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HERMITIAN FORMS OVER ODD DIMENSIONAL ALGEBRAS

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0. Introduction

Let L be an odd degree extension field of the field K, $\operatorname{char} K \neq 2$. Let U^* denote the natural extension map from $W(K)$ to $W(L)$ where $W(K)$, resp. $W(L)$ denotes the Witt group of quadratic forms over K, resp. L. It is well-known that U^* is injective [**4**, p. 198]. In fact Springer [**10**] proved a stronger theorem, namely that if ϕ is anisotropic over K then it remains anisotropic on extension to L. Rosenberg and Ware [**8**] proved that if L is a Galois extension then the image of U^* is precisely the subgroup of $W(L)$ fixed by the Galois group of L over K, this Galois group having a natural action on $W(L)$. See [**4**, p. 214] and [**3**] for a quick proof. See also Dress [**1**] who extended these results to equivariant forms. In this article we investigate the corresponding map U^* when we replace the field L by a central simple K-algebra of odd degree and indeed more generally by any finite dimensional K-algebra which becomes odd-dimensional on factoring out by the radical. Our algebras are equipped with an involution of the second kind, i.e. one which is non-trivial on the centre, and we replace quadratic forms by hermitian forms with respect to the involution. We show that U^* is injective for all the algebras mentioned above and that a weaker version of Springer's theorem holds for central simple algebras of odd degree provided we make a suitable restriction on the nature of the involution. We show that the analogue of the Rosenberg–Ware result is valid for hermitian forms over odd-dimensional Galois *field* extensions but that for central simple algebras of odd degree a result as nice as the Rosenberg–Ware one cannot hold. Indeed the group of all K-automorphisms of such an algebra which commute with the involution fixes all of the Witt group. However the map U^* is not surjective in general even for division algebras of odd degree.

1. Basic definitions

Let K be a field of characteristic not two and $^-$ a non-trivial involution on K. Then $K = K_0(\sqrt{d})$ for some $d \in K_0$, K_0 being the fixed field of the involution. Let A be a central simple algebra with an involution of the second kind, i.e. one which restricts to $^-$ on K. We again use $^-$ to denote this involution. (Up to isomorphism A will be $M_n D$, the algebra of all $n \times n$ matrices with entries in a division algebra D over K. Since A and D are equivalent in the Brauer group of K it follows from [**9**, Cor. 8.3, p. 306] that D also admits an involution of the second kind. Let $\hat{}$ denote this involution on D. It does not follow that the involution $^{-}$ on A restricts to $\hat{}$ on D. However by the classification of involutions [9, ch. 8, section 7] it follows that for $X \in M_nD$, $\bar{X} = S^{-1}\hat{X}^tS$ for some $S \in M_nD$ with $S = \hat{S}^t$.

We write $W(K,^-)$, *resp* $W(A,^-)$, for the Witt group of non-singular hermitian forms over $(K,^-)$, resp. $(A,^-)$, these being defined in the usual way, the forms over $(A,^-)$ being defined on finitely generated A-modules. $W(K,^-)$ has a ring structure as the product of two hermitian forms over $(K,^-)$ and may be defined via the tensor product of the underlying spaces in the same fashion as for the product of quadratic forms. $W(A,^-)$ is an additive abelian group but also has a $W(K,^-)$-module structure in a natural way. The map U^* is easily seen to be a $W(K,^-)$-module homomorphism. (Given $\psi: V \times V \to K$, $U^*\psi: V \otimes_K A \times V \otimes_K A \to A$ is induced by $U^*\psi(x \otimes \alpha, y \otimes \beta) = \bar{\alpha}\psi(x,y)\beta$).

2. Injectivity of U^* and Springer's theorem

Lemma 1. *The Witt class in $W(K,^-)$ of a non-singular hermitian form on an odd-dimensional K-vector space cannot be a zero divisor in the ring $W(K,^-)$.*

Proof. This fact about zero-divisors is well known for quadratic forms [9, p. 54]. There is a natural imbedding [9, p. 348] $S: W(K,^-) \to W(K_0)$ under which a form over $(K,^-)$ with diagonalization $\langle a_1, a_2, \ldots, a_m \rangle$ maps to the form over K_0 of twice the dimension and with diagonalization $\langle a_1, a_2, \ldots, a_m, -da_1, -da_2, \ldots, -da_m \rangle$. (Recall that $K = K_0(\sqrt{d})$). Note that this form is a product $\langle 1, -d \rangle \langle a_1, \ldots, a_m \rangle$ of forms over K_0. The above map is not a ring homomorphism but is a $W(K_0)$-module homomorphism.

Now if $\langle a_1, \ldots, a_m \rangle$ for m odd is a zero divisor in $W(K,^-)$ then $\langle a_1, \ldots, a_m \rangle \langle b_1, \ldots, b_n \rangle = 0$ in $W(K,^-)$ for some form $\langle b_1, \ldots, b_n \rangle$ over $(K,^-)$. Passing down to K_0 we obtain a product $\langle 1, -d \rangle \langle a_1, \ldots, a_m \rangle \langle b_1, \ldots, b_n \rangle = 0$ in $W(K_0)$. Thus we have that $\langle a_1, \ldots, a_m \rangle$ as a form over K_0 is a zero divisor in $W(K_0)$ as $\langle 1, -d \rangle \langle b_1, \ldots, b_n \rangle$ is non-zero in $W(K_0)$ for $\langle b_1, \ldots, b_n \rangle$ non-zero in $W(K,^-)$. Since $W(K_0)$ can have no odd-dimensional zero divisors the lemma is proven.

Lemma 2. *The reduced trace map $T: A \to K$, A being a central simple K-algebra with an involution of the second kind, induces a well-defined $W(K,^-)$-module homomorphism $T^*: W(A,^-) \to W(K,^-)$.*

Proof. Recall that the reduced trace on A may be defined by tensoring on a splitting field L so that $A \otimes_K L$ is isomorphic to a matrix ring M_mL for some m and then taking the usual matrix trace in M_mL. The value of $T(a)$ for $a \in A$ lies in K and is independent of the choice of splitting field L [9, pp. 296–7]. Provided that we can choose a splitting field L which carries an involution extending that on K it follows that $T(\bar{a}) = \overline{T(a)}$ for all $a \in A$ since the involution on M_mL is similar to conjugate transpose. Any hermitian form $\phi: M \times M \to A$ then leads to a form $T\phi: M \times M \to K, (x,y) \mapsto T(\phi(x,y))$. It is now straightforward to check that T induces a well-defined $W(K)$-module homomorphism $T^*: W(A,^-) \to W(K,^-)$.

It remains to show that A has a splitting field L carrying an involution extending that on K. Now $(A, {}^{-})$ is isomorphic to $(M_mD, {}^{-})$ where $\bar{X}=S^{-1}\hat{X}^tS$ for $X \in M_nD$, $S=\hat{S}^t$, and $\hat{}$ is an involution on D which extends that on K. Since any splitting field of D is a splitting field of A it suffices to show the existence of a splitting field L of D carrying an involution extending that on K. To do this choose a subfield E of D maximal with respect to the property of being invariant under the involution $\hat{}$. By taking $x \notin K$, $\hat{x}=x$ we can certainly obtain proper extension fields of K invariant under $\hat{}$). Let C_DE be the centralizer of E in D which is a division algebra and is involution-invariant since E is. If $C_DE=E$ we take $L=E$. If not then, by induction on the degree, C_DE has an involution-invariant subfield M splitting C_DE. This contradicts the maximality of E since M contains E properly.

Theorem 1. *The map $U^*: W(K,{}^{-}) \to W(A,{}^{-})$ is injective for any odd degree central simple algebra with an involution of the second kind.*

Proof. Let $\phi: A \times A \to K$, $\phi(x,y)=T(\bar{x}y)$ be the trace form of $(A,{}^{-})$ over $(K,{}^{-})$. If ψ represents an element of $W(K,{}^{-})$ then the composite map $(T^* \circ U^*)(\psi)=\phi\psi$ where $\phi\psi$ is the product in the ring $W(K,{}^{-})$. To see this choose a K-basis for A and write out the matrix for $(T^* \circ U^*)(\psi)$, ψ being written in diagonal form.

If ψ is in the kernel of U^* then $(T^* \circ U^*)(\psi)=0$ in $W(K,{}^{-})$ (so that $\phi\psi=0$ in $W(K,{}^{-})$. But ϕ is odd-dimensional and so, by Lemma 1, cannot be a zero divisor in $W(K,{}^{-})$. Thus $\psi=0$ in $W(K,{}^{-})$ and U^* is injective.

We can, in fact, obtain a more general theorem as follows:

Theorem 2. *Let A be a finite-dimensional K-algebra with an involution ${}^{-}$ of the second kind. If J denotes the Jacobson radical of A and A/J is odd-dimensional over K then the map $U^*: W(K,{}^{-}) \to W(A,{}^{-})$ is injective.*

Proof. J is the intersection of all the maximal right ideals of A or equivalently the intersection of all the maximal left ideals [2, p. 196]. Thus J is involution-invariant and there is a well-defined involution on A/J which we again denote by ${}^{-}$.

Since A is finite-dimensional A is the inverse limit of A/J^n so that A is complete in the J-adic topology. We may thus apply the reduction principle of [9, Theorem 4.4, p. 253]. This gives an isomorphism $W(A,{}^{-}) \to W(A/J,{}^{-})$ and since the isomorphism is compatible with the extension maps U^* it suffices to prove that $U^*: W(K,{}^{-}) \to W(A/J,{}^{-})$ is injective.

Now A/J is isomorphic to a direct sum of simple algebras, i.e. $A/J=A_1 \oplus A_2 \oplus \cdots \oplus A_r$. The involution ${}^{-}$ on A/J either preserves components or interchanges pairs of isomorphic components. For the interchanging involution the corresponding Witt group is trivial [9, p. 245]. Thus $W(A/J,{}^{-})$ is a direct sum $\sum_{i=1}^{s} W(A_i,{}^{-})$ for some $s \leqq r$. The map $U^*: W(K,{}^{-}) \to W(A/J,{}^{-})$ thus amounts to

$$W(K,{}^{-}) \to \sum_{i=1}^{s} W(A_i,{}^{-})$$

$$\psi \mapsto \sum_{i=1}^{s} U_i^*(\psi)$$

for maps $U_i^*: W(K,^-)\to W(A_i,^-)$, each A_i being a simple K-algebra with an involution of the second kind. U^* will be injective if at least one of the maps U_i^* is injective. Since A/J is odd-dimensional at least one A_i is odd-dimensional. The centre of this A_i is a field L which either equals K or is an odd degree extension of K. The map U_i^* for this i must be injective because of Theorem 1 and the fact that $W(K,^-)\to W(L,^-)$ is injective for an odd degree extension field L of K. This completes the proof.

We now try to obtain an analogue of Springer's theorem **[10]** that an anisotropic quadratic form over K remains anisotropic over L where L is an odd degree extension of K. His proof goes through equally well if hermitian forms over K and L are used instead of quadratic forms. However his technique does not apply when we extend a hermitian form over $(K,^-)$ to one over $(A,^-)$ where A is non-commutative. (A hermitian form $h: M\times M\to A$ is *isotropic* if $h(x,x)=0$ for some $x\neq 0$. Otherwise h is said to be *anisotropic.*) We will obtain a weaker version of Springer's theorem in this situation.

First some terminology is needed. A form ψ is said to be *strongly anisotropic* provided $m\times\psi$, the orthogonal sum of m copies of ψ, is anisotropic for each positive integer m. If ψ fails to be strongly anisotropic it is said to be weakly isotropic. (This definition makes sense for both quadratic and hermitian forms over fields or central simple algebras.)

An involution on a central simple algebra A is said to be *congenial* if A admits an involution invariant maximal subfield L such that the extended involution on $A\otimes_K L$ corresponds to conjugate transpose on $M_n L$.

Theorem 3. *Let A be a central simple K-algebra of odd degree and with an involution of the second kind. Assume that this involution is congenial. Let ψ be a strongly anisotropic hermitian form over $(K,^-)$. Then $U\psi$ is anisotropic as a form over $(A,^-)$.*

Proof. Suppose $U\psi$ is isotropic over $(A,^-)$. For the involution-invariant maximal subfield L the extended form $U\psi\otimes L$ over $A\otimes_K L$ is isotropic. But $A\otimes_K L$ is isomorphic to $M_n L$ and the involution is congenial. Hence, by Morita theory **[6]**, the form $U\psi\otimes L$ corresponds to the form $n\times U_1\psi$ over $(L,^-)$. Here $U_1\psi$ denotes the extension to L of the form ψ. Since Morita theory preserves the property of being anisotropic it follows that $U_1\psi$ is weakly isotropic as a form over $(L,^-)$. Hence ψ is weakly isotropic over $(K,^-)$ since Springer's theorem holds for hermitian forms under odd degree field extensions. This completes the proof.

Comment 1. Theorem 3 is not true unless we make an assumption of congeniality for the involution even if we assume that A is a division algebra.

Example. The following is a well-known cyclic division algebra of degree three. The base field $K=\mathbb{Q}(w)$ where $w^2+w+2=0$, $\mathbb{Q}$ being the rationals. We take $w=\theta+\theta^2+\theta^4$

where θ is a primitive seventh root of unity. Now $\mathbb{Q}(w)$ lies inside the field of complex numbers $\mathbb{C}$ and complex conjugation restricts to give an involution on $\mathbb{Q}(w)$. (Note that $\bar{w}=-1-w$).

Let $\alpha=\theta+\theta^6$, $\beta=\theta^2+\theta^5$, $\gamma=\theta^3+\theta^4$. Then $L=K(\alpha)$ is a cubic extension of K with Galois group $\{1,\sigma,\sigma^2\}$ where $\alpha^\sigma=\beta$, $\beta^\sigma=\gamma$, $\gamma^\sigma=\alpha$. Let D be the cyclic algebra $(L/K,\sigma,b)$ where $b=\bar{w}/w$. A typical element of D is an expression $x_0+x_1u+x_2u^2$ where $u^3=b$, $x_i\in L$ for $i=0,1,2$ and $ux=x^\sigma u$ for $x\in L$. Let $^-$ be the involution on D induced by $\bar{u}=u^{-1}$. (The involution on K extends to L in a natural way.) This involution on D is easily seen to be congenial.

Now let $p=1+3\alpha$ and define a new involution $*$ on D by $x^*=p\bar{x}p^{-1}$ for each $x\in D$. The form $\langle 1,1,1\rangle$ over $(K,^-)$ is strongly anisotropic. However on extension to $(D,*)$ the form $\langle 1,1,1\rangle$ becomes, by scaling, equivalent to the form $\langle p,p,p\rangle$ over $(D,^-)$. This latter form is isotropic since $p+\bar{u}pu+\bar{u}^2pu^2=p+p^{\sigma^2}+p^\sigma=T(p)$ where T is the trace map from L to K. It is easy to check that $T(p)=0$ since $p=1+3\alpha$. Thus $U\psi$ is isotropic as a form over $(D,*)$.

Comment 2. One may ask whether the full Springer theorem is true for congenial involutions, i.e. whether the word strongly can be removed in Theorem 2. This is certainly not the case for central simple algebras in general since taking $A=M_nK$ with conjugate transpose as involution any anisotropic form ψ over $(K,^-)$ *with* $n\times\psi$ isotropic will give $U\psi$ isotropic. It seems unlikely that the full Springer theorem would hold even for division algebras with congenial involution but we have not produced a counter-example.

3. The image of U^*

We first show that the Rosenberg–Ware theorem remains valid for *hermitian* forms under odd degree Galois *field* extensions.

Let L be an odd degree Galois extension of K and assume that there is an involution $^-$ of the second kind on K which extends to an involution of the second kind on L. We also use $^-$ to denote this involution. Letting L_0 and K_0 be the fixed fields of the involutions on L and K respectively we have that L_0 is a Galois extension of K_0. Let G and G_0 be the Galois group of L over K and L_0 over K_0 respectively. Now $L=L_0K$ and the restriction to L_0 of each element of G yields an isomorphism between G and G_0. We identify G and G_0 via this isomorphism.

The Galois groups G and G_0 act naturally on $W(L,^-)$ and $W(L_0)$ respectively. Recall that this action is defined as follows:

Given $\psi: V\times V\to L_0$, a symmetric bilinear form over L_0 we define a new L_0-vector space V^g for $g\in G_0$ by letting $V^g=V$ as an additive abelian group but with a new scalar multiplication defined by $\lambda * v=\lambda^{g^{-1}}v$ for $\lambda\in L_0$, $v\in V$. The form $\psi^g: V^g\times V^g\to L_0$ is defined by $\psi^g(v,w)=\psi(v,w)^g$. A similar definition may be given for the G-action on $W(L,^-)$.

Theorem 4. *The map* $U_1^*: W(K,^-)\to W(L,^-)$ *is injective with image* $W(L,^-)^G$, *i.e. the subgroup of elements of* $W(L,^-)$ *fixed by* G.

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Proof. The fact that U_1^* is injective follows from what we have done earlier. To find the image of U_1^* consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & W(L,^-) & \xrightarrow{S_L} & W(L_0) & \xrightarrow{V_L} & W(L) \\ & & \uparrow U_1^* & & \uparrow U_0^* & & \uparrow U^* \\ 0 & \longrightarrow & W(K,^-) & \xrightarrow{S_K} & W(K_0) & \xrightarrow{V_K} & W(K) \end{array}$$

The maps U_1^*, U_0^*, U^* are the natural extension maps, S_L and S_K are the injective maps described at the start of Section 2, and V_L and V_K are the extension maps to the quadratic extensions L and K of L_0 and K_0 respectively. The exactness of the rows is well-known [**9**, p. 348]. Also the maps U_1^*, U_0^*, U^* are injective.

It is evident that $(S_L\psi)^g = S_L(\psi^g)$ for $g \in G$. (Recall that we identify G and G_0). Also ψ^g is isometric to ψ if and only if $(S_L\psi)^g$ is isometric to $S_L\psi$. It is clear also that ψ is fixed by G if and only if $S_L\psi$ is fixed by G_0. Thus the image of U_1^* is contained in $W(L,^-)^G$ since the image of U_0^* is $W(L_0)^{G_0}$ by the Rosenberg–Ware theorem. Given an arbitrary element of $W(L,^-)^G$ a diagram chasing argument shows that it must be in the image of U_1^*. This completes the proof.

Now we investigate whether a similar result holds for the image of $U^*: W(K,^-) \to W(A,^-)$ when A is a central simple K-algebra of odd degree. Let G denote the group of all automorphisms of the algebra A which act as the identity map on K, and also which commute with the involution $^-$ on A. This last condition is needed to ensure that the form ψ^g is hermitian symmetric for $g \in G$, ψ a hermitian form over $(A,^-)$. (The definition of ψ^g is analogous to the field case earlier.)

Theorem 5. *Let A be a central simple K-algebra of odd degree and with an involution of the second kind. Then G fixes all of $W(A,^-)$.*

Proof. Let $g \in G$. Then, since all automorphisms of A are inner by the Skolem–Noether theorem, there exists a unit $u \in A$ such that $g(x) = uxu^{-1}$ for all $x \in A$. The condition that g commutes with $^-$ implies that $\bar{u}u$ belongs to K.

Now $U^*: W(K,^-) \to W(A,^-)$ is injective and $\bar{u}u$ is trivially a hermitian square in A. Hence $\bar{u}u$ must be a hermitian square in K, i.e. $\bar{u}u = \bar{\beta}\beta$ for some $\beta \in K$. Thus $g(x) = uxu^{-1} = \bar{z}xz$ where $z = \beta^{-1}\bar{u}$. It follows that ψ^g is isometric to ψ for each hermitian form ψ over $(A,^-)$, and hence the whole of $W(A,^-)$ is fixed by G.

Comment 1. The group G is infinite and $g \in G$ will have finite order m if and only if d^m is an element of K.

Comment 2. If the analogue of the Rosenberg–Ware result were to hold for this group G then U^* would have to be surjective. This is certainly not the case in general.

Example. Let K be an algebraic number field with an involution of the second kind. Let D be a cyclic division algebra over K admitting an involution of the second kind. Hermitian forms over $(D,^-)$ are classified up to isometry by rank, determinant and a set of signatures one at each real prime $\not{p}$ of K_0 at which D becomes a full matrix ring over $K_{\not{p}}$. Here $K_{\not{p}} = K \otimes_{K_0} (K_0)_{\not{p}}$ and $(K_0)_{\not{p}}$ is the completion at prime $\not{p}$. Note that $K_{\not{p}}$ is isomorphic to $\mathbb{C}$, the complex numbers. The classification was done originally by Landherr [5]. See also [7] and [9, ch. 10] for a modern treatment. Forms exist with any prescribed set of invariants, modulo the obvious relationships that must exist between these invariants. Let us assume that our involution is definite [9, p. 376]. i.e. that at each real prime of the kind mentioned above the involution on $D_{\not{p}} = D \otimes_K K_{\not{p}}$ corresponds to conjugate transpose on $M_n\mathbb{C}$. It is easy to see that the form $\langle 1 \rangle$ over $(D,^-)$ has signature n at each prime $\not{p}$ and hence that any element in the image of $U^*: W(K,^-) \rightarrow W(D,^-)$ must have each of its signatures divisible by n. Since $W(D,^-)$ contains components $\mathbb{Z}$ corresponding to signatures it is clear that U^* is not surjective.

Comment 3. It is possible to impose strong restrictions on the nature of the algebra A and on the base field K which ensures that the map U^* is surjective. In general it is not at all clear that the image of U^* can be described in a concise way in any kind of analogue of the Rosenberg–Ware result.

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