7

Asymptotic simplicity

The analysis of the conformal structure of exact solutions carried out in Chapter 6 exhibited a number of common features among the various spacetimes considered. The most conspicuous one is that they all admit a smooth conformal extension which attaches a boundary to the spacetime. This *conformal boundary* represents *points at infinity*. It is natural to ask whether this property is shared by a larger class of spacetimes. This question leads to the notion of *asymptotic simplicity*. In formulating this notion one tries to strike a delicate balance: the definition should be strong enough so that it excludes clearly *pathological* situations, but at the same time it should leave enough room to include interesting spacetimes that go beyond the obvious explicit examples. The original definition of asymptotic simplicity is due to Penrose (1963, 1964, 1965). This definition has had a lasting influence on the field of mathematical relativity, in general, and in the applications of conformal methods to the analysis of global properties of spacetimes, in particular.

7.1 Basic definitions

The following definition of asymptotic simplicity is adapted from Hawking and Ellis (1973):

Definition 7.1 (***asymptotically simple spacetimes***) *A spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *is said to be* ***asymptotically simple*** *if there exists a smooth, oriented, time-oriented, causal*[1] *spacetime* $(\mathcal{M}, \boldsymbol{g})$ *and a smooth function* Ξ *on* $\mathcal{M}$ *such that:*

(i) $\mathcal{M}$ *is a manifold with boundary* $\mathscr{I} \equiv \partial\mathcal{M}$.
(ii) $\Xi > 0$ *on* $\mathcal{M} \setminus \mathscr{I}$, *and* $\Xi = 0$, $\mathbf{d}\Xi \neq 0$ *on* $\mathscr{I}$.

[1] A ***causal spacetime*** is one in which there exist no closed timelike or null (i.e. causal) curves; see also Chapter 14.

(iii) There exists an embedding $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ *such that* $\varphi(\tilde{\mathcal{M}}) = \mathcal{M} \setminus \mathscr{I}$ *and*

$$\varphi^* \boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}.$$

(iv) Each null geodesic of $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *acquires two distinct endpoints on* $\mathscr{I}$*.*

The spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is called the ***physical spacetime***, while $(\mathcal{M}, \boldsymbol{g})$ is known as the ***unphysical spacetime***. The boundary $\mathscr{I}$ is generally known as ***conformal infinity***. In the cases where $\mathscr{I}$ corresponds to a null hypersurface one calls it ***null infinity***. More informally, $\mathscr{I}$ is also called *scri*[2] – a shortened version of *script I*. In a slight abuse of notation, one usually identifies $\tilde{\mathcal{M}}$ and $\mathcal{M} \setminus \mathscr{I}$ so that one writes $\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}$; see, for example, the examples discussed in Chapter 6. In what follows phrases like "***at infinity***" are to be understood as meaning *in a suitable neighbourhood* of $\mathscr{I}$ in $\mathcal{M}$.

Definition 7.1 allows for a non-vanishing matter content. Spacetimes for which, in addition, one has that $R_{ab} = 0$ in a neighbourhood of $\mathscr{I}$ in $\varphi^{-1}(\mathcal{M})$ are sometimes called ***asymptotically empty and simple***.

Remarks

(a) **Restriction on the conformal class.** Definition 7.1 imposes restrictions only on the conformal class of the admissible spacetimes $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. It does not single out any specific conformal representation; that is, it does not provide a *canonical* unphysical spacetime $(\mathcal{M}, \boldsymbol{g})$.

(b) **Conformal infinity is a hypersurface.** The boundary $\mathscr{I}$ as introduced in point (i) in Definition 7.1 is a well-defined three-dimensional hypersurface of $\mathcal{M}$ with normal given by $\mathbf{d}\Xi$. In particular, sets where $\mathbf{d}\Xi = 0$ – such as spatial infinity i^0 and the timelike infinities $i^{\pm}$ of the Minkowski spacetime – are excluded from $\mathscr{I}$. Points of this type, if present, will still be regarded as belonging to the conformal boundary but will be treated separately.

(c) **Conformal infinity is at infinity.** Points (ii) and (iii) of Definition 7.1 ensure that the boundary $\mathscr{I}$ shares the key properties of the null infinity of the Minkowski, de Sitter and anti-de Sitter spacetimes. To see that this is the case one needs to analyse the behaviour of null geodesics. The transformation behaviour of null geodesics under conformal rescalings has already been discussed in Section 5.5. In what follows, let $\tilde{\mathrm{s}}$ and s denote, respectively, $\tilde{\boldsymbol{g}}$-affine and $\boldsymbol{g}$-affine parameters of a null geodesic $\gamma \subset \tilde{\mathcal{M}}$. It follows then that $\tilde{\mathrm{s}}$ and s are related to each other by the equation

$$\frac{\mathrm{d}\tilde{\mathrm{s}}}{\mathrm{d}\mathrm{s}} = \frac{1}{\Xi^2}.$$

Without loss of generality, one can choose the unphysical affine parameter s to vanish at $\mathscr{I}$; that is, $\Xi = O(\mathrm{s}^{\alpha})$ along the null geodesic with $\alpha > 0$.

[2] Remarkably, the word *scri* is pronounced in the same way as the Polish word *scraj* meaning *boundary*.

Now, as $\mathbf{d}\Xi \neq 0$ at $\mathscr{I}$, one concludes that, in fact, $\alpha \geq 1$. Hence, $\tilde{s} \to \infty$ as $\Xi \to 0$ – that is, from the *physical point of view (as measured by the affine parameter* $\tilde{s}$*) the null geodesic never reaches the conformal boundary* $\mathscr{I}$. Thus, the conformal boundary lies at infinity from the perspective of the physical metric $\tilde{\boldsymbol{g}}$.

(d) **Smoothness of the conformal extension and decay.** As will be discussed in Chapter 10 the *smoothness assumption* in Definition 7.1 imposes a sharp decay behaviour on the gravitational field at infinity – in particular, it leads to what is known as the ***peeling behaviour*** of the Weyl tensor. There are variations of Definition 7.1 in which the smoothness requirement is relaxed to admit conformal extensions of class C^k for some suitable positive integer k; see Penrose and Rindler (1986). The physical relevance of these weaker regularity conditions is a delicate technical point which cannot be satisfactorily assessed by just looking at specific examples. These weaker regularity conditions lead to a different asymptotic behaviour of the gravitational field.

(e) **Matter and causal nature of null infinity.** As already mentioned, Definition 7.1 allows for the presence of matter in the spacetime. If the energy-momentum tensor of the matter models has a suitable decay at infinity, then the causal nature of $\mathscr{I}$ is fixed by the sign of the cosmological constant λ: it is spacelike if $\lambda < 0$, null if $\lambda = 0$ and timelike if $\lambda > 0$; see Theorem 10.1.

(f) **The completeness requirement.** Point (iv) in Definition 7.1 is a completeness condition which, in particular, excludes spacetimes such as the Schwarzschild solution in which there exist null geodesics which do not reach $\mathscr{I}$ – not only those falling into the black hole region, but also those lying in the ***photon sphere*** at $r = 3m$; see Wald (1984).

(g) **Regular solutions which are not asymptotically simple.** That a spacetime is smooth and geodesically complete is not a guarantee that it admits a smooth conformal extension. An example of this is given by the so-called *Nariai spacetime* described by

$$\tilde{\mathcal{M}} = \mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2), \qquad \tilde{\boldsymbol{g}} = \mathbf{d}t \otimes \mathbf{d}t - \cosh^2 t \mathbf{d}\psi \otimes \mathbf{d}\psi - \boldsymbol{\sigma}, \tag{7.1}$$

which is a solution to the vacuum Einstein field equations with $\lambda = -1$. In addition to being geodesically complete, the Nariai spacetime is also *globally hyperbolic*; see Section 14.1. Remarkably, *the Nariai spacetime does not even admit a patch of a conformal boundary.* To see this, assume one has a conformal extension with the required properties. The standard conformal transformation laws imply that

$$\tilde{C}_{abcd}\tilde{C}^{abcd} = \Xi^4 C_{abcd}C^{abcd}.$$

Thus, if the solution admits a smooth conformal extension, then $\tilde{C}_{abcd}\tilde{C}^{abcd} = 0$. On the other hand, a direct computation with the line

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element (7.1) shows that $\tilde{C}_{abcd}\tilde{C}^{abcd} = \text{constant} \neq 0$. This is a contradiction, and accordingly there cannot exist a piece of conformal boundary which is C^2. This argument is adapted from Friedrich (2015a); an alternative topological argument has been given in Beyer (2009a).

In order to consider spacetimes for which the completeness condition (iv) in Definition 7.1 does not hold, one introduces the further notion of ***weakly asymptotically simple spacetimes***, that is, spacetimes whose *asymptotic region* is diffeomorphic to that of an asymptotically simple spacetime. More precisely, one has the following:

Definition 7.2 (***weakly asymptotically simple spacetimes***) *A spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *is said to be* ***weakly asymptotically simple*** *if there exists an asymptotically simple spacetime* $(\tilde{\mathcal{M}}', \tilde{\boldsymbol{g}}')$ *and a neighbourhood* $\mathcal{U}'$ *of* $\mathscr{I}' \equiv \partial\mathcal{M}'$ *such that* $\varphi^{-1}(\mathcal{U}') \cap \tilde{\mathcal{M}}'$ *is isometric to an open subspace* $\tilde{\mathcal{U}}$ *of* $\tilde{\mathcal{M}}$.

Basic examples of asymptotically simple spacetimes have been given in Chapter 6. Notoriously, all the given examples are time independent. More generally, it can be shown that stationary solutions to the vacuum equations $R_{ab} = 0$ with a suitable behaviour at infinity are at least weakly asymptotically simple; see Damour and Schmidt (1990) and Dain (2001b). Thus, it is natural to ask whether there are *dynamic* solutions to the Einstein field equations. At the level of exact solutions, the closest examples are given by the spacetimes known as ***boost-rotation symmetric spacetimes*** – see, for example, Bičák and Schmidt (1989), Bičák (2000) and Griffiths and Podolský (2009) – and in particular the so-called ***C-metric*** – see Ashtekar and Dray (1981). All these exact solutions contain some pathologies (e.g. naked singularities, piercing of null infinity) which prevent them from being true examples of asymptotically simple spacetimes.

A detailed discussion of the properties of asymptotic simple spacetimes requires the conformal Einstein field equations and is deferred to Chapter 10.

7.2 Other related definitions

The definition of asymptotic simplicity makes neither reference to nor restricts the behaviour of the conformal spacetime $(\mathcal{M}, \boldsymbol{g})$ at spatial infinity. Several authors have introduced more refined definitions of asymptotic simplicity in which further requirements on the behaviour of the gravitational field at null infinity are prescribed, as in, for example, Persides (1979), or at spatial infinity, as in the concept of ***asymptotically empty and flat spacetime at null and spatial infinity*** of Ashtekar and Hansen (1978); see also Persides (1980). Similar ideas have been pursued by a number of authors in an attempt to analyse the structure of timelike infinity; see, for example, Persides (1982a,b), Porrill (1982), Cutler (1989) and Herberthson and Ludwig (1994).

The aim of the definitions mentioned in the previous paragraph and similar other proposals for the analysis of the asymptotic structure of spacetime is to identify a *minimal* number of assumptions on the asymptotic structure which, in turn, can be used to develop a formalism to construct physical and geometrical notions of interest. A critique to this approach is that, a priori, they do not provide any information on the genericity of the assumptions or on the size of the class of spacetimes they contain. Moreover, it is not clear how these spacetimes can be constructed. As pointed out in Geroch (1976), pages 3–4:

> Conditions too strong will have the effect of eliminating solutions which would seem clearly to represent isolated systems; conditions too weak may have the effect of admitting too many solutions, or what is worse, may result in a structure which is so weak that potentially useful aspects of the asymptotic behaviour of one's fields are lost in a sea of bad behaviour ⋯ There are no correct or incorrect definitions, only more or less useful ones.

The point of view pursued in this book is that rather than making assumptions on the nature of spatial, null or timelike infinity, the structures of the conformal boundary of a spacetime should arise as a result of the evolution of some initial data set for the Einstein field equations.

7.3 Penrose's proposal

Asymptotically empty and simple spacetimes (i.e. spacetimes with a vanishing cosmological constant and matter suitably decaying at infinity) play an important role in the approach to the analysis of ***isolated systems*** in general relativity put forward by Penrose (1963, 1964). The notion of an isolated system is a convenient idealisation of astrophysical systems where the effects of the cosmological expansion are ignored. This notion allows one to define concepts of clear physical interest such as the total energy of a system or its mass loss due to gravitational radiation. Intuitively, an isolated system should behave asymptotically like the Minkowski spacetime. Penrose, based on earlier work by Bondi et al. (1962) and Sachs (1962b), takes this idea further; see also Friedrich (2002, 2004):

Penrose's proposal. *Far fields of isolated gravitating systems behave like those of asymptotically simple spacetimes in the sense that they can be smoothly extended to null infinity after a suitable conformal rescaling.*

In other words, if a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ describes an isolated system, then it should be weakly asymptotically simple.

As pointed out in Remark (d) earlier in the chapter, the requirement of smoothness results in a very definite decay behaviour of the gravitational field at infinity. Whether this behaviour is actually realised in solutions to the Einstein equations, and if so to what extent, is a delicate question which will be analysed in later chapters of this book.

7.4 Further reading

The literature on asymptotic simplicity and other definitions of asymptotic flatness is dauntingly vast and is best accessed through reviews. There are a good number of references covering various periods and aspects of the topic. Penrose (1964, 1967) gives an overview of the early ideas and results on asymptotic simplicity; Geroch (1976) provides a good discussion on the physical motivation of the study of isolated systems in general relativity, the notion of conserved quantities and asymptotic symmetries; Schmidt (1978), Newman and Tod (1980) and Ashtekar (1980, 1984) provide alternative discussions of these topics and Friedrich (1992, 1998a, 1999) provides reviews of the notion of asymptotic simplicity from the point of view of the conformal Einstein field equations and the construction of global solutions. More recent reviews of the topic can be found in Frauendiener (2004), Ashtekar (2014) and Friedrich (2015a).