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ON τ-COMPLETELY DECOMPOSABLE MODULES

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For a hereditary torsion theory τ, a module A is called τ-completely decomposable if it is a direct sum of modules that are the τ-injective hull of each of their non-zero submodules. We give a positive answer in several cases to the following generalised Matlis' problem: Is every direct summand of a τ-completely decomposable module still τ-completely decomposable? Secondly, for a commutative Noetherian ring R that is not a domain, we determine those torsion theories with the property that every τ-injective module is an essential extension of a (τ-injective) τ-completely decomposable module.

1. INTRODUCTION

The torsion-theoretic version of completely decomposable modules has been mentioned and studied by several authors, such as Bueso, Jara and Torrecillas [**1**], García [**9**], Masaike and Horigome [**12**], Mohamed, Müller and Singh [**14, 15**].

The aim of the present paper is to present some applications of the recently reconsidered τ-complemented modules, introduced by Golan [**10**] and afterwards studied by Smith, Viola-Prioli and Viola-Prioli [**18, 19, 20**], in the study of τ-completely decomposable modules. We shall show and use the result that every τ-completely decomposable module is τ-complemented as well as its converse under some extra hypotheses.

The final goal is to discuss the following two problems:

PROBLEM 1. Is every direct summand of a τ-completely decomposable module still τ-completely decomposable?

PROBLEM 2. For some particular classes of rings, characterise the torsion theories with the property that every τ-injective module is an essential extension of a τ-injective τ-completely decomposable module.

The first one is the torsion-theoretic version of a classical question raised by Matlis [**13**], namely: *is every direct summand of a direct sum of indecomposable injective modules still a direct sum of indecomposable injective modules?* The problem is still open in the general case, even if the answer is known to be positive in a number of cases, some of them mentioned by us and generalised for a hereditary torsion theory in Section 3.

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The second one is discussed in Section 4 and is related to the following problem: *for some particular torsion theories τ, characterise the rings with the property that every τ-injective module is an essential extension of a τ-injective τ-completely decomposable module.* This was previously studied by Masaike and Horigome [**12**], Bueso, Jara and Torrecillas [**1**].

Let us now give some basic notation and definitions, mainly following [**10**]. Throughout this paper, we denote by R an associative ring with non-zero identity and all modules will be left unital R-modules. We denote by τ a hereditary torsion theory on the category R-Mod of left R-modules and by $T_\tau(A)$ the unique maximal τ-torsion submodule of a module A.

A submodule B of a module A is said to be *τ-dense (τ-closed)* in A if A/B is τ-torsion (τ-torsionfree). A non-zero module A is said to be *τ-cocritical* if A is τ-torsionfree and each of its non-zero submodules is τ-dense in A.

A module A is called *τ-complemented* if every submodule of A is τ-dense in a direct summand of A [**18**, p. 1309]. A module A is said to be *τ-injective* if it is injective with respect to every monomorphism having a τ-torsion cokernel. For any module A, $E(A)$ and $E_\tau(A)$ denote the injective and the τ-injective hull of A respectively. If B is a submodule of a module A, $B \trianglelefteq A$ denotes the fact that A is an essential extension of B.

In this paper, a non-zero module that is the τ-injective hull of each of its non-zero submodules is called *minimal τ-injective*. If $A = \bigoplus_{i\in I} A_i$ is a (finite) direct sum of minimal τ-injective submodules, then A is said to be *(finitely) τ-completely decomposable* [**12**, p. 77]. This extends for a torsion theory the terminology of completely decomposable module in the sense of Faith and Walker [**8**], that is, a direct sum of indecomposable injective modules.

A module is called τ-Noetherian if it has ACC on τ-closed submodules. A torsion theory τ is called: (1) *stable* if the class of τ-torsion modules is closed under injective hulls; (2) *Noetherian* if for every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R the union of which is τ-dense in R, there exists a positive integer k such that I_k is τ-dense in R; (3) *perfect* if it is Noetherian and the localisation functor $Q_\tau : R\text{-Mod} \to R\text{-Mod}$ is exact.

For additional terminology on modules and torsion theories the reader is referred to [**6, 10**].

2. Preliminary results

In this section we set the scene with some needed properties of τ-complemented τ-injective modules.

Let us show first that τ-completely decomposable modules are some other examples of τ-complemented modules, besides the immediate examples of semisimple, uniform,

τ-torsion or τ-cocritical modules [**18**, p. 1311].

PROPOSITION 2.1. *Every τ-completely decomposable module is τ-complemented.*

PROOF: Let A be a τ-completely decomposable module and let $A \xrightarrow{f} B \longrightarrow 0$ be an exact sequence of modules with B τ-torsionfree. We show that the sequence splits. Then the result will follow by [**18**, Proposition 1.6].

Let $A = \bigoplus_{i \in I} A_i$, where each A_i is a minimal τ-injective submodule of A. We may suppose that f is non-zero. Denote $f_i = f|_{A_i}$ for every $i \in I$.

Now let $i \in I$. Then $f_i(A_i)$ is τ-torsionfree. Since A_i is minimal τ-injective, A_i is either τ-torsion or τ-torsionfree. If A_i is τ-torsion or $f_i = 0$, then $f_i(A_i) = 0$. Suppose now that A_i is τ-torsionfree and $f_i \neq 0$. Then $f_i(A_i) \cong A_i$, because A_i is τ-cocritical. Let $J = \{j \in I \mid f(A_j) \neq 0\}$. Then $B = f(A) = \sum_{j \in J} f(A_j)$. It follows that there exists a subset $K \subseteq J$ such that $B = \bigoplus_{k \in K} f(A_k)$ [**10**, Proposition 14.11]. Now let $g : B \to A$ be the homomorphism defined by $g = \bigoplus_{k \in K} f_k^{-1}$. Then $fg = 1_B$, tht is, the above sequence splits. ∎

We shall continue with a few other useful results concerning τ-complemented modules.

PROPOSITION 2.2. *The following statements are equivalent for a τ-injective module A:*

(i) *A is τ-complemented;*

(ii) *Every τ-injective submodule of A is a direct summand;*

(iii) *A has no proper essential τ-injective submodule.*

PROOF: (i)$\Longrightarrow$(ii). If A is τ-complemented and B is a τ-injective submodule of A, then B is τ-dense in a direct summand D of A. But B is τ-injective, hence it is a direct summand of D and, consequently, of A.

(ii)$\Longrightarrow$(iii). Clear.

(iii)$\Longrightarrow$(i). Assume (iii) and let B be a submodule of A. If $E_\tau(B) = A$, we are done. Assume further that $E_\tau(B)$ is a proper submodule of A. Then it is not essential in A. Let D be a complement of $E_\tau(B)$ in A. Since $E_\tau(B) \cap D = 0$, we have $E_\tau(B) \cap E_\tau(D) = 0$, hence $D = E_\tau(D)$. Then $E_\tau(B) \oplus E_\tau(D) = E_\tau(B) \oplus D \trianglelefteq A$. Since $E_\tau(B) \oplus D$ is τ-injective, we have $E_\tau(B) \oplus D = A$. Thus, B is τ-dense in the direct summand $E_\tau(B)$ of A. Hence, A is τ-complemented. ∎

PROPOSITION 2.3. *Let $(A_i)_{i \in I}$ be a family of τ-complemented τ-injective modules. Then $E_\tau\left(\bigoplus_{i \in I} A_i\right)$ is τ-complemented.*

PROOF: We show that $A = E_\tau\left(\bigoplus_{i \in I} A_i\right)$ has no proper essential τ-injective submod-

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ule. Then the result will follow by Proposition 2.2.

Let D be an essential τ-injective submodule of A and let $i \in I$. Then $D \cap A_i \neq 0$ and $E(D) = E(A)$. Since D is τ-injective, it follows that $E(A)/D$ is τ-torsionfree, hence A/D is τ-torsionfree. Then $A_i/(D \cap A_i) \cong (D + A_i)/D$ is τ-torsionfree. Thus, $D \cap A_i$ is τ-closed in the τ-injective module A_i, hence it is τ-injective. Now let $0 \neq a_i \in A_i$. Then there exists $r_i \in R$ such that $0 \neq r_i a_i \in D$, hence $r_i a_i \in D \cap A_i$. It follows that $D \cap A_i \trianglelefteq A_i$. By Proposition 2.2, $D \cap A_i = A_i$. Then $\bigoplus_{i \in I} A_i \trianglelefteq D \trianglelefteq A$, we have $D = A$. ∎

It is known that a module is τ-complemented if and only if it is a direct sum of a τ-torsion module and a τ-torsionfree τ-complemented module [**18**, Theorem 1.8]. Using this result we obtain the following characterisation of τ-complemented τ-injective modules.

THEOREM 2.4. *Let R be a ring that has ACC on τ-closed left ideals and let A be an R-module. Then A is τ-complemented τ-injective if and only if $A = B \oplus C$, where B is τ-torsion τ-injective and C is the τ-injective hull of a τ-torsionfree τ-completely decomposable module.*

PROOF: Suppose first that A is τ-complemented τ-injective. Then $A = T_\tau(A) \oplus C$, where C is τ-torsionfree τ-complemented τ-injective [**18**, Theorem 1.8]. We may suppose that $C \neq 0$. The hypothesis on R allows us to write $E(C) = \bigoplus_{i \in I} E_i$ as a direct sum of indecomposable injective R-modules [**10**, Proposition 20.17]. For every $i \in I$, denote $D_i = C \cap E_i$. Then for every $i \in I$, we have $0 \neq D_i \trianglelefteq E_i$, hence $\bigoplus_{i \in I} D_i \trianglelefteq \bigoplus_{i \in I} E_i = E(C)$. It follows that $\bigoplus_{i \in I} D_i \trianglelefteq C$. By Proposition 2.2, $C = E_\tau\Big(\bigoplus_{i \in I} D_i\Big)$. Clearly, $\bigoplus_{i \in I} D_i$ is τ-torsionfree. We still have to show that each D_i is minimal τ-injective. Let $i \in I$. Since $E_i/D_i \cong (C + E_i)/C \subseteq E(C)/C$ is τ-torsionfree, it follows that D_i is τ-injective. Now by Proposition 2.2 and the fact that C is τ-complemented τ-injective, D_i is also τ-complemented τ-injective. Since D_i is uniform, it has to be minimal τ-injective, because otherwise, if F were a non-zero proper τ-injective submodule of D_i, then F would be a direct summand by Proposition 2.2.

Suppose now that $A = B \oplus C$, where B is τ-torsion τ-injective and C is the τ-injective hull of a τ-torsionfree τ-completely decomposable module. Then B is τ-complemented τ-injective. Now use Proposition 2.3. ∎

Direct sum decomposition theorems for τ-torsion τ-injective modules or even for τ-injective modules were studied in [**1, 12, 14, 15**] and for τ-complemented modules in [**18**]. For instance, we have the following two characterisation theorems:

THEOREM 2.5. [**12**, Theorem 1] *The following statements are equivalent:*

(i) *Every τ-torsion τ-injective R-module is τ-completely decomposable.*

(ii) *R has ACC on τ-dense left ideals.*

Theorem 2.6. [**18**, Theorem 3.9] *The following statements are equivalent for a τ-complemented module M:*

(i) *M is a direct sum of a τ-torsion module and τ-cocritical modules.*

(ii) *R has ACC on left ideals of the form $\mathrm{Ann}_R\, x$, where $x \in M/T_\tau(M)$.*

For τ-complemented τ-injective modules we give the following result.

Theorem 2.7. *Let R be a ring that has ACC both on τ-dense and τ-closed left ideals. Then an R-module A is τ-complemented τ-injective if and only if A is τ-completely decomposable.*

Proof: Suppose that A is τ-complemented τ-injective. Then by Theorem 2.4, $A = B \oplus C$, where B is τ-torsion τ-injective and C is the τ-injective hull of a τ-torsionfree τ-completely decomposable module. Since R has ACC on τ-dense left ideals, every τ-torsion τ-injective R-module has a τ-complete decomposition [**10**, Proposition 41.6]. Under the both hypotheses on R, direct sums of τ-injective modules are τ-injective [**10**, Proposition 41.10]. Now the result follows.

For the converse use Proposition 2.1 and [**10**, Proposition 41.10]. ∎

Remarks. (1) Theorem 2.7 will allow us to use, over rings with ACC both on τ-dense and τ-closed left ideals, the nicer properties of τ-complemented τ-injective modules when working with τ-completely decomposable modules. On the other hand, it refines a part of Theorem 2.5 for the more general class of τ-complemented τ-injective modules.

(2) Clearly, Theorem 2.7 holds for a left Noetherian ring, but there also exist non-left Noetherian rings satisfying ACC both on τ-dense and τ-closed left ideals [**16**, Example 28].

We have seen that every τ-completely decomposable module is τ-complemented. Now we are able to give an example of a τ-complemented module which is not τ-completely decomposable.

Example 2.8. Consider the polynomial ring $R = K[X_1, \dots, X_m]$ ($m \geqslant 2$), where K is a field. Denote by τ_D the Dickson torsion theory, that is, the hereditary torsion theory generated by the class of all simple modules [**4**]. Since K is τ_D-torsion and $K[X_1]$ is τ_D-cocritical, $K \oplus K[X_1]$ is a τ_D-complemented R-module [**18**, Corollary 1.5]. On the other hand, $K \oplus K[X_1]$ cannot be τ_D-injective, because $K[X_1]$ is not (for instance, by [**3**, Theorem 2.5]). Having noted that R is Noetherian, Theorem 2.7 shows that $K \oplus K[X_1]$ is not a τ_D-completely decomposable R-module.

3. Direct summands of τ-completely decomposable modules

Previously established results will allow us to give partial answers to Problem 1 stated in the introduction. Among classical questions to be asked on a class of modules there is the following one:

If $M = \bigoplus_{i \in I} M_i$ *is a direct sum of modules of a class* $\mathcal{A}$*, is a direct summand* N *of* M *still a direct sum of modules of the class* $\mathcal{A}$*?*

The problem has been raised for various classes of modules, ranging for instance from the class of countably generated modules (see Cohen and Kaplansky [**2**]) to the class of uniserial modules (see Dung and Facchini [**5**]), having complete or partial answers, such as a complete positive answer in the former case or some partial positive answers in the latter.

This is apparently an open question if $\mathcal{A}$ is either the class of all modules with local endomorphism rings or the class of all indecomposable injective modules [**7**, p. 267]. For the former, the answer is positive if each M_i is countably generated [**7**]. For the latter, raised by Matlis [**13**], the answer is positive in several cases, such as: R left Noetherian [**13**], M injective [**8**] or even M quasi-injective [**11**], I finite [**17**], N countably generated [**8**] or N injective [**13**].

We shall consider here the class $\mathcal{A}$ consisting of all minimal τ-injective modules, that incidently are known to have local endomorphism rings (for instance, by [**12**, Lemma 1]), and we shall give an affirmative answer in several cases (including the torsion-theoretic versions of the above ones, but not only them) to the previous question, reformulated as:

Is a direct summand N *of a* τ*-completely decomposable module* M *still a* τ*-completely decomposable module?*

A positive answer was given by Masaike and Horigome for N τ-injective [**12**, Remark, p. 81]. Now we can easily give the following result.

Corollary 3.1. *Let* R *be a ring that has ACC both on* τ*-dense and* τ*-closed left ideals and let* A *be a* τ*-completely decomposable module. Then any direct summand of* A *is* τ*-completely decomposable.*

Proof: By Theorem 2.7, A is τ-complemented τ-injective. Now let B be a direct summand of A. Then B is τ-injective and if C is a τ-injective submodule of B, then by Proposition 2.2 C is a direct summand of A, hence of B. Thus, again by Proposition 2.2, B is τ-complemented τ-injective. Finally, use again Theorem 2.7 to obtain that B is τ-completely decomposable. ∎

Consider now the following condition on a module A [**6**, p. 16]:

(C_2) *Every submodule isomorphic to a direct summand of* A *is a direct summand of* A.

Among the modules satisfying (C_2) we mention continuous modules (that can be seen as extending modules with (C_2)) and, in particular, quasi-injective modules [**6**, p. 16].

The next theorem is the main result of this section.

Theorem 3.2. *Let* A *be a* τ*-completely decomposable module that satisfies* (C_2)*. Then any direct summand of* A *is* τ*-completely decomposable.*

PROOF: Let $A = \bigoplus_{i \in I} A_i$, where each A_i is a minimal τ-injective submodule of A and let B be a non-zero proper direct summand of A. Since each A_i is uniform and B is not essential in A, there exists $k \in I$ such that $B \cap A_k = 0$ [**6**, p. 38]. By Zorn's Lemma, there exists a maximal subset $J \subseteq I$ such that $B \cap \left(\bigoplus_{j \in J} A_j\right) = 0$. Let $p : A \to \bigoplus_{i \in I \setminus J} A_i$ be the natural projection. Then the restriction $p|_B$ is a monomorphism, we have $p(B) \cong B$. Since A satisfies (C_2), $p(B)$ is a direct summand of A and, consequently, of $\bigoplus_{i \in I \setminus J} A_i$. If $p(B) \neq \bigoplus_{i \in I \setminus J} A_i$, then there exists $h \in I \setminus J$ such that $p(B) \cap A_h = 0$ [**6**, p. 38], we have $B \cap \left(A_h \oplus \left(\bigoplus_{j \in J} A_j\right)\right) = 0$, which contradicts the maximality of J. Hence $p(B) = \bigoplus_{i \in I \setminus J} A_i$. Thus B is τ-completely decomposable. ∎

In order to obtain some interesting consequences of Theorem 3.2, we need the following proposition.

PROPOSITION 3.3.

(i) *Every τ-torsionfree τ-completely decomposable module is quasi-injective.*

(ii) *Every τ-complemented τ-injective module is quasi-injective.*

PROOF: (i) Let A be a τ-torsionfree τ-completely decomposable module, say $A = \bigoplus_{i \in I} A_i$, where each A_i is a minimal τ-injective submodule of A. Also let $f : E(A) \to E(A)$ be a homomorphism. We may suppose that $f \neq 0$. Following the proof of Proposition 2.1, there exists a subset $K \subseteq I$ such that $f(A) = \sum_{k \in K} f(A_k)$, where each A_k is τ-cocritical τ-injective, that is, τ-torsionfree minimal τ-injective. Since $f(A_k) \cap A \neq 0$, we may choose a non-zero element $x \in f(A_k) \cap A$. Then there exists a finite subset $J \subseteq I$ such that $Rx \subseteq f(A_k) \cap \left(\bigoplus_{i \in J} A_i\right)$. But $\bigoplus_{i \in J} A_i$ is τ-injective, hence $E_\tau(Rx) \subseteq \bigoplus_{i \in J} A_i$. Clearly, $E_\tau(Rx)$ is τ-injective τ-cocritical. But $f(A_k) \cap E_\tau(Rx) \neq 0$, hence $f(A_k) = E_\tau(Rx)$. Thus, $f(A_k)$ is a submodule of A. Therefore, $f(A) \subseteq A$ and, consequently, A is quasi-injective.

(ii) Let A be a τ-complemented τ-injective module. Also, let B be a submodule of A, $f : B \to A$ a homomorphism and $i : B \to A$ the inclusion. Since $E_\tau(B)/B$ is τ-torsion and A is τ-injective, there exists a homomorphism $g : E_\tau(B) \to A$ extending f. Since A is τ-complemented τ-injective, by Proposition 2.2 there exists a submodule D of A such that $A = E_\tau(B) \oplus D$. If $h = g \oplus 1_D : A \to A$, then $hi = f$. Thus, A is quasi-injective. ∎

COROLLARY 3.4. *Let A be a τ-completely decomposable module. If one of the following extra conditions on A holds:*

(i) *A is continuous;*

(ii) *A is τ-torsionfree;*

(iii) *A is τ-injective;*

(iv) *A is finitely τ-completely decomposable;*

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then any direct summand of A is τ-completely decomposable.

PROOF: If (i) holds, apply Theorem 3.2. If (ii) holds, use Proposition 3.3 and the result for (i). If (iii) holds, apply Propositions 2.1 and 3.3 and the result for (i). If (iv) holds, note that the class of τ-injective modules is closed under finite direct sums and apply the result for (iii). ∎

For completeness, we shall give one more result, whose proof follows the idea of the corresponding one given for indecomposable injective modules [**8**, Proposition 6.2]. In order to complete the proof, an auxiliary lemma is needed.

LEMMA 3.5. *Let A be a τ-completely decomposable module and let B be a τ-injective submodule of A. Then B is a τ-complemented direct summand of A.*

PROOF: By Proposition 2.1, A is τ-complemented. Then by [**18**, Theorem 1.8], [**20**, Theorem 4] and Proposition 2.3, it follows easily that $E_\tau(A)$ is τ-complemented. Then by Proposition 2.2, B is a direct summand of $E_\tau(A)$ and, consequently, of A. Use again Proposition 2.2 to get immediately that B is τ-complemented. ∎

THEOREM 3.6. *Let A be a τ-completely decomposable module. Then:*

(i) *If B is a direct summand of A and C is a finitely generated submodule of B, then B contains a finitely τ-completely decomposable τ-injective hull of C.*

(ii) *Any countably generated direct summand of A is τ-completely decomposable.*

PROOF: Let $A = \bigoplus_{i\in I} A_i$, where each A_i is a minimal τ-injective submodule of A.

(i) Since C is finitely generated, there exists a finite subset $J \subseteq I$ such that $C \subseteq \bigoplus_{j\in J} A_j$. Then $E_\tau(C) \subseteq \bigoplus_{j\in J} A_j$ and by Lemma 3.5, $E_\tau(C)$ is a direct summand of $\bigoplus_{j\in J} A_j$. Then by Corollary 3.4, $E_\tau(C)$ is finitely τ-completely decomposable. Now let $p : A \to B$ be the canonical projection. Then $p|_C$ and thus $p|_{E_\tau(C)}$ is a monomorphism. Hence $p\big(E_\tau(C)\big) \cong E_\tau(C)$ is a finitely τ-completely decomposable τ-injective hull of C.

(ii) Let D be a countably generated direct summand of A and let $d_1, \dots, d_n, \dots$ be a countable generating set of D. By (i), for each $n \geqslant 1$, there exists a finitely τ-completely decomposable module D_n with $d_1, \dots, d_n \in D_n$. Then each D_n is τ-injective. By Lemma 3.5, each D_n is a direct summand of A and, consequently, of D. But $D = \cup_{n\geqslant 1} D_n$. Setting $D_0 = 0$, we have $D \cong \bigoplus_{n\geqslant 0} D_{n+1}/D_n$, a direct sum of finitely τ-completely decomposable modules. Thus, D is τ-completely decomposable. ∎

4. ESSENTIAL EXTENSIONS OF τ-COMPLETELY DECOMPOSABLE MODULES

This section deals with Problem 2 from the introduction. For a commutative Noetherian ring R that is not a domain, we determine those torsion theories on R-Mod having the property that every τ-injective module is an essential extension of a (τ-injective) τ-completely decomposable module or equivalently of a τ-complemented τ-injective module (see Theorem 2.7).

Previously, for some particular hereditary torsion theories τ, Masaike and Horigome [**12**], Bueso, Jara and Torrecillas [**1**] established conditions on the ring R under which every τ-injective module is an essential extension of a τ-injective τ-completely decomposable module, but the flavor of their work is different. Thus, they proved the following results, the second one refining the first one:

THEOREM 4.1. [**12**, Theorem 2] *Let τ be a perfect torsion theory. The following conditions are equivalent:*

(i) *Every τ-injective R-module is an essential extension of a τ-injective τ-completely decomposable R-module.*

(ii) *R has ACC on τ-dense left ideals and the ring of quotients R_τ of R at τ is left semiartinian.*

THEOREM 4.2. [**1**, Proposition 2.4] *Let τ be a Noetherian torsion theory. The following conditions are equivalent:*

(i) *Every τ-injective R-module is an essential extension of a τ-injective τ-completely decomposable R-module.*

(ii) *R has ACC on τ-dense left ideals and R is τ-semiartinian.*

For the rest of this section we shall assume the ring R to be commutative.

We mention first an auxiliary result, whose proof is straightforward, and that will be used freely onwards.

LEMMA 4.3. ([**3**, Lemma 2.1]) *If A is a τ-cocritical R-module, then:*

(i) *$\operatorname{Ann}_R a = \operatorname{Ann}_R A$ for every non-zero element $a \in A$;*

(ii) *$\operatorname{Ann}_R A$ is a prime ideal of R;*

(iii) *$R/\operatorname{Ann}_R A$ is τ-cocritical.*

In what follows, let $\mathcal{P}$ be a non-empty set of minimal prime ideals of R. Denote by $\mathcal{A}_\mathcal{P}$ the class of all modules isomorphic to factor modules U/V, where U and V are ideals of R containing a non-zero prime ideal $q \notin \mathcal{P}$. Then $\mathcal{A}_\mathcal{P}$ generates a hereditary torsion theory, that will be denoted by $\tau_\mathcal{P}$.

PROPOSITION 4.4. *Let p be a non-zero prime ideal of R such that R/p is a $\tau_\mathcal{P}$-Noetherian R-module. Then R/p is $\tau_\mathcal{P}$-cocritical if and only if $p \in \mathcal{P}$.*

PROOF: Suppose first that R/p is $\tau_\mathcal{P}$-cocritical. If $p \notin \mathcal{P}$, then R/p is $\tau_\mathcal{P}$-torsion by the definition of $\tau_\mathcal{P}$, a contradiction. Hence $p \in \mathcal{P}$.

Suppose now that $p \in \mathcal{P}$. Assume that R/p is $\tau_\mathcal{P}$-torsion. Then R/p contains a non-zero submodule $A \cong U/V$, where U and V are ideals of R containing a non-zero prime ideal $q \notin \mathcal{P}$. We have $q \setminus p \neq \emptyset$, say $r \in q \setminus p$. Hence, if $0 \neq a \in A$, then $r \in \operatorname{Ann}_R a = p$, a contradiction. Therefore, R/p is not $\tau_{\mathcal{P}}$-torsion. Since R/p is a $\tau_{\mathcal{P}}$-Noetherian, there exists an ideal t of R such that $p \subseteq t$ and R/t is $\tau_{\mathcal{P}}$-cocritical [**10**, Proposition 20.3]. But then $t = \operatorname{Ann}_R(R/t)$ is a prime ideal and if $t \neq p$, then $t \notin \mathcal{P}$, hence R/t is $\tau_{\mathcal{P}}$-torsion, a contradiction. Thus, $t = p$ and R/p is $\tau_{\mathcal{P}}$-cocritical. □

THEOREM 4.5. *Let R be Noetherian, but not a domain. Then the following statements are equivalent:*

(i) *Every τ-injective module is an essential extension of a (τ-injective) τ-completely decomposable R-module.*

(ii) *τ is either the improper torsion theory χ (that is, every module is τ-torsion) or $\tau = \tau_{\mathcal{P}}$ for some non-empty set $\mathcal{P}$ of minimal prime ideals of R.*

PROOF: By the hypotheses and Theorem 2.7, τ-completely decomposable modules and τ-complemented τ-injective modules coincide.

(i)$\Longrightarrow$ (ii) Suppose that τ is proper. Then there exists a τ-cocritical module A [**10**, p. 486]. It follows that $E_\tau(A) \cong E_\tau(R/p)$ for some non-zero prime ideal p of R [**3**, Proposition 2.3].

We show first that p is a minimal prime ideal. Suppose the contrary. Then there exists a prime ideal q of R such that $q \subset p$. Since R/p is τ-torsionfree, R/q is τ-torsionfree. Moreover, R/q cannot be τ-cocritical, because otherwise $R/p \cong (R/q)/(p/q)$ would be τ-torsion. On the other hand, $E_\tau(R/q)$ is an essential extension of a τ-complemented τ-injective module B. Since B is uniform, it has to be minimal τ-injective, because otherwise, if D were a non-zero proper τ-injective submodule of B, then D would be a direct summand by Proposition 2.2. Furthermore, B is also τ-torsionfree and, consequently, τ-cocritical τ-injective. Since $B \trianglelefteq E_\tau(R/q)$, there exists a non-zero element $b \in B \cap R/q$. We have $\operatorname{Ann}_R B = \operatorname{Ann}_R b = q$ and R/q is τ-cocritical, a contradiction. Therefore, p is minimal.

Denote by $\mathcal{P}$ the set of all minimal prime ideals s of R such that $E_\tau(R/s)$ is τ-cocritical. Note that $\mathcal{P}$ is non-empty, since $p \in \mathcal{P}$.

Let us now show that τ-torsion and $\tau_{\mathcal{P}}$-torsion modules coincide.

Let M be a τ-torsion module. By the hypotheses on R, every torsion theory is stable, hence we have $E_\tau(M) = E(M) = \bigoplus_{i \in I} E(R/p_i)$, where each $p_i \notin \mathcal{P}$ is a (non-zero) prime ideal of R. Then $R/p_i \in \mathcal{A}_{\mathcal{P}}$, thus it follows immediately that M is $\tau_{\mathcal{P}}$-torsion. Hence, every τ-torsion module is $\tau_{\mathcal{P}}$-torsion.

Now let $N \in \mathcal{A}_{\mathcal{P}}$. Then $N \cong U/V$, where U and V are ideals of R containing a (non-zero) prime ideal $t \notin \mathcal{P}$. Suppose that R/t is τ-torsionfree. By hypothesis, $E_\tau(R/t)$ is an essential extension of a τ-complemented τ-injective module C. Repeating the above arguments, it follows that R/t is τ-cocritical, which contradicts the choice of t. Then R/t is τ-torsion. Hence R/V and, consequently, $N \cong U/V$ is τ-torsion. Thus, every $\tau_{\mathcal{P}}$-torsion module is τ-torsion. Therefore, $\tau = \tau_{\mathcal{P}}$.

(ii)$\Longrightarrow$ (i) Suppose first that $\tau = \chi$, that is, every module is τ-torsion. Then every module is τ-complemented and the result follows.

Suppose now that $\tau = \tau_{\mathcal{P}}$, for some non-empty set $\mathcal{P}$ of minimal prime ideals of R. Let A be a τ-injective module. By the stability of τ we may write $A = T_\tau(A) \oplus C$, where C is τ-torsionfree τ-injective [**10**, Proposition 8.6]. Clearly, $T_\tau(A)$ is τ-complemented τ-injective, hence τ-completely decomposable. By the hypotheses on R, we have $E(C) = \bigoplus_{i\in I} E(R/p_i)$, where each p_i is a (non-zero) prime ideal of R. Then $E(R/p_i)$ is τ-torsionfree, hence $p_i \in \mathcal{P}$ for every $i \in I$. Now let $i \in I$. By Proposition 4.4, R/p_i is τ-cocritical, we have $E_\tau(R/p_i)$ is minimal τ-injective. Thus, $\bigoplus_{i\in I} E_\tau(R/p_i)$ is τ-completely decomposable.

We have $C \cap E_\tau(R/p_i) \neq 0$. Then $E_\tau(R/p_i)/(C \cap E_\tau(R/p_i)$ is both τ-torsion, because $E_\tau(R/p_i)$ is τ-cocritical, and τ-torsionfree, because

$$E_\tau(R/p_i)/\big(C \cap E_\tau(R/p_i)\big) \cong \big(C + E_\tau(R/p_i)\big)/C \subseteq E(C)/C\,.$$

Hence $E_\tau(R/p_i) \subseteq C$ and thus $\bigoplus_{i\in I} E_\tau(R/p_i) \trianglelefteq C$. Now A is an essential extension of the τ-completely decomposable module $T_\tau(A) \oplus \Big(\bigoplus_{i\in I} E_\tau(R/p_i)\Big)$. ∎

Corollary 4.6. *Let R be Noetherian, but not a domain. Consider $\mathcal{P}$ to be the set of all minimal prime ideals of R. Then every $\tau_{\mathcal{P}}$-injective module A is isomorphic to an essential extension of*

$$\Big(\bigoplus_{i\in I} E(R/p_i)\Big) \oplus \Big(\bigoplus_{j\in J} E_{\tau_{\mathcal{P}}}(R/q_j)\Big)\,,$$

where each p_i and each q_j is a prime ideal of R. Moreover, each $p_i \notin \mathcal{P}$ and each $q_j \in \mathcal{P}$.

Proof: By Theorem 4.5, A is an essential extension of a $\tau_{\mathcal{P}}$-completely decomposable module, that is, of a direct sum of minimal $\tau_{\mathcal{P}}$-injective modules. First, every $\tau_{\mathcal{P}}$-torsion minimal $\tau_{\mathcal{P}}$-injective module is of the form $E_{\tau_{\mathcal{P}}}(D)$ for some uniform $D \in \mathcal{A}_{\mathcal{P}}$ [**3**, Proposition 2.2]. Since R is commutative Noetherian, $E_{\tau_{\mathcal{P}}}(D) = E(D)$ and then, $E_{\tau_{\mathcal{P}}}(D) \cong E(R/p)$ for some prime ideal p of R. Secondly, every τ-torsionfree minimal τ-injective module is isomorphic to $E_\tau(R/q)$ for some prime ideal q of R [**3**, Proposition 2.3]. Hence, every $\tau_{\mathcal{P}}$-injective module A is isomorphic to an essential extension of $\Big(\bigoplus_{i\in I} E(R/p_i)\Big) \oplus \Big(\bigoplus_{j\in J} E_{\tau_{\mathcal{P}}}(R/q_j)\Big)$, where each p_i and each q_j is a prime ideal of R. Moreover, each $E_{\tau_{\mathcal{P}}}(R/q_j)$ is $\tau_{\mathcal{P}}$-cocritical, hence each R/q_j is $\tau_{\mathcal{P}}$-cocritical. Then by Proposition 4.4, each $q_j \in \mathcal{P}$ and each $p_i \notin \mathcal{P}$. ∎

Remark. The hypothesis on R not to be a domain is essential in Theorem 4.5. Indeed, suppose that R is a domain and consider $\tau = \tau_{\mathcal{P}}$ for some non-empty set $\mathcal{P}$ of minimal prime ideals of R. Clearly $E_\tau(R)$ is τ-torsionfree and $\operatorname{Ann}_R x = 0$ for every $0 \neq x \in E_\tau(R)$. Suppose that $E_\tau(R)$ is an essential extension of a τ-completely decomposable (or equivalently τ-complemented τ-injective) module A. Then, since A is uniform, A has to be minimal τ-injective, hence τ-cocritical. It follows that $\mathrm{Ann}_R A = p$, where $p \in \mathcal{P}$, because R/p is τ-cocritical by Proposition 4.4. Hence $\mathrm{Ann}_R A \neq 0$, a contradiction. Thus, $E_\tau(R)$ is a τ-injective module that is not an essential extension of any τ-completely decomposable module.

References

[1] J.L. Bueso, P. Jara and B. Torrecillas, 'Decomposition of injective modules relative to a torsion theory', *Israel J.Math.* **52** (1985), 266–272.

[2] I.S. Cohen and I. Kaplansky, 'Rings for which every module is a direct sum of cyclic modules', *Math. Z.* **54** (1951), 97–101.

[3] S. Crivei, 'On τ-injective hulls of modules', *Publ. Math. Debrecen* **61** (2002), 145–155.

[4] S.E. Dickson, 'A torsion theory for abelian categories', *Trans. Amer. Math. Soc.* **121** (1966), 223–235.

[5] N.V. Dung and A. Facchini, 'Direct summands of serial modules', *J. Pure Appl. Algebra* **133** (1998), 93–106.

[6] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, *Extending modules*, Pitman Research Notes in Mathematics Series **133** (Longman Scientific and Technical, Harlow, 1994).

[7] A. Facchini, *Module theory. Endomorphism rings and direct sum decompositions in some classes of modules* (Birkhäuser Verlag, Basel, Boston, Berlin, 1998).

[8] C. Faith and E. Walker, 'Direct sum representations of injective modules', *J. Algebra* **5** (1967), 203–221.

[9] J.L. García, *Condiciones de finitud relativas a topologías aditivas*, Ph.D. thesis (in Spanish) (Universidad de Murcia, 1983).

[10] J.S. Golan, *Torsion theories* (Longman Scientific and Technical, New York, 1986).

[11] U.S. Kahlon, 'Problem of Krull-Schmidt-Remak-Azumaya-Matlis', *J. Indian Math. Soc. (N.S.)* **35** (1971), 255–261.

[12] K. Masaike and T. Horigome, 'Direct sum of σ-injective modules', *Tsukuba J. Math.* **4** (1980), 77–81.

[13] E. Matlis, 'Injective modules over noetherian rings', *Pacific J. Math.* **8** (1958), 511–528.

[14] S. Mohamed, B.J. Müller and S. Singh, 'A note on the decomposition of σ-injective module', *Comm. Algebra* **12** (1984), 663–672.

[15] S. Mohamed and S. Singh, 'Decomposition of σ-injective modules', *Comm. Algebra* **9** (1981), 601–611.

[16] S.S. Page and Y. Zhou, 'Direct sums of injective modules and chain conditions', *Canad. J. Math.* **46** (1994), 634–647.

[17] D.W. Sharpe and P. Vámos, *Injective modules* (Cambridge University Press, London, New York, 1972).

[18] P.F. Smith, A.M. Viola-Prioli and J.E. Viola-Prioli, 'Modules complemented with respect to a torsion theory', *Comm. Algebra* **25** (1997), 1307–1326.

[19] P.F. Smith, A.M. Viola-Prioli and J.E. Viola-Prioli, 'On μ-complemented and singular modules', *Comm. Algebra* **25** (1997), 1327–1339.

[20] A.M. Viola-Prioli and J.E. Viola-Prioli, 'The smallest closed subcategory containing the μ-complemented modules', *Comm. Algebra* **28** (2000), 4971–4980.

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