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UNIQUENESS IN THE CAUCHY PROBLEM FOR THE HEAT EQUATION

by SOON-YEONG CHUNG

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We relax the growth condition in time for uniqueness of solutions of the Cauchy problem for the heat equation as follows: Let $u(x, t)$ be a continuous function on $\mathbb{R}^n \times [0, T]$ satisfying the heat equation in $\mathbb{R}^n \times (0, t)$ and the following:

(i) There exist constants $a > 0, 0 < \alpha < 1$, and $C > 0$ such that

$$|u(x, t)| \leq C \exp\left[\left(\frac{a}{t}\right)^{\alpha} + a|x|^2\right] \quad \text{in } \mathbb{R}^n \times (0, T).$$

(ii) $u(x, 0) = 0$ for $x \in \mathbb{R}^n$.

Then $u(x, t) \equiv 0$ on $\mathbb{R}^n \times [0, T]$.

We also prove that the condition $0 < \alpha < 1$ is optimal.

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1. Introduction

In this paper we deal with uniqueness of solutions of the Cauchy problem for the heat equation

$$\begin{cases} (\partial_t - \Delta)u(x, t) = f & \text{in} \quad \mathbb{R}^n \times (0, T) \quad (T > 0), \\ u(x, 0) = \varphi(x) & \text{for} \quad x \in \mathbb{R}^n. \end{cases}$$

For the uniqueness problem, it suffices, by linearity, to consider only the homogeneous case $f \equiv \varphi \equiv 0$.

It is well known that the temperature of the infinite rod is not uniquely determined by its initial temperature (see [**1, 3, 7, 8**]).

In fact, a very sharp counterexample will be given in the last part of this paper.

With additional growth conditions there are uniqueness theorems. The following is the famous uniqueness theorem which was originally given by Tychonoff.

Theorem A ([8, 9, 10, 11]). *Let $u(x, t)$ be a continuous function on $\mathbb{R}^n \times [0, T]$ satisfying*

$$(\partial_t - \Delta)u(x, t) = 0 \quad in \quad \mathbb{R}^n \times (0, T)$$

and for some constants $a > 0$ *and* $C > 0$

$$|u(x, t)| \le C \exp a|x|^2 \quad on \quad \mathbb{R}^n \times [0, T].$$

Then $u(x, 0) = 0$ *implies* $u(x, t) \equiv 0$ *on* $\mathbb{R}^n \times [0, T]$.

For most uniqueness theorems the solution $u(x, t)$ must be uniformly bounded with respect to the t variable. However, sometimes we need a uniqueness theorem with a milder condition on time.

Some authors have had interests in this direction and have relaxed the condition on t. For example, Shapiro [**8**] has shown:

Theorem B ([8]). *Let* $u(x, t)$ *be a solution of the heat equation in the strip* $0 < t < c$ *and bounded in every substrip of the form* $0 < t_0 \le t < c$. *Suppose that*

(i) $\|u(x, t)\|_\infty = o(t^{-1})$ *as* $t \to 0$;

(ii) $\lim_{t \to 0} u(x, t) = 0$ *except possibly for a countable set* E;

(iii) $\liminf_{t \to 0} t^{1/2} u(x, t) = 0$ *for every* x *in* E.

Then $u(x, t) \equiv 0$ *in the strip* $0 < t < c$.

On the other hand, Chung and Kim [**2**] showed the following:

Theorem C ([2]). *Let* $u(x, t)$ *be a solution of the heat equation in* $\mathbb{R}^n \times (0, T)$ *satisfying:*

(i) *There exist constants* $k > 0$ *and* $C > 0$ *such that*

$$|u(x, t)| \le C \exp k\left(|x|^2 + \frac{1}{t}\right), \quad 0 < t < T;$$

(ii) $\lim_{t \to 0+} \int u(x, t)\phi(x)dx = 0$ *for every* C^∞ *function* $\phi(x)$ *such that for every* $h > 0$,

$$\sup_{\substack{x \in \mathbb{R}^n \\ \alpha}} \frac{|\partial^\alpha \phi(x)| \exp 2k|x|}{h^{|\alpha|}\alpha!} < \infty.$$

Then $u(x, t) \equiv 0$ *on* $\mathbb{R}^n \times [0, T]$.

In the above two theorems they relaxed the growth condition on time so that the uniqueness classes determined by them are larger than that of Theorem A. In fact, Theorem C gives a much larger uniqueness class than others. But, nevertheless, the hypotheses are not so natural that one can apply them effectively. The hypothesis (ii) is stronger than $u(x, 0) = 0$, since there exists a nonzero temperature function satisfying (i) and $u(x, 0) = 0$ (see Section 4).

The purpose of this paper is to prove the following theorem:

Theorem 3.1. *Let $u(x, t)$ be a continuous function on $\mathbb{R}^n \times [0, T]$ satisfying the heat equation in $\mathbb{R}^n \times (0, T)$ and the following:*

(i) *There exist constants $a > 0, 0 < \alpha < 1$, and $C > 0$ such that*

$$|u(x,t)| \le C \exp\left[\left(\frac{a}{t}\right)^{\alpha} + a|x|^2\right] \quad in \quad \mathbb{R}^n \times (0, T).$$

(ii) $u(x, 0) = 0$ *for* $x \in \mathbb{R}^n$.

Then $u(x, t) \equiv 0$ on $\mathbb{R}^n \times [0, T]$.

Moreover, we prove the uniqueness of the solution with weaker initial condition, not imposing the continuity of $u(x, t)$ at $t = 0$.

Besides these results we prove that the condition $0 < \alpha < 1$ in the above is optimal with a counterexample showing that if we take $\alpha = 1$ then the theorem is no longer true.

To prove the main theorem we heavily make use of the generalized function theory. In particular, we depend largely on the nonquasianalytic properties of the ultradistributions of Gevrey type.

2. Notation and basic results

We introduce briefly the ultradistributions of Gevrey type which will be very useful later. See [6] and also [5] for more details.

Definition 2.1. Let Ω be an open subset of $\mathbb{R}^n$ and $\varphi \in C^\infty(\Omega)$. Then we say that φ belongs to $\mathcal{E}_{(s)}(\Omega)$ for $s > 1$ if for any compact subset K of Ω and for every $h > 0$ there exists a constant $C = C(K, h) > 0$ such that

$$\sup_{x \in K} |\partial^\alpha \varphi(x)| \le C h^{|\alpha|} \alpha!^s, \quad \alpha \in \mathbb{N}_0^n$$

where $\mathbb{N}_0$ is the set of nonnegative integers, and we use the multi-index notations $|\alpha| = \alpha_1 + \cdots + a_n$ and $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$, $\partial_j = \frac{\partial}{\partial_{x_j}}$, $j = 1, 2, \cdots, n$, for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$.

We denote by $\mathcal{D}_{(s)}(\Omega)$ the subspace of $\mathcal{E}_{(s)}(\Omega)$ which consists of functions with compact support in Ω. The topologies of these spaces are defined as follows:

(i) $\{\phi_j(x)\} \in \mathcal{E}_{(s)}(\Omega)$ converges to zero in $\mathcal{E}_{(s)}(\Omega), s > 1$, if for any compact subset K of Ω and for every $h > 0$,

$$\sup_{\substack{x \in K \\ \alpha \in \mathbf{N}_0^n}} \frac{|\partial^\alpha \phi_j(x)|}{h^{|\alpha|}\alpha!^s} \to 0 \quad \text{as} \quad j \to \infty.$$

(ii) $\{\phi_j(x)\} \in \mathcal{D}_{(s)}(\Omega)$ converges to zero in $\mathcal{D}_{(s)}, s > 1$, if there is a compact set K of Ω such that $\operatorname{supp}\phi_j \subset K$, $j = 1, 2, \cdots$, and $\phi_j \to 0$ in $\mathcal{E}_{(s)}(\Omega)$.

As usual, we denote by $\mathcal{D}'_{(s)}(\Omega)$ (by $\mathcal{E}'_{(s)}(\Omega)$) the strong dual space of $\mathcal{D}_{(s)}(\Omega)$ (of $\mathcal{E}_{(s)}(\Omega)$, respectively) and we call its elements the ultradistributions of Gevrey type.

We have the inclusion

$$\mathcal{D}'(\Omega) \subset \mathcal{D}'_{(s)}(\Omega), \quad \mathcal{E}'(\Omega) \subset \mathcal{E}'_{(s)}(\Omega), \quad s > 1,$$

where $\mathcal{D}'(\Omega)$ and $\mathcal{E}'(\Omega)$ are the space of Schwartz distributions and the space of Schwartz distributions with compact support respectively. It is well known that $\mathcal{E}'_{(s)}(\Omega)$ consists of the ultradistributions in $\mathcal{D}'_{(s)}(\Omega)$ with compact support in Ω like the space $\mathcal{E}'(\Omega)$ in $\mathcal{D}'(\Omega)$. In fact, since there exist cut off functions and partitions of unity in $\mathcal{D}_{(s)}(\Omega)$, $s > 1$, the properties of the ultradistributions in $\mathcal{D}'_{(s)}(\Omega)$ are very similar to those of Schwartz distributions. In particular, the concepts of the support, convolution, etc. are defined very similarly and naturally.

We note here that in view of the topology on $\mathcal{D}_{(s)}(\Omega)$, u belongs to $\mathcal{D}'_{(s)}(\Omega)$ if and only if for every compact subset K of Ω there exist constants $h > 0$ and $C > 0$ such that

$$|u(\phi)| \le C \sup_{\substack{x \in K \\ \alpha \in \mathbf{N}_0^n}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|}\alpha!^s}$$

for every $\phi \in D_{(s)}(\Omega)$ with support in K.

On the other hand, define the partial differential operators of class (s) of infinite order as follows:

$$p(\partial) = \sum_\alpha a_\alpha \partial^\alpha, \quad |a_\alpha| \le CL^{|\alpha|}/\alpha!^s$$

for some $L > 0$ and $C > 0$. Then the mappings

$$p(\partial) : \mathcal{D}_{(s)}(\Omega) \to \mathcal{D}_{(s)}(\Omega), \quad \mathcal{E}_{(s)}(\Omega) \to \mathcal{E}_{(s)}(\Omega)$$

$$p(\partial) : \mathcal{D}'_{(s)}(\Omega) \to \mathcal{D}'_{(s)}(\Omega), \quad \mathcal{E}'_{(s)}(\Omega) \to \mathcal{E}'_{(s)}(\Omega)$$

are continuous.

From now on we denote by $E(x, t)$ the fundamental solution of the heat equation:

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0 \\ 0, & t < 0. \end{cases}$$

Then the following can be obtained by some tedious calculation.

Proposition 2.2. *Let $g(x)$ be a continuous function on $\mathbb{R}^n$ satisfying that for some constants $a > 0$ and $C > 0$*

$$|g(x)| \leq C \exp a|x|^2, \quad x \in \mathbb{R}^n.$$

Then $G(x,t) = g(x) * E(x,t)$ is a well defined C^∞ function in $\mathbb{R}^n \times (0, 1/4a)$ and satisfies

(i) $(\partial_t - \Delta)G(x,t) = 0, \quad 0 < t < 1/4a,$

(ii) $|G(x,t)| \leq C \exp(2a|x|^2), \quad 0 < t < 1/8a,$

(iii) *$G(x,t) \to g(x)$ uniformly on each compact subset of $\mathbb{R}^n$ as $t \to 0+$.*

Here, $*$ denotes the convolution with respect to x variable.

The following lemma is very useful later. In fact, this is the main tool used to prove the main theorem.

Lemma 2.3. *For any $L > 0, s > 0$ and for a small $\varepsilon > 0$ there exist functions $v(t), w(t) \in C_0^\infty(\mathbb{R})$ and a differential operator $p(d/dt)$ of infinite order such that*

$$p(d/dt)v(t) = \delta(t) + w(t); \tag{2.1}$$

$$\operatorname{supp} v \subset [0, \varepsilon], \quad \operatorname{supp} w \subset [\varepsilon/2, \varepsilon]; \tag{2.2}$$

$$p(d/dt) = \sum_{k=0}^{\infty} a_k (d/dt)^k, \quad |a_k| \leq Ch^k/k!^s; \tag{2.3}$$

for some positive constants C and h and

$$|v(t)| \leq C \exp\left[-(cL/t)^{1/(s-1)}\right]. \tag{2.4}$$

where δ is the Dirac measure and c is a constant depending only on s.

Proof. We set

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{itx} dt}{p(it)}, \quad \text{where } p(\zeta) = (1+\zeta)^2 p_1(\zeta), \quad p_1(\zeta) = \prod_{q=1}^{\infty} (1 + L\zeta/q^s).$$

The function $u(x)$ is the inverse Fourier transform of $1/p(it)$. Notice that $1/(1 + i\lambda t)$ is the Fourier transform of the function

$$u_\lambda(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Hence $u(x)$ is the convolution of a sequence of functions $u_\lambda(x)$ with

$$\lambda = 1, 1, L, L/2^s, L/3^s, \cdots.$$

This implies the properties $u(x) = 0$ for $x < 0, u(x) \geq 0$ for $x \geq 0$,

$$\int_{-\infty}^{\infty} u(x)dx = 1, \quad \text{and} \quad p(d/dx)u(x) = \delta(x).$$

Further, for $x > 0$ and $q \in \mathbb{N}$,

$$\begin{aligned} \left| \left(\frac{d}{dx} \right)^q u(x) \right| &= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \frac{(it)^q e^{itx}}{p(it)} dt \right| \\ &\leq \max_t \frac{t^q}{|p_1(it)|} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{|1+it|^2} \\ &= \frac{1}{2} \max_t t^q \prod_{j=1}^{q} \left| \frac{j^s}{j^s + iLt} \right| \\ &\leq \frac{1}{2} \cdot \frac{q!^s}{L^q} \end{aligned}$$

Thus if we use this and

$$u(x) = \int_0^x \frac{(x-y)^{q-1}}{(q-1)!} u^{(q)}(y)dy$$

then we have for each $q \in \mathbb{N}$ and $x > 0$

$$\begin{aligned} |u(x)| &\leq \frac{x^q}{q!} \max_{t \in \mathbf{R}} |u^{(q)}(t)| \\ &\leq \frac{q!^{s-1} x^q}{2L^q} \leq 2^{s-2} e^{-\frac{s-1}{2}\left(\frac{L}{x}\right)^{1/s-1}}, \end{aligned}$$

since $\inf_k \frac{k!}{t^k} \leq 2e^{-t/2}$ for $t > 0$. This implies that $u(x)$ satisfies the inequality (2.4).

In order to estimate a_k, we write

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$$\ln p_1(\zeta) = \sum_{q=1}^{\infty} \ln(1 + L\zeta/q^s) \approx \int_1^{\infty} \ln(1 + L\zeta/q^s)dq$$

for real $\zeta > 0$. By substitution $q = (L\zeta)^{1/s}t$, we get

$$\ln p_1(\zeta) \leq C(L\zeta)^{1/s} \int_0^{\infty} \ln(1 + t^{-s})dt = C_1(L\zeta)^{1/s}$$

The above estimate obviously holds also for $p(\zeta)$, with a different constant C_1. For complex ζ,

$$|p(\zeta)| \leq p(|\zeta|) \leq \exp(C_1|L\zeta|^{1/s}).$$

From the Cauchy equality

$$a_k = \frac{1}{2\pi} \oint_{\{|\zeta|=R\}} \frac{p(\zeta)}{\zeta^{k+1}} d\zeta$$

with $R = k^s/L$, it follows

$$|a_k| \leq \frac{1}{R^k} \max_{|\zeta|=R} |p(\zeta)| \leq \frac{L^k}{k^{ks}} e^{C_1 k} \leq \frac{(CL)^k}{k!^s}.$$

Therefore, the estimate (2.3) holds.

By multiplying $u(x)$ with a function in $\mathcal{E}_{(s)}(\mathbb{R})$ which is equal to 1 in $(-\infty, \varepsilon/2]$ and equal to 0 in $[\varepsilon, \infty)$ the function $v(x)$ can be obtained. By the definition of $p(\zeta)$ in (2.5) we can easily see that it is an entire function of order $1/s$ such that

$$p(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k, \quad |a_k| \leq CL^k/k!^s.$$

3. Main theorems

We give here a much better uniqueness theorem than those in **[2, 3, 4, 8]**.

Theorem 3.1. *Let $u(x, t)$ be a continuous function on $\mathbb{R}^n \times [0, T]$ satisfying*

(i) $(\partial_t - \Delta)u(x, t) = 0$ *in* $\mathbb{R}^n \times (0, T]$.

(ii) $|u(x, t)| \leq C\exp[(a/t)^{\alpha} + a|x|^2]$ *in* $\mathbb{R}^n \times (0, T]$ *for some constants* $a > 0, 0 < \alpha < 1$, *and* $C > 0$.

(iii) $u(x, 0) = 0$ *on* $\mathbb{R}^n$.

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Then $u(x, t)$ is identically zero on $\mathbb{R}^n \times [0, T]$. Here T may be ∞.

Proof. In view of Theorem A given by Tychonoff in the introduction we have only to show that $u \equiv 0$ on $\mathbb{R} \times [0, T_0]$ for sufficiently small $T_0 > 0$.

Let $s = \frac{1}{2}(1 + \frac{1}{\alpha}) > 1$. Then according to Lemma 2.3 we can choose C_0^∞ functions $v(t)$, $w(t)$ on $\mathbb{R}$ and a differential operator $p(d/dt)$ of infinite order such that

$$p(d/dt)v(t) = \delta(t) + w(t); \tag{3.1}$$

$$\operatorname{supp} v \subset [0, T_0], \quad \operatorname{supp} w \subset [T_0/2, T_0]; \tag{3.2}$$

$$p(d/dt) = \sum_{k=1}^{\infty} a_k (d/dt)^k, \quad |a_k| \le Ch^k/k!^{2s}; \tag{3.3}$$

$$|v(t)| \le C \exp\left[-(2a/t)^{1/(2s-1)}\right], \tag{3.4}$$

where $T_0 > 0$ is a small number such that $2T_0 < \min(T, 1/16a)$.

Define two functions $G(x, t)$ and $H(x, t)$ by

$$G(x, t) = \int_0^T u(x, t + \tau) v(\tau) d\tau,$$

and

$$H(x, t) = -\int_0^T u(x, t + \tau) w(\tau) d\tau.$$

Then it follows from (3.2), (3.4) and the condition (ii) that the integrals converge and are continuous functions on $\mathbb{R}^n \times [0, T_0]$. Moreover, they satisfy the heat equation and the growth condition

$$|G(x, t)| \le C \exp a|x|^2 \tag{3.5}$$

and

$$|H(x, t)| \le C \exp a|x|^2 \tag{3.6}$$

for all $(x, t) \in \mathbb{R}^n \times [0, T_0]$. If we take $g(x) = G(x, 0)$ and $h(x) = H(x, 0)$ then Proposition 2.2 implies that $g * E$ and $h * E$ are continuous functions on $\mathbb{R}^n \times [0, T_0]$ which converge to $g(x)$ and $h(x)$ respectively as $t \to 0+$. In view of Theorem A in the introduction and the growth conditions (3.5), (3.6) it follows that

$$G(x, t) = g(x) * E(x, t), \quad H(x, t) = h(x) * E(x, t) \tag{3.7}$$

on $\mathbb{R}^n \times [0, T_0]$.

It is clear that for the differential operator $p(-\Delta)=\sum_{k=0}^{\infty}a_k(-\Delta)^k$ is a differential operator of class (s) which acts continuously on $\mathcal{D}'_{(s)}(\mathbb{R}^n)$. Thus, if we define

$$u=p(-\Delta)g(x)+h(x), \tag{3.8}$$

then u belongs to $\mathcal{D}'_{(s)}(\mathbb{R}^n)$.

On the other hand, since $u(x,t)$ is uniformly continuous on each compact subset of $\mathbb{R}^n\times[0,T)$, the initial condition $u(x,0)=0$ implies that $u(x,t)$ converges uniformly to 0 on each compact subset K of $\mathbb{R}^n$ as $t\to 0+$.

Applying the differential operator $p(-\partial/\partial t)$ in (3.1) we have

$$p(-\partial/\partial t)G(x,t)=p(-\Delta)G(x,t)=u(x,t)-H(x,t) \tag{3.9}$$

in $\mathbb{R}^n\times(0,T_0)$. Then we note that $u=p(-\Delta)g+h$ is a $\mathcal{D}'_{(s)}$-limit (or weak limit) of $p(-\Delta)G(x,t)+H(x,t)$, since $G(x,t)\to g(x)$ and $H(x,t)\to h(x)$ uniformly on each compact subset of $\mathbb{R}^n$. Therefore, we have

$$u=\lim_{t\to 0+}u(x,t)\quad\text{in}\quad \mathcal{D}'_{(s)}(\mathbb{R}^n).$$

From this we obtain that

$$u(\varphi)=\lim_{t\to 0}\int u(x,t)\varphi(x)dx=0,\qquad \varphi\in\mathcal{D}_{(s)}(\mathbb{R}^n), \tag{3.10}$$

which implies that $u=0$ as an element of $\mathcal{D}'_{(s)}(\mathbb{R}^n)$.

Now we show that $p(-\Delta)[g*E]=[p(-\Delta)g]*E$. To do this we need the following estimate for $E(x,t)$:

$$|\partial_x^\alpha E(x,t)|\le C^{|\alpha|}t^{-(n+|\alpha|)/2}\alpha!^{\frac{1}{2}}\exp[-|x|^2/8t],\qquad t>0 \tag{3.11}$$

for some constant $C>0$. We shall prove this for $n=1$ for simplicity. Since $E(z,t)$ is entire holomorphic the derivatives of $E(x,t)$ can be evaluated by Cauchy's integral formula

$$\partial_x^k E(x,t)=\frac{k!}{2\pi i}\int_{\Gamma_R}\frac{E(z,t)}{(z-x)^{k+1}}dz,$$

where Γ_R is a circle of radius R in the complex plane $\mathbb{C}$ with centre at x. Then we find

$$|\partial_x^k E(x,t)|\le\frac{k!}{\sqrt{4\pi t}R^k}\exp\left[\frac{-\bar{x}^2+R^2}{4t}\right] \tag{3.12}$$

where $\bar{x}=x-R$ or $x+R$. Let us choose R so that $\exp[R^2/4t]/R^k$ would attain its minimum. This is realized for $R = \sqrt{2kt}$, so that (3.12) reduces to

$$|\partial_x^k E(x,t)| \le \frac{1}{\sqrt{4\pi}}(e/2)^{k/2} t^{-(1+k)/2} k!k^{-(k/2)} \exp[-\bar{x}^2/4t].$$

The last factor may be estimated as follows.

$$\exp[-\bar{x}^2/4t] \le \exp[-(|x|-R)^2/4t] \le \exp\left[-\frac{|x|^2}{8t}+\frac{R^2}{4t}\right].$$

Since $R^2 = 2kt$ we have

$$|\partial_x^k E(x,t)| \le C^k t^{-(1+k)/2} k!^{\frac{1}{2}} \exp\left[-\frac{|x|^2}{8t}\right]$$

for some constant $C > 0$, which gives (3.11).

On the other hand, we have for each $x \in \mathbb{R}^n$ and $0 < t < T_0$

$$\begin{aligned} &\sum_{k=0}^{\infty}\int |g(y)a_k(-\Delta_x)^k E(x-y,t)|\,dy \\ &\le \sum_{k=0}^{\infty}\int \{\exp a|y|^2\}\cdot|a_k|\cdot|\Delta_x^k E(x-y,t)|\,dy \qquad (3.13)\\ &\le \sum_{k=0}^{\infty} C_1\frac{(2a)^k}{k!^{2s}}C_2^k t^{-(n+2k)/2}(2k)!^{\frac{1}{2}}\int \exp\left[a|y|^2-\frac{|x-y|^2}{8t}\right]dy. \end{aligned}$$

The last inequality in the above is obtained in view of (3.3) and (3.11). Since $k!^2 \le (2k)! \le 4^k(k!)^2$ and $2s-1=\frac{1}{\alpha}>1$ we estimate (3.13) as follows.

$$\sum_{k=0}^{\infty}\frac{(t^{-1}C_3)^k}{k!^{2s-1}}\cdot t^{-n}\int \exp\left[a|y|^2-\frac{|y|^2}{16t}+\frac{|x|^2}{8t}\right]dy \le t^{-n}\exp\left[\frac{C_3}{t}+\frac{|x|^2}{8t}\right]\cdot\int \exp\left[\left(a-\frac{1}{16t}\right)|y|^2\right]dy.$$

Since $0 < t < T_0$ and $T_0 < \frac{1}{32a}$ the last integral is finite, so that (3.13) reduces to a finite number depending on x and t. Therefore, the Lebesgue dominated convergence theorem implies that

$$\begin{aligned} g * p(-\Delta)E &= g * \sum_{k=0}^{\infty} a_k(-\Delta_x)^k E \\ &= \sum_{k=0}^{\infty} g * [a_k(-\Delta_x)^k E] \\ &= \sum_{k=0}^{\infty} a_k(-\Delta_x)^k [g * E] \\ &= p(-\Delta)[g * E]. \end{aligned}$$

But since $\{p(-\Delta)g\} * E = g * p(-\Delta)E$ from the definition of differential operator of class (s) acting on the ultradistributions we have $p(-\Delta)[g * E] = [p(-\Delta)g] * E$.

Then it follows from (3.7) and (3.9) that

$$\begin{aligned} u(x,t) &= p(-\Delta)G(x,t) + H(x,t) \\ &= p(-\Delta)g * E + h * E \\ &= [p(-\Delta)g + h] * E \\ &= u * E \equiv 0, \end{aligned}$$

which completes the proof.

In the above proof the continuity of $u(x, t)$ at $t = 0$ can be weakened. In fact, the continuity was used only to derive (3.10), which means that $u(x, t)$ weakly converges to 0 in $\mathcal{D}'_{(s)}$. Thus we can obtain the following uniqueness theorem without continuity on $t = 0$.

Theorem 3.2. *If $u(x, t)$ satisfies the conditions in $\mathbb{R}^n \times (0, T)$:*

(i) $(\partial_t - \Delta)u(x, t) = 0$,

(ii) $|u(x, t)| \le C\exp[(a/t)^\alpha + a|x|^2]$ *for some constants* $a > 0$, $0 < \alpha < 1$ *and* $C > 0$,

(iii) $\lim_{t\to 0+} \int u(x, t)\varphi(x)dx = 0$, $\varphi \in \mathcal{D}_{(s)}(\mathbb{R}^n)$ *where* $s = \frac{1}{2}\left(1 + \frac{1}{\alpha}\right)$, *then* $u(x, t) \equiv 0$ *in* $\mathbb{R}^n \times [0, T)$.

4. Example

In this section we show that the uniqueness class in the previous section is the optimal one. In particular, it will be shown that the growth condition on the time variable is optimal.

For the space variable it is well known (see [3]) that for every $\varepsilon > 0$ there exists a C^∞ function $u(x, t) \not\equiv 0$ satisfying the following:

(i) $(\partial_t - \Delta)u(x, t) = 0$ in $\mathbb{R}^n \times (0, T)$.

(ii) u is continuous on $\mathbb{R}^n \times [0, T)$.

(iii) $|u(x, t)| \le C_\varepsilon \exp|x|^{2+\varepsilon}$ on $\mathbb{R}^n \times (0, T)$.

(iv) $u(x, 0) = 0$ on $\mathbb{R}^n$.

This shows that Theorem 3.1 is not true if in that condition we replace $\exp(a|x|^2)$ by $\exp(a|x|^{2+\varepsilon})$, $\varepsilon > 0$.

Now we will show that Theorem 3.1 is also no longer true if we replace the condition $0 < \alpha < 1$ by $\alpha = 1$. To see this let D_N be a domain in the complex plane $\mathbb{C}$ given by

$$D_N = \{z \in \mathbb{C} \mid z = x + yi, \quad x \geq N, -\pi \leq y \leq \pi\}, \quad N \geq 0$$

and C_N be the boundary of D_N. Define a function $u(x, t)$ on $\mathbb{R} \times (0, \infty)$ by

$$u(x, t) = \frac{1}{2\pi i} \int_{C_N} E(x - \zeta, t) \exp(e^{\zeta}) d\zeta, \tag{4.1}$$

where the integral is taken counterclockwise.

Since the function $\exp(e^{\zeta})$ decreases very rapidly as $\operatorname{Re}\zeta \to \infty$ on the curve C_N the integral converges and $u(x, t)$ satisfies

$$(\partial_t - \Delta)u(x, t) = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty).$$

Also, Cauchy's integral theorem implies that $u(x, t)$ is independent of $N \geq 0$. Since the integral

$$\frac{1}{2\pi} \int_{C_N} |\exp(e^{\zeta})| \, |d\zeta|$$

is finite it follows that

$$|u(x, t)| \leq C(N) \sup_{\zeta \in D_N} |E(x - \zeta, t)|.$$

Writing $\zeta = \xi + i\eta$ we obtain

$$\begin{aligned} \sup_{\zeta \in D_N} |E(x - \zeta, t)| &= \frac{1}{\sqrt{4\pi t}} \sup_{\substack{N \leq \xi \\ |\eta| \leq \pi}} \left| \exp\left[-\frac{(x - \xi)^2 - \eta^2}{4t} \right] \right| \\ &= \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{\pi^2}{4t} \right) \sup_{N \leq \xi} \exp\left[-\frac{(x - \xi)^2}{4t} \right], \end{aligned}$$

since $\zeta = \xi + i\eta$ implies $N \leq \xi$ and $|\eta| \leq \pi$ by its definition.

Then it follows that for some constant $a > 0$

$$|u(x, t)| \leq C(N) \exp\left(\frac{a}{t} \right) \exp\left[-\frac{d(x, D_N)^2}{4t} \right] \tag{4.2}$$

where d is the Euclidean distance.

Thus we have

$$|u(x, t)| \leq C(N) \exp\left(\frac{a}{t} \right) \quad \text{in} \quad \mathbb{R}^n \times (0, \infty). \tag{4.3}$$

Let $r > 0$ and $x \leq r$. Since the integral (4.1) is independent of N we may choose a sufficiently large $N > 0$ so that $4a < (N - r)^2$. Then by (4.2) we obtain

$$\sup_{x \leq r} |u(x, t)| \leq C(N) \exp\left[\frac{4a - (N - r)^2}{4t}\right], \quad t > 0. \tag{4.4}$$

The right hand side of (4.4) converges to 0 as $t \to 0+$ and $u(x, t)$ converges uniformly to 0 as $t \to 0+$ in every half-line $(-\infty, r], r > 0$. Therefore, we can conclude that $u(x, t)$ is continuous on $\mathbb{R}^n \times [0, \infty)$ and $u(x, 0) = 0$.

Now it remains to show that $u(x, t) \not\equiv 0$.

To do this we suppose that $u(x, t) \equiv 0$ in $\mathbb{R} \times [0, \infty)$. Then we obtain from (4.1) that

$$\int_{C_N} \exp\left[-\frac{(x - \zeta)^2}{4t}\right] \cdot \exp(e^{\zeta}) d\zeta \equiv 0$$

in $\mathbb{R}^n \times [0, \infty)$. Applying the Lebesgue dominated convergence theorem we can see that

$$\int_{C_N} \exp(e^{\zeta}) d\zeta = 0. \tag{4.5}$$

Since the integral (4.1) does not depend on $N \geq 0$ we may choose $N = 0$. Then (4.5) can be written as

$$\begin{aligned} 0 &= -\int_0^{\infty} \exp(-e^{\zeta}) d\zeta - i \int_{-\pi}^{\pi} \exp(e^{yi}) dy + \int_0^{\infty} \exp(-e^{\zeta}) d\zeta \\ &= -2i \int_0^{\pi} e^{\cos y} \cos(\sin y) dy. \end{aligned}$$

But the integral $e^{\cos y} \cos(\sin y) > 0$ on $[0, \pi]$, which leads a contradiction. Thus we can conclude that $u(x, t) \not\equiv 0$.

Remark. In [1] they gave an example with the same estimate as (4.3). But that was more complicated than the above one.

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Department of Mathematics
Sogang University
Seoul 121-742
Korea
E-mail address: sychung@ccs.sogang.ac.kr