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Elementary abelian operator groups

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Suppose G is a finite solvable p'-group admitting the elementary abelian p-group A as an operator group. If $n = \max\left\{\text{nilpotent length of } C_G(X) \mid X \in A^{\#}\right\}$ and $|A| \geq p^{n+2}$, then the nilpotent length of G is n.

1. Introduction

Suppose A is an elementary abelian p-group of order p^m acting as an operator group on the finite p'-group G. If $m \geq 3$ and $C_G(X)$ is nilpotent for each non-identity element X in A, then Ward [8] showed that G is nilpotent. More recently, Ward [9] proved that if G is solvable, $m \geq 4$, and the derived group of $C_G(X)$ is nilpotent for each non-identity element X in A, then G' is nilpotent. The principal result (Theorem 3.1) of the present paper asserts that if G is solvable, n is the maximum of the nilpotent lengths of $C_G(X)$ where X runs through the non-identity elements of A, and $m \geq n + 2$, then the nilpotent length of G is n. Using this result, an easy argument shows that if G is solvable, $C_G(X)$ is supersolvable for each non-identity element X in A, and $m \geq 4$, then G is super-solvable. Examples are given showing the necessity of the inequalities $m \geq n + 2$ and $m \geq 4$ in these results.

These theorems depend on a rather complicated technical result (Theorem 2.4) proved in §2 about the upper nilpotent series of a finite solvable group G which admits an operator group A where

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$(|G|, |A|) = 1$. The main results are proved in §3 and examples are given in §4.

2. Notation and preliminary results

All groups considered in this paper are finite. If G is a group, $F_0(G) = 1$ and $F_{n+1}(G)/F_n(G) = F(G/F_n(G))$ equals the largest normal nilpotent subgroup of $G/F_n(G)$. If G is solvable, $l(G)$ is the smallest non-negative integer n such that $F_n(G) = G$. The rest of the notation agrees with [2]. We now prove a number of technical results needed for the main theorems.

THEOREM 2.1. *Suppose P is a p-group which admits the group G as an operator group. Assume Q is a normal p'-subgroup of G which centralizes every G-invariant proper subgroup of P but $[P, Q] \neq 1$. Then P is a special p-group and any proper G-invariant subgroups of P are contained in P' .*

Proof. This follows immediately from Theorem C of [6].

LEMMA 2.2. *Let P be a p-subgroup of the group G . Assume $P \leq F_2(G)$ but $P \nleq F_1(G)$. Then for some prime $q \neq p$, the Sylow q-subgroup of $F_1(G)$ is not centralized by P .*

Proof. Let H be a Hall p'-subgroup of $F_1(G)$. If S is a Sylow p-group of $F_2(G)$, then $HC_S(H)$ is a normal nilpotent subgroup of G . Hence $C_S(H) \leq F_1(G)$. This implies that $[H, P] \neq 1$. Since H is nilpotent, the desired result follows immediately.

LEMMA 2.3. *Let G be a solvable group and H a subgroup of G . Assume that $P_1, \ldots, P_n$ $(n > 1)$ are subgroups of H and $p_1, \ldots, p_n$ are primes satisfying the following conditions:*

(a) P_i *is a* p_i*-group if* $1 \leq i \leq n$;

(b) $p_i \neq p_{i+1}$ *if* $1 \leq i \leq n-1$;

(c) $P_{i+1} < N_G(P_i)$ *if* $1 \leq i \leq n-1$;

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(d) $P_i \leq F_i(G)$ *if* $1 \leq i \leq n$ *;*

(e) $[P_n, P_{n-1}, \ldots, P_2, P_1] \neq 1$.

Then $l(H) \geq n$.

Proof. Clearly $F_i(H) \geq F_i(G) \cap H \geq P_i$ for $1 \leq i \leq n$. Suppose $l(H) < n$. Then $F_{n-1}(H) = H$. Since $H/F_{n-2}(H)$ is nilpotent and $(|P_n|, |P_{n-1}|) = 1$, we obtain $[P_n, P_{n-1}] \leq F_{n-2}(H) \cap P_{n-1}$. Now P_{n-1} normalizes P_{n-2} and P_{n-2} is a p'_{n-1}-subgroup of $F_{n-2}(H)$. Thus $[P_n, P_{n-1}, P_{n-2}] \leq F_{n-3}(H) \cap P_{n-2}$. Continuing in this way, we eventually obtain

$$[P_n, P_{n-1}, \ldots, P_2, P_1] \leq F_0(H) \cap P_1 = 1 ,$$

which is a contradiction. Thus $l(H) \geq n$.

THEOREM 2.4. *Suppose* A *is an operator group on the solvablè group* G *where* $(|A|, |G|) = 1$. *Assume* $l(G) = n > 0$. *Then there are primes* $p_1, \ldots, p_n$ *and* A*-invariant subgroups* $P_1, \ldots, P_n$ *in* G *such that:*

(a) P_i *is a* p_i*-group if* $1 \leq i \leq n$ *;*

(b) $p_i \neq p_{i+1}$ *if* $1 \leq i \leq n-1$ *;*

(c) $P_i \leq N_G(P_j)$ *if* $1 \leq j \leq i \leq n$ *;*

(d) $P_i \leq F_i(G)$ *but* $P_i \nleq F_{i-1}(G)$ *if* $1 \leq i \leq n$ *;*

(e) $[P_{i+1}, P_i] = P_i$ *if* $1 \leq i \leq n$ *;*

(f) if Q *is an* A*-invariant proper subgroup of* P_n *, then* $Q \leq F_{n-1}(G)$ *;*

(g) if $1 \leq i \leq n-1$ *and* Q *is a proper subgroup of* P_i *which is invariant under* $A \prod_{i<j} P_j$ *, then* $[P_{i+1}, Q] \leq F_{i-1}(G)$.

Proof. If $n = 1$, simply let P_1 be a minimal A-invariant subgroup of G . Suppose next $n = 2$. Let p_2 be a prime dividing $|G/F_1(G)|$.

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By [3, Corollary 2, p. 124], there is an A-invariant Sylow p_2-subgroup P in G. Let P_2 be minimal with respect to: $P_2 \leq P$, $P_2 \nleq F_1(G)$, and P_2 is A-invariant. By Lemma 2.2, there is a prime $p_1 \neq p_2$ such that P_2 does not centralize the Sylow p_1-subgroup of $F_1(G)$. Choose P_1 to be minimal with respect to: P_1 is a p_1-subgroup of $F_1(G)$, $N_G(P_1) \geq P_2$, P_1 is A-invariant, and $[P_2, P_1] \neq 1$. Since $[P_1, P_2, P_2] = [P_1, P_2]$ from [3] and $[P_1, P_2]$ is A-invariant and normalized by P_2, we must have $[P_2, P_1] = P_1$. This proves the theorem for $n \leq 2$. We now assume $n > 2$ and proceed by induction on n.

By [3], there is an A-invariant Carter subgroup C of $F_2(G)$. Let $N = N_G(C)$. $N \cap F_2(G) = C$ and, by the Fitting argument, $G = F_2(G)N$. Since $F_2(G)/F_1(G)$ is nilpotent, $F_2(G) = F_1(G)C$. Suppose now $2 \leq i \leq n$. Then from $G = F_2(G)N$ follows $F_i(G) = F_2(G)(F_i(G) \cap N)$. Since $l(F_i(G)/F_2(G)) = i - 2$ and $F_i(G) \cap N \cap F_2(G) = C \leq F_1(N)$, we find that $l(F_i(G) \cap N) \leq i - 1$. Hence $F_i(G) \cap N \leq F_{i-1}(N)$. Conversely, $F_{i-1}(N)F_2(G)$ is normal in $NF_2(G) = G$ and

$$l(F_{i-1}(N)F_2(G)) = l(F_{i-1}(N)CF_1(G)) = l(F_{i-1}(N)F_1(G)) \leq i .$$

This implies that $F_{i-1}(N) \leq F_i(G) \cap N$. Hence $F_i(G) \cap N = F_{i-1}(N)$ for $2 \leq i \leq n$. A consequence of this is that $l(N) = n - 1$.

By induction, there are primes $q_1, \ldots, q_{n-1}$ and A-invariant subgroups $Q_1, \ldots, Q_{n-1}$ in N satisfying *(a)* through *(g)* for N. For $3 \leq i \leq n$, let $p_i = q_{i-1}$ and $P_i = Q_{i-1}$. From the fact that $F_j(G) \cap N = F_{j-1}(N)$ for $2 \leq j \leq n$, it follows that $P_3, \ldots, P_n$ satisfy the required conditions with respect to G. It remains to chose P_1 and P_2.

Now $P_3 \leq F_2(N) \leq F_3(G)$ but $P_3 \nleq F_1(N) = F_2(G) \cap N$. Lemma 2.2 applied to $G/F_1(G)$ yields that for some prime $p_2 \neq p_3$, P_3 does not centralize the Sylow p_2-subgroup of $F_2(G)/F_1(G)$. Now $F_2(G) = F_1(G)C$ and C is nilpotent. Thus if S is the Sylow p_2-subgroup of C, S is invariant under $AP_3 \ldots P_n$ and $[S, P_3] \nleq F_1(G)$. Let P_2 be minimal

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with respect to: $P_2 \leq S$, P_2 is invariant under $AP_3 \ldots P_n$, and $[P_2, P_3] \nleq F_1(G)$. Since $[P_2, P_3]$ is invariant under $AP_3 \ldots P_n$ and $[P_2, P_3, P_3] = [P_2, P_3]$, [3], we must have $[P_2, P_3] = P_2$. It now only remains to choose P_1 .

$P_2 \leq F_2(G)$ but $P_2 \nleq F_1(G)$. Hence there is a prime $p_1 \neq p_2$ such that P_2 does not centralize the Sylow p_1-subgroup of $F_1(G)$. Then there is a group p_1 which is minimal with respect to: P_1 is a p_1-subgroup of $F_1(G)$, P_1 is invariant under $AP_2 \ldots P_n$, and $[P_1, P_2] \neq 1$. Since $[P_1, P_2]$ is invariant under $AP_2 \ldots P_n$ and $[P_1, P_2, P_2] = [P_1, P_2] \neq 1$, we must have $[P_1, P_2] = P_1$. $P_1, \ldots, P_n$ now satisfy *(a)* through *(g)* and the theorem is proved.

COROLLARY 2.5. *In Theorem 2.4, let* $Q_i = P_i/(P_i \cap F_{i-1}(G))$ *for* $1 \leq i \leq n$. *Then* Q_n *is elementary abelian and is transformed irreducibly by* A . *If* $1 \leq i \leq n-1$, *then* Q_i *is a special* p_i*-group and any proper subgroups of* Q_i *which are invariant under* $A \prod_{i<j} P_j$ *are contained in* Q_i' .

Proof. This follows from Theorem 2.1 and from *(f)* and *(g)* in Theorem 2.4.

LEMMA 2.6. *Suppose* G, A, n, P_i , *and* p_i *for* $1 \leq i \leq n$ *have the same meaning as in Theorem 2.4. Assume that every* A*-invariant proper subgroup of* G *has nilpotent length* $< n$. *For* $1 \leq i \leq n$, *let* $T_i = P_i \cap F_{i-1}(G)$, $Q_i = P_i/T_i$, *and* $C_i = C_A(Q_i)$. *Then* $G = P_1P_2 \ldots P_n$ *and* $[P_j, C_i] = 1$ *if* $1 \leq i \leq j \leq n$.

Proof. $P_1P_2 \ldots P_n$ is A-invariant and, from Lemma 2.3, $l(P_1P_2 \ldots P_n) \geq n$. Hence $P_1P_2 \ldots P_n = G$. Let $H_i = C_{P_i}(C_i)$. Since $C_i \trianglelefteq C$, H_i is A-invariant. From $[P_i/T_i, C_i] = 1$ and $(|P_i|, |C_i|) = 1$ follows $H_iT_i = P_i$. Since H_n is A-invariant,

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$T_n \leq F_{n-1}(G)$, and $P_n \nleq F_{n-1}(G)$, Theorem 2.4 *(f)* implies that $H_n = P_n$. Assume now $1 \leq i < n$ and $[P_j, C_{i+1}] = 1$ if $i+1 \leq j \leq n$. $[P_i, C_i, P_{i+1}] \leq F_{i-1}(G)$ and $[P_{i+1}, P_i, C_i] = [P_i, C_i] \leq F_{i-1}(G)$. The 3 Subgroups Lemma yields $[P_{i+1}, C_i, P_i] \leq F_{i-1}(G)$. It follows from this that $[P_{i+1}, C_i] \leq C_{P_{i+1}}(Q_i)$. Let $K = C_{P_{i+1}}(Q_i)$. Then $KF_{i-1}(G)$ is normalized by $F_{i-1}(G)P_iP_{i+1} \ldots P_n = G$. Since $KF_{i-1}(G)/F_{i-1}(G)$ is nilpotent, we must have $K \leq F_i(G)$. A consequence of this is that $[P_{i+1}, C_i] \leq T_{i+1}$. Hence $C_i \leq C_{i+1}$. Then $[P_j, C_i] = 1$ if $i+1 \leq j \leq n$. It follows from this that H_i is normalized by $P_{i+1}P_{i+2} \ldots P_n$. Theorem 2.4 *(g)* now implies that either $H_i = P_i$ or $[P_{i+1}, H_i] \leq F_{i-1}(G)$. Since $P_i = H_iT_i$, $T_i \leq F_{i-1}(G)$, and $[P_{i+1}, P_i] \nleq F_{i-1}(G)$, we cannot have $[P_{i+1}, H_i] \leq F_{i-1}(G)$. Thus $H_i = P_i$ and the lemma is proved.

3. The main results

Throughout this section we assume A is an elementary abelian group of order $p^m > 1$ which acts as an operator group on the p'-group G .

THEOREM 3.1. *Assume G is solvable and let* $n = \max\left\{l\left(C_G(X)\right) \mid X \in A^{\#}\right\}$. *If $m \geq n+2$, then $l(G) = n$.*

Proof. Suppose G is a counter-example of minimal order. Then if H is an A-invariant proper subgroup of G , we must have $l(H) \leq n$. Also if H is an A-invariant non-identity normal subgroup of G , then $l(G/H) \leq n$. This implies that $l(G) = n + 1$.

Let $P_1, \ldots, P_{n+1}$ be the A-invariant subgroups of G guaranteed by Theorem 2.4. Let $T_i = P_i \cap F_{i-1}(G)$, $Q_i = P_i/T_i$, and $C_i = C_A\left(Q_i\right)$ for $1 \leq i \leq n+1$. Now $C_{Q_{n+1}}(X)$ is A-invariant for $X \in A^{\#}$. Using Corollary 2.5, we see that $X \in A^{\#}$ implies $C_{Q_{n+1}}(X) = 1$ or Q_{n+1} .

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Hence if B_{n+1} is a complement to C_{n+1} in A, we see that $\left\langle C_{Q_{n+1}}(X) \mid X \in B_{n+1}^{\#}\right\rangle = 1$. Hence, by [4, Theorem 6.2.4], B_{n+1} must be cyclic. This implies $|A : C_{n+1}| \leq p$.

By Lemma 2.6, $C_i \leq C_{i+1}$ if $1 \leq i \leq n$. Let B_i be a complement of C_i in C_{i+1}. Let $X \in B_i^{\#}$ and $R = C_{Q_i}(X)$. R is A-invariant and is also invariant under $C_{P_j}(X) = P_j$ for $i < j \leq n+1$. $\left\{C_{P_j}(X) = P_j\right.$ from Lemma 2.6.$\left.\right\}$ Corollary 2.5 implies that R is one of the groups 1, Q_i', or Q_i. $R \neq Q_i$ since $X \notin C_i$. Hence we have shown that $\left\langle C_{Q_i}(X) \mid X \in B_i^{\#}\right\rangle \leq Q_i' \neq Q_i$. From [4, Theorem 6.2.4] it follows that B_i is cyclic. Hence $|C_{i+1} : C_i| \leq p$.

From $|A : C_{n+1}| \leq p$, $|A| \geq p^{n+2}$, and $|C_{i+1} : C_i| \leq p$ for $1 \leq i \leq n$, we obtain $|C_i| \geq p^i$ for $1 \leq i \leq n$. Hence there is a non-identity element X in C_1. Then Lemma 2.6 implies that $[P_i, X] = 1$ for $1 \leq i \leq n+1$. Hence $C_G(X) = G$. But $l(G) = n + 1$ and $l\left(C_G(Y)\right) \leq n$ for all $Y \in A^{\#}$. This contradiction finishes the proof.

LEMMA 3.2. *Assume* $m \geq 3$ *and* $C_G(X)$ *is abelian for all* $X \in A^{\#}$. *Then* G *is abelian.*

Proof. Let G be a minimal counter-example. Then if H is an A-invariant non-identity normal subgroup of G, G/H must be abelian. It follows from this that G' is a minimal A-invariant normal subgroup of G. From [8], G is nilpotent. Since $G' \cap Z(G) \neq 1$, we must have $G' \leq Z(G)$. Then any subgroup of G' is normal in G. This implies that A transforms G' irreducibly. Thus $C_{G'}(X) = 1$ or $= G'$ for each $X \in A^{\#}$. Let $C = C_A(G')$ and let B be a complement to C in A. Then

$\langle C_{G'}(X) \mid X \in B^{\#} \rangle = 1$. Hence B must be cyclic and so $|C| \geq p^2$.

Since C is not cyclic, $G = \langle C_G(X) \mid X \in C^{\#} \rangle$. Let $X, Y \in C^{\#}$, $H = C_G(X)$, and $K = C_G(Y)$. H and K are both abelian and $[H, K, \langle X \rangle] \leq [G', \langle X \rangle] = 1$ and $[\langle X \rangle, H, K] = 1$. The 3 Subgroups Lemma implies $[K, \langle X \rangle, H] = 1$. Now K is A-invariant and so $K = [K, \langle X \rangle] C_K(X)$. But $C_K(X) \leq H$ and H is abelian. Thus $[K, H] = [K, \langle X \rangle, H] = 1$. It follows that G is abelian.

THEOREM 3.3. *Assume* $m \geq 4$, G *is solvable, and* $C_G(X)$ *is supersolvable for all* $X \in A^{\#}$. *Then* G *is supersolvable.*

Proof. Suppose G is a counter-example of minimal order. If H is an A-invariant non-identity normal subgroup of G , then G/H is supersolvable. It follows from this that $D(G) = 1$ and there is only one minimal A-invariant normal subgroup of G . Therefore $F(G)$ is an elementary abelian q-group for some prime q . From Theorem 2.1, $l(G) = 2$. Hence $G/F(G)$ is a nilpotent q'-group. Now if $G/F(G)$ were abelian of exponent dividing $q - 1$, then from [1, Theorem 6.1], G would be supersolvable. Thus for some prime $r \neq q$, there is an A-invariant r-subgroup R in G such that either R is non-abelian or the exponent of R does not divide $q - 1$. Then $RF(G)$ is an A-invariant subgroup of G and $RF(G)$ is not supersolvable. Thus $RF(G) = G$.

Let $C = C_A(R)$ and let B be a complement to C in A . I assert that $|B| \geq p^3$. Suppose to the contrary that $|B| < p^3$. Then $|C| \geq p^2$ and so $F(G) = \langle C_{F(G)}(X) \mid X \in C^{\#} \rangle$. Thus there would be an $X \in C^{\#}$ such that $C_{F(G)}(X) \neq 1$. Now $C_R(X) = R$ and A is abelian. Thus $C_{F(G)}(X)$ is invariant under AR . By Maschke's Theorem, there is an AR-invariant complement K to $C_{F(G)}(X)$ in $F(G)$. Since $F(G)$ is abelian, K and $C_{F(G)}(X)$ are normal in $RF(G) = G$. Since there is only one minimal A-invariant normal subgroup in G , we must have $K = 1$. Then $C_{F(G)}(X) = F(G)$ which implies $G = F(G)R = C_G(X)$ is supersolvable.

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Thus $|B| \geq p^3$. Now let $X \in B^{\#}$. $C_R(X) \neq R$ and so $C_R(X)F(G)$ is a proper A-invariant subgroup of G. Thus $C_R(X)F(G)$ must be supersolvable. It follows from this that $C_R(X)$ is abelian of exponent dividing $(q-1)$. Lemma 3.2 now implies that R is abelian. Since $R = \left\langle C_R(X) \mid X \in B^{\#} \right\rangle$, the exponent of R must divide $(q-1)$ and the theorem is proved.

4. Examples

1. Let A be an elementary abelian p-group of order p^{n+1} where $n \geq 1$. Then by [5], there is an odd order p'-group G on which A operates in a fixed-point-free manner and such that $l(G) = n + 1$. If $X \in A^{\#}$, then $C_G(X)'$ admits a fixed-point-free abelian operator group of order p^n. By [7], this implies that $l\left(C_G(X)\right) \leq n$. Hence the requirement $m \geq n + 2$ is necessary in Theorem 2.1.

2. Let G be a non-abelian group of order 27 and exponent 3. Let a and b be any elements generating G. Then there are automorphisms x and y of G such that $a^x = a$, $b^x = b^{-1}$, $a^y = a^{-1}$, and $b^y = b$. x and y generate an elementary abelian group A of order 4. $C_G(Z)$ has order 3 for all $Z \in A^{\#}$ but G is not abelian. Thus the requirement $m \geq 3$ is necessary in Lemma 3.2.

3. Let p, q, r, and s be four distinct odd primes such that $q \equiv 1 \pmod{rs}$ and $r \equiv 1 \pmod{s}$. (For example, $p = 5$, $q = 43$, $r = 7$, and $s = 3$ would be satisfactory.) Let A be elementary abelian of order p^3. Using the methods of [5], it is possible to construct a solvable group G such that:

(a) A acts in a fixed-point-free manner on G;

(b) $l(G) = 3$;

(c) $F_1(G)$ is an elementary abelian q-group;

(d) $F_2(G)/F_1(G)$ is an elementary abelian r-group;

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(e) $G/F_2(G)$ is an elementary abelian s-group.

Now if $X \in A^{\#}$, then $C_G(X)$ admits a fixed-point-free operator group of order p^2. Thus, by [6], $l\left(C_G(X)\right) \le 2$. From (c), (d), and (e), it follows that $C_G(X)$ is supersolvable. However, $l(G) = 3$, and so G is not supersolvable. Thus $m \ge 4$ is necessary in Theorem 3.3.

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