpha = (\underline{T}_\varepsilon;\ \varepsilon < \Sigma)$. The construction of X_α and T_α is similar to that in Proposition 3 except that here we define

$$I_\alpha = \left\{\alpha < \lambda;\ A_\alpha \cap \left(L_\alpha - \bigcup \left\{\underline{T}_\beta;\ \beta \in \bigcup \mathfrak{M}[\sigma]\right\}\right) \neq \varnothing\right\}.$$

The sets $\mathbb{S}_\sigma$ and $\mathfrak{Y}_\sigma$ are as before. The construction of $y^\sigma(\gamma)$ and $s^\sigma(\gamma)$ is similar except that here we define

$$\begin{aligned} L_\sigma(\gamma) = \Big(L_\sigma - &\bigcup \left\{\underline{T}_\beta;\ \beta \in \bigcup \mathfrak{M}[\sigma]\right\}\Big) \\ &- \left(\bigcup \mathbb{S}_\sigma \cup \bigcup \mathcal{A}\left[\bigcup \mathbb{S}_\sigma\right] \cup \{s^\sigma(\nu);\ \nu < \gamma\}\right). \end{aligned}$$

The sets X_α and T_α have all the required properties. We present only the proof that $|T_\beta \cap T_\alpha| < \mu$ for each β less than α. Suppose $\varepsilon < \Sigma$ and let $\sigma(\varepsilon)$ be the unique σ less than μ' such that $\varepsilon \in M_\sigma$. If $\sigma(\varepsilon) < \sigma < \mu'$ then $\underline{T}_\varepsilon \subseteq \bigcup \{\underline{T}_\beta;\ \beta \in \bigcup \mathfrak{M}[\sigma]\}$ and $\underline{T}_\varepsilon \cap L_\sigma(\gamma) = \varnothing$ for all γ less than μ_σ. Hence $\underline{T}_\varepsilon \cap S_\sigma = \varnothing$ for each σ with $\sigma(\varepsilon) < \sigma < \mu'$. Therefore, $\underline{T}_\varepsilon \cap T_\alpha \subset \bigcup \{S_\sigma;\ \sigma \leqslant \sigma(\varepsilon)\}$ and $|\underline{T}_\varepsilon \cap T_\alpha| < \mu$ as required.

The family $\mathfrak{T} = (T_\alpha;\ \alpha < \Sigma^+)$ is an almost disjoint (Σ^+, μ) decomposition of Σ and each member of $\mathfrak{T}$ is a μ-transversal of some μ-sized subfamily of $\mathcal{A}$.

We are now in a position to prove that $RS_\theta(\mu, \mathcal{A}) = S_\theta(\mu, \Sigma)$.

PROOF OF THEOREM. Write $\mathcal{A} = (A_\alpha;\ \alpha < \lambda)$. Clearly, $RS_\theta(\mu, \mathcal{A}) \leqslant S_\theta(\mu, \Sigma)$. Hence

(a) if $\theta < \mu$ or if $\mu' \neq \Sigma'$, then $RS_\theta(\mu, \mathcal{A}) \leqslant \Sigma$.

(b) If $\mu' = \Sigma'$ then $RS(\mu, \mathcal{A}) \leqslant \Sigma^+$.

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To show that these upper bounds are the values of $RS_\theta(\mu, \mathscr{A})$ we construct, in each case, a 'suitably large' family $\mathfrak{T}$ of μ-partial representing sets of $\mathscr{A}$ such that $\delta(\mathfrak{T}) \leqslant \theta$.

Case 1. $\kappa = \Sigma$ and $\mu < \kappa$. It is clear that $\mathscr{A}[\mu]$ possesses a pairwise disjoint (κ, μ) family of representing sets. This suffices if either $\theta < \mu$ or $\mu' \neq \kappa'$. Next, suppose $\theta = \mu$ and $\mu' = \kappa'$. Then κ is singular and we choose $\langle \kappa_\sigma; \sigma < \mu' \rangle$ to be a strictly increasing κ-sequence. Let $\langle \mu_\sigma; \sigma < \mu' \rangle$ be a μ-sequence. Inductively define an almost disjoint family $(T_\alpha; \alpha < \kappa^+)$ of μ-sized representing sets of $\mathscr{A}[\mu]$ as follows. Suppose that $\alpha < \kappa^+$ and the members of $\mathfrak{T}_\alpha = (T_\beta; \beta < \alpha)$ have been defined. Write $\mathfrak{T}_\alpha = (\underline{T}_\varepsilon; \varepsilon < \kappa)$ (Repetitions occur if $\alpha < \kappa$). To define T_α inductively define a pairwise disjoint family of subsets of κ with $|S_\sigma| = \mu_\sigma$ for all σ less than μ' as follows. Given σ less than μ' choose S_σ to be a μ_σ-sized representing set of the almost disjoint (μ_σ, κ) family

$$\left(A_\nu - \left(\bigcup \{\underline{T}_\varepsilon; \varepsilon < \kappa_\sigma\} \cup \bigcup \{S_\delta; \delta < \sigma\}\right); \nu < \mu_\sigma\right),$$

and set $T_\alpha = \bigcup\{S_\sigma; \sigma < \mu'\}$. The set T_α will do. Then $\mathfrak{T} = (T_\alpha; \alpha < \kappa^+)$ is an almost disjoint (κ^+, μ) family of μ-partial representing sets of $\mathscr{A}$ and the result follows in this case.

Case 2. $\kappa = \Sigma$ and $\mu = \kappa$. The proof is immediate from Balanda [1]. Let $\mathfrak{T}$ be a family of κ-sized representing sets of $\mathscr{A}[\kappa]$ with $\delta(\mathfrak{T}) \leqslant \theta$ and $|\mathfrak{T}| = S_\theta(\kappa, \kappa)$. The family $\mathfrak{T}$ consists of μ-partial representing sets of $\mathscr{A}$ and the result follows in this case.

Case 3. $\Sigma > \kappa$. In this case we use the lemmas and propositions above. First suppose that $\mu < \kappa'$. A simple application of Zorn's Lemma shows there is a family $\mathfrak{T}$ of μ-partial representing sets of $\mathscr{A}$ that is maximal with respect to $\delta(\mathfrak{T}) \leqslant \theta$. Lemmas 1 and 2 guarantee that $|\mathfrak{T}| \geqslant \Sigma$ if $\theta < \mu$ or if $\mu' \neq \Sigma'$, and $|\mathfrak{T}| \geqslant \Sigma^+$ if $\theta = \mu$ and $\mu' = \Sigma'$. Next, suppose that $\mu \geqslant \kappa'$. Propositions 3 and 4 show that there exists a $(S_\theta(\mu, \Sigma), \mu)$ family $\mathfrak{T}$ with $\delta(\mathfrak{T}) \leqslant \theta$ such that each member of $\mathfrak{T}$ is a μ-transversal of a μ-sized subfamily of $\mathscr{A}$.

This completes the proof of the Theorem.

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