Glasgow Math. J. **54** (2012) 599–603.
doi:10.1017/S0017089512000171.

ON KELLOGG'S THEOREM FOR QUASICONFORMAL MAPPINGS

DAVID KALAJ
University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinjski put b.b. 81000 Podgorica, Montenegro
e-mail: davidk@ac.me

(Received 22 May 2011; revised 28 October 2011; accepted 21 December 2011; first published online 30 March 2012)

Abstract. We give some extensions of classical results of Kellogg and Warschawski to a class of quasiconformal (q.c.) mappings. Among the other results we prove that a q.c. mapping f, between two planar domains with smooth $C^{1,\alpha}$ boundaries, together with its inverse mapping f^{-1}, is $C^{1,\alpha}$ up to the boundary if and only if the Beltrami coefficient μ_f is uniformly α Hölder continuous ($0 < \alpha < 1$).

2010 *Mathematics Subject Classification.* Primary 30C62, Secondary 30C20.

1. Introduction and notation.

1.1. Quasiconformal mappings. Let D and Ω be subdomains of the complex plane $\mathbf{C}$.

We say that a function $w : D \to \mathbf{C}$ is ACL (absolutely continuous on lines) in the region D, if for every closed rectangle $R \subset D$ with sides parallel to the x and y-axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R. Such a function has of course, partial derivatives w_x, w_y a.e. in D.

A sense-preserving homeomorphism $w: D \to \Omega$, where D and Ω are subdomains of the complex plane $\mathbf{C}$, is said to be K-quasiconformal (K-q.c.), with $K \geqslant 1$, if w is ACL in D and

$$|w_{\bar{z}}| \leqslant k|w_z| \quad \text{a.e. on } D,$$

where $k = (K-1)/(K+1)$ (cf. the Ahlfors book [**1**, pp. 23–24]). See also the book of Lehto and Virtanen [**3**] for good setting of quasiconformal mappings.

It is well known that an orientation preserving quasiconformal mapping $f : \Omega \mapsto \Omega' \subset \mathbb{C}$ of a planar domain is a solution to the Beltrami equation:

$$f_{\bar{z}}(z) = \mu(z) f_z(z), \tag{1.1}$$

where $\mu(z)$, a measurable function in Ω with $\|\mu\|_\infty < 1$, is called the Beltrami coefficient or the complex dilatation of f. We recommend the book [**2**] of Astala Iwaniec and Martin, where authors establish all the classical results in a modern setting and discuss future development and applications of the theory of the planar Beltrami equation.

The starting point of this note is the following classical result.

PROPOSITION 1.1 (Kellogg ($n = 1$) see [**6**, **8**] and Warshawski ($n > 1$), [**13**, **14**]). *Let $n \in \mathbb{N}$, $0 < \alpha \leqslant 1$. If Ω and Ω' are Jordan domains having $\mathscr{C}^{n,\alpha}$ boundaries and ω is a conformal mapping of Ω onto Ω', then $\omega^{(n)} \in \mathscr{C}^{\alpha}(\Omega)$ and $(\omega^{-1})^{(n)} \in \mathscr{C}^{\alpha}(\Omega')$.*

For a function $\xi \in C^{\alpha}(\Omega)$, i.e. a function $\xi : \Omega \to \mathbf{C}$ satisfying the condition

$$\mathbf{Lip}_{\alpha}(\xi) := \sup_{z\neq w, z, w\in\Omega} \frac{|\xi(z) - \xi(w)|}{|z - w|^{\alpha}} < \infty$$

we say that is a uniformly $\alpha-$Hölder continuous function. From now one, instead of $\omega^{(n)} \in \mathscr{C}^{\alpha}(\Omega)$ we write $\omega \in \mathscr{C}^{n,\alpha}(\Omega)$. In the similar way, we define the class $\mathscr{C}^{n,\alpha}(\Omega)$ of non-necessarily conformal mappings. The theorem of Kellogg and of Warshawski has been extended in various directions, see for example the extension to minimal surfaces by Nitsche [**10**], and to q.c. harmonic mappings w.r. hyperbolic metric by Tam and Wan [**12**, Theorem 5.5.]. For some other extensions and quantitative Lipschitz constants, we refer to the papers [**7**] and [**9**].

In this note is presented the following extension of Kellogg theorem to the class of quasiconformal mappings.

THEOREM 1.2. *Let f be a quasiconformal mapping between two domains Ω and Ω' of the complex plane having $\mathscr{C}^{1,\alpha}$ compact boundaries. Then the following conditions are equivalent:*

(1) *μ_f is uniformly $\alpha-$Hölder continuous in Ω*
(2) *$\mu_{f^{-1}}$ is uniformly $\alpha-$Hölder continuous in Ω'*
(3) *$f \in \mathscr{C}^{1,\alpha}(\Omega)$ and $f^{-1} \in \mathscr{C}^{1,\alpha}(\Omega')$.*

REMARK 1.3. If f is conformal then $\mu_f = \mu_{f^{-1}} \equiv 0$ and therefore μ_f and $\mu_{f^{-1}}$ are trivial Hölder continuous functions, and consequently Theorem 1.2 is an extension of Kellogg's theorem. The condition that Ω is a planar domain having $\mathscr{C}^{1,\alpha}$ compact boundary means that the boundary is consisted of a finite number of mutually disjoint $\mathscr{C}^{1,\alpha}$ Jordan curves.

EXAMPLE 1.4. [**2**, p. 391]. Let $\mathbf{U} = \{z : |z| < 1\}$ be the unit disk. Let $g(z) = -z\log|z|^2$, where $|z| \leqslant r = e^{-2}$. Then $g : r\mathbf{U} \to 4r\mathbf{U}$ is a homeomorphism and

$$\frac{\partial g}{\partial \bar{z}} = -\frac{z}{\bar{z}}, \quad \frac{\partial g}{\partial z} = -1 - \log|z|^2, \quad \mu_g(z) = \frac{z}{\bar{z}(1 + \log|z|^2)}.$$

Thus, g is quasiconformal with continuous Beltrami coefficient, and yet g is not Lipschitz. The mapping $f(z) = \frac{1}{4r}g(rz)$ is a q.c. mapping of the unit disk onto itself with a continuous Beltrami coefficient, but g is not Lipschitz neither locally Lipschitz. Thus, the condition that μ_f is Hölder continuous in Theorem 1.2 is important even for local Lipschitz behaviour of a solution to the Beltrami equation $f_{\bar{z}} = \mu(z)f_z$.

2. Proof of Theorem 1.2.

We need the following propositions:

PROPOSITION 2.1. [**2**, Corollary of Theorem 15.0.7] *Let $n \geqslant 0$ be an integer and suppose that $f \in W^{1,2}_{loc}(\Omega, \Omega')$ is a quasiconformal solution to Beltrami equation with $\mu \in C^{n,\alpha}_{loc}(\Omega)$. Then f belongs to $C^{n+1,\alpha}_{loc}(\Omega)$.*

PROPOSITION 2.2. (Mori's theorem, [**1**, p. 47]) *If* $w : \mathbf{U} \to \mathbf{U}$, $w(0) = 0$, *is a* K *quasiconformal mapping of the unit disk onto itself, then*

$$|w(z_1) - w(z_2)| \leqslant 16|z_1 - z_2|^{1/K}, \quad z_1, z_2 \in \mathbf{U}.$$

Mori's theorem for q.c. selfmappings of the unit disk has been generalised in various directions in the plane and in the space. See for example, the papers [**4**] and [**5**].

We now prove the following lemma.

LEMMA 2.3. *Let* f *be a* K*-q.c. mapping of the unit disk* $\mathbf{U}$ *onto itself such that the Beltrami coefficient* μ_f *is Hölder continuous in* $\mathbf{U}$ *with the power* $0 < \alpha < 1$. *Then* $\mu_{f^{-1}}$ *is uniformly* α*−Hölder continuous and* $f, f^{-1} \in \mathscr{C}^{1,\alpha}(\mathbf{U})$.

REMARK 2.4. Under the condition of Lemma 2.3, the function f is $\mathscr{C}^{1,\alpha}_{loc}(\mathbf{U})$, $\alpha < 1$, ([**2**, Theorem 15.0.7]) but the last fact of course does not imply that f is $\mathscr{C}^{1,\alpha}(\mathbf{U})$.

Proof. Since μ is Hölder continuous, there exists a constant C such that $|\mu(z) - \mu(w)| \leqslant C|z - w|^{\alpha}$ $z, w \in \mathbf{U}$. Thus, μ has a continuous extension to the boundary of the unit disk. Now we can choose an α-Hölder continuous extension of μ in $\mathbf{C}$. For example, let

$$\hat{\mu}(z) = \begin{cases} \mu(z), & \text{if } |z| \leqslant 1; \\ \mu(z/|z|), & \text{if } |z| > 1. \end{cases}$$

First of all $\|\hat{\mu}\|_\infty = \|\mu\|_\infty < 1$. It is clear that it is enough to consider the cases $|z| > 1$ and $|w| > 1$. If $|z| > 1$, we have

$$\begin{aligned} |\hat{\mu}(z) - \hat{\mu}(w)| &= |\mu(z/|z|) - \mu(w/|w|)| \\ &\leqslant C|z/|z| - w/|w||^{\alpha} \\ &\leqslant 2^{\alpha} C|z - w|^{\alpha}. \end{aligned}$$

Namely for $z = Re^{it}$, $w = re^{is}$, $R \geqslant r$,

$$|z/|z| - w/|w||^2 = 2(1 - \cos(t - s))$$

and

$$\begin{aligned} |z - w|^2 &= (R - r)^2 + 2Rr(1 - \cos(t - s)) \\ &= (1 - \cos(t - s)) \left(\frac{(R - r)^2}{(1 - \cos(t - x))} + 2Rr \right) \\ &\geqslant (1 - \cos(t - s)) \left(\frac{(R - r)^2}{2} + 2Rr \right) \\ &= (1 - \cos(t - s)) \cdot \frac{(R + r)^2}{2} \\ &\geqslant (1 - \cos(t - s))/2. \end{aligned}$$

Similarly can be treated the case $|w| > 1$.

Let $\hat{f}$ be a quasiconformal solution to Beltrami equation $\hat{f}_{\bar{z}} = \hat{\mu}(z)\hat{f}_z$ (for the existence of $\hat{f}$, we refer to the Ahlfors book [**1**, Chapter V]). By Proposition 2.1, for $n = 0$, $\hat{f}$ is $\mathscr{C}^{1,\alpha}_{loc}(\mathbf{C})$. Let $\Omega = \hat{f}(\mathbf{U})$. Then Ω is a Jordan domain with $\mathscr{C}^{1,\alpha}$ boundary. Let φ be a

https://doi.org/10.1017/S0017089512000171 Published online by Cambridge University Press

conformal mapping of Ω onto the unit disk such that $\varphi(\hat{f}(0)) = f(0)$ and $\varphi(\hat{f}(1)) = f(1)$. Take $g(z) = \varphi(\hat{f}(z))$. For $z \in \mathbf{U}$, we have $\mu_g(z) = \mu_{\hat{f}}(z) = \hat{\mu}(z) = \mu(z)$. We infer that $\mu_{f\circ g^{-1}} = 0$ for $z \in \mathbf{U}$ (see e.g. [**1**, p. 10]) and in view of the above normalisation we obtain $f(z) = g(z)$ for $z \in \mathbf{U}$. By Kellogg's theorem, φ has $\mathscr{C}^{1,\alpha}$ extension to the boundary. It follows that f has $\mathscr{C}^{1,\alpha}$ extension to the boundary. Further, since $\mu_{f^{-1}} = -\mu_f \circ f^{-1}$, because of Mori's theorem, it follows that $\mu_{f^{-1}}$ is $\frac{\alpha}{K}-$Hölder continuous. As above we construct a mapping $\widehat{f^{-1}}$ such that $\widehat{f^{-1}} \in \mathscr{C}^{1,\alpha/K}_{loc}(\mathbf{C})$. Thus as above, we find out that $f^{-1} \in \mathscr{C}^{1,\alpha/K}(\mathbf{U})$. In particular, f^{-1} is Lipschitz continuous. By using the last fact instead of Mori's theorem, we find out that $\mu_{f^{-1}}$ is $\alpha-$ Hölder continuous. By applying again the previous procedure, we obtain that $f^{-1} \in \mathscr{C}^{1,\alpha}(\mathbf{U})$ as desired. □

Proof of Theorem 1.2. Prove the direction 1) $\Rightarrow$ 3). It is clear that it is enough to show that f has $\mathscr{C}^{1,\alpha}$ extension in some neighbourhood of a fixed boundary point $t \in \partial\Omega$. Since $\partial\Omega \in C^{1,\alpha}$, we can find a Jordan domain D_1 with the boundary γ_1 such that t is an interior point of $\gamma_1 \cap \partial\Omega$. Let Φ be a conformal mapping of the unit disk onto D_1 such that $\Phi(t) = t$ and assume that $\Phi^{-1}(\partial D_1 \cap \partial\Omega)$ contains the chord e^{is}, $-\epsilon < s < \epsilon$. Let $V \subset \mathbf{U}$ be a Jordan domain with $C^{1,\alpha}$ boundary containing the chord e^{is}, $-\epsilon/2 < s < \epsilon/2$. Then by Kellogg's theorem, the domain $D = \Phi(V)$ is a Jordan domain with $\mathscr{C}^{1,\alpha}$ Jordan boundary γ such that t is an interior point of $\gamma \cap \partial\Omega$.

Then $D' = f(D)$ is a Jordan domain with boundary γ' containing the point $t' = f(t)$ in an open arc of $\partial\Omega' \cap \gamma'$. Let ψ be a conformal mapping of the unit disk onto D and let φ be a conformal mapping of D' onto the unit disk. Then $f_1 = \varphi \circ f \circ \psi$ is a q.c. mapping of the unit disk onto itself having $\alpha-$ Hölder Beltrami coefficient μ_{f_1}. Namely by [**1**, p. 9] we have

$$\mu_{f_1}(z) = \mu_{f\circ\psi}(z) = \mu_f \circ \psi(z) \cdot \left(\frac{|\psi'(z)|}{\psi'(z)}\right)^2.$$

On the other hand, by Kellogg's theorem ψ and its inverse ψ^{-1} is $\mathscr{C}^{1,\alpha}$ up to the boundary. Further for $w = \psi(z)$ and $w' = \psi(z')$

$$\begin{aligned}
|\mu_{f_1}(z) - \mu_{f_1}(z')| &\leqslant |\mu_f(w) - \mu_f(w')| + 2|\mu_f(w')| \cdot \left|\frac{|\psi'(z)|}{\psi'(z)} - \frac{|\psi'(z')|}{\psi'(z')}\right| \\
&\leqslant C|w - w'|^\alpha + 2\|\mu_f\|_\infty \|1/\psi'\|_\infty \cdot |\psi'(z) - \psi'(z')| \\
&\leqslant \left(C\|\psi'\|_\infty^\alpha + 2\|\mu_f\|_\infty \|1/\psi'\|_\infty \mathrm{Lip}_\alpha(\psi')\right) |z - z'|^\alpha.
\end{aligned}$$

The conclusion of the theorem now follows from Lemma 2.3 and Kellogg's theorem to the conformal mapings ψ and φ near the points t and t' respectively.

The implication 3) $\Rightarrow$ 1) of the theorem is obvious, because under the condition 3), we have $0 < c \leqslant |f_{\bar{z}}| < |f_z| < C < \infty$. Now $\mu_f = f_{\bar{z}}/f_z$ is uniformly $\alpha-$Hölder continuous, because $f_{\bar{z}}$ and f_z are uniformly $\alpha-$Hölder continuous. The proof of 2) $\Leftrightarrow$ 3) is the same. □

By following the lines of the proof of Theorem 1.2, having in mind Proposition 2.1 for $n \geqslant 2$, and Proposition 1.1, the following extension of theorem of Warschawski can be proved.

THEOREM 2.5. *Let $n \in \mathbf{N}$. Let f be a quasiconformal mapping between two domains Ω and Ω' of the complex plane having $\mathscr{C}^{n,\alpha}$ compact boundaries. Then the following conditions are equivalent:*

(1) $\mu_f \in \mathscr{C}^{n,\alpha}(\Omega)$
(2) $\mu_{f^{-1}} \in \mathscr{C}^{n,\alpha}(\Omega')$
(3) $f \in \mathscr{C}^{n+1,\alpha}(\Omega)$ *and* $f^{-1} \in \mathscr{C}^{n+1,\alpha}(\Omega')$.

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