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MATHEMATICAL PROBLEMS IN TRANSONIC FLOW

BY
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ABSTRACT. We present an outline of the problem of irrotational compressible flow past an airfoil at speeds that lie somewhere between those of the supersonic flight of the Concorde and the subsonic flight of commercial airlines. The problem is simplified and the important role of modifying the equations with physics terms is examined.

The subject of my talk is transonic flow. But before I can talk about transonics I must talk about flight in general. There are two ways of staying off the ground — by rocket propulsion (the Buck Rogers mode) or by using the forces that permit gliding or for that matter sailing. Every form of man's flight is some compromise between these two extremes.

The one that most mathematicians begin their learning on is incompressible, steady (no time dependence), irrotational, flow governed by (i) Conservation of mass with $\boldsymbol{q}$ the velocity, (what goes in comes out) div $\boldsymbol{q} = 0$, (ii) Irrotationality, curl $\boldsymbol{q} = 0$. The pressure is given by Bernoulli's law, $p = p(|\boldsymbol{q}|)$.

These equations are equivalent to the Cauchy–Riemann equations and so lots of problems can be solved. But there is an anomaly — there is no drag and for that matter often no lift i.e. no net force on the object which for our purposes and from here on is a cross section of a wing. By taking a nonsymmetric cross section that has a cusp at the end and requiring that the flow has a finite velocity we obtain a flow with lift but still no drag.

Very slight modifications are necessary to take compressibility into account. The equations become: (i) Conservation of mass, div $\rho\boldsymbol{q} = 0$, (ii) Irrotationality, curl $\boldsymbol{q} = 0$. Density $\rho = \rho(|q|)$ is given by Bernoulli's law. Pressure is a function of density and hence speed. The boundary layer can be treated separately and the flow can be regarded as flow at zero viscosity past an infinite "airfoil" with a boundary layer inside (see Figure 1). We get a very good description of the flow at low speeds and a fairly accurate measure of drag and lift, pretty well confirmed in wind tunnels.

Let me remind you that we keep the airfoil stationary. Thus its actual speed is the speed q_∞ at ∞.

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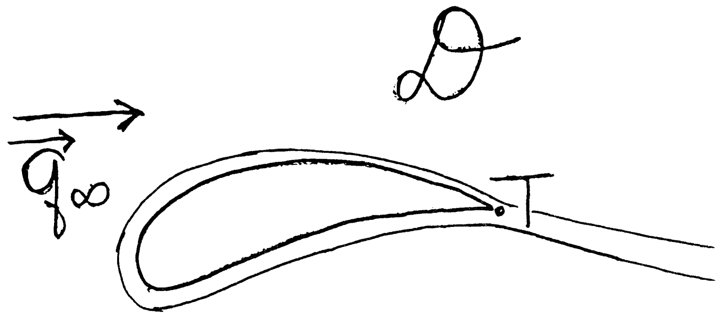


Figure 1

Cusped nonsymmetric airfoil with boundary layer.

The system of equations reduces to

$$(c^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (c^2 - v^2)\phi_{yy} = 0 \tag{1}$$

with $\boldsymbol{q} = \nabla\phi$; $c = c(|q|)$ is the local speed of sound.

By setting $v = 0$ (always possible at a point) we see that the equation is elliptic (like the Laplacian) or hyperbolic (like the wave equation) depending on whether $c^2 > q^2$ (subsonic) or $c^2 < q^2$ (supersonic).

If u and v are very small c is nearly constant and we have the Laplacian and incompressible flow. The property that determines that things are nice and soluble is that over a range of speeds at ∞ the equation is elliptic everywhere. Problems arise however as that speed increases and supersonic zones appear. The equation is of mixed type and the smoothness associated with elliptic problems disappears. The airfoil problem I am describing is not the only case where mixed equations arise but it is not exactly your everyday equation like the Laplacian or the wave equation. It is what is called quasilinear and where the flow is supersonic it can be expected to have shocks. Nonlinear mixed equations also arise in elasticity and in geometry.

Elliptic Case. The first studies on equation (1) were made by perturbing about the zero speed problem, that is the Laplacian case, but the big step forward for the subsonic case came with the whole theory of nonlinear elliptic equations in two space variables. This was developed in the forties and early fifties and many contributions went back and forth from analysis to the two big applications — subsonic fluid dynamics and minimal surfaces.

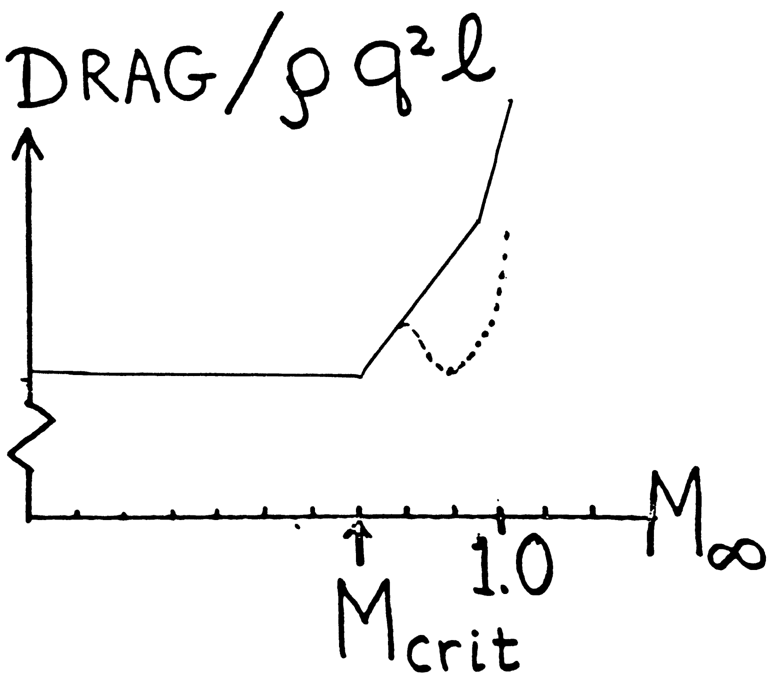
But even as long ago as the mid-thirties, engineers, in particular Busemann [2], following some wind tunnel studies, were raising questions about the supersonic case. A bit was known from simple examples and from one dimensional gas dynamics about the completely supersonic case (the "Concorde" case, flight is more rocket-like). At that time wind tunnels were very inaccurate; computations were impossible — not enough power or memory so no one really knew what happened when the flow went supersonic.

Wind tunnel pictures show very clearly how unsteady the flows are but these pictures nevertheless give us some idea of what the underlying steady flow is like and where the shocks are.

First let me point out that elliptic implies smoothness and hyperbolic implies that singularities propagate. Recall that in one variable $u_{tt} - u_{xx} = 0$, $u = f(x - t) + g(x + t)$, so singularities run on the curves $x - t =$ constant, $x + t =$ constant. Nonlinear hyperbolic also implies the possibility in fact the probability that shocks will propagate and interact.

Examining the experiments carefully one finds that shocks occur at moderate speeds in such a way as to close off the supersonic region. At higher speeds this shock moves off to the tail. Finally, if the speed at ∞ is supersonic there is a bow shock and a tail shock. At all times the supersonic region is marked by the fact that one sees time dependent singularities jumping around in them.

We now ask (1) What is the mathematical theory behind these phenomena? i.e. what do we know about the boundary value problem satisfied by the flow? (2) What phenomena are important to engineers? What they want to know is the size of the force retarding the object, i.e. the drag.



——— normal profile
– – – – supercritical profile

Figure 2

We all know that the Concorde is expensive to fly and carries few passengers and a lot of fuel. This means the energy needed to overcome the drag caused by the bow shock is very high. So we should all fly slowly in a region where there is only boundary layer or modest shock drag. But how slowly do we have to fly?

In Figure 2 we have a graph of drag versus Mach number (ratio of speed to sound speed) for a typical airfoil. When and why does the drag increase? The answer: when shocks appear. Why? Because across a shock lots of things are conserved but there is a gain of entropy and that leads to drag. Shocks cannot appear in an elliptic problem but do appear as we have seen in mixed, that is to say, transonic problems. Could we get rid of them? How do we get an airfoil that does not have a shock at some ambient cruising speed? One way is to make it stay subsonic. The only way to do that, as you increase the speed at ∞, is to let the wing become thinner and thinner. That is no good for flying passengers but it does reveal that it is possible. The right way to do it was discovered experimentally by Whitcomb and mathematically by Paul Garabedian [1]. Garabedian and Korn's earlier design was tested first by Kacprzynski in Canada, [7].

Lighthill [8] had shown that there were special cases where separation of variables could be used theoretically to find airfoils. To each subsonic speed at infinity there was one airfoil with a supersonic region and no shock. Garabedian abandoned Lighthill's series method and made use of new computational power to find airfoils by a very beautiful mathematical technique. However it is not beloved in the machine shop because it is somewhat mysterious and involves extending the flow analytically into complex space and working with a hyperbolic system.

The idea is to treat x, y as complex variables so that there are four real variables and then the equations become complex hyperbolic equations. Written in characteristic form these are

$$\begin{aligned} y_\xi + \lambda_+ x_\xi &= 0, \\ y_\eta + \lambda_- x_\eta &= 0, \\ u_\xi - \lambda_- v_\xi &= 0, \\ u_\eta - \lambda_+ v_\eta &= 0, \end{aligned}$$

$$\lambda_\pm = \frac{uv \pm c\sqrt{q^2 - c^2}}{c^2 - u^2};$$

ξ, η are characteristic variables. Luckily, for computations one can reduce the problem to three real variables.

In the complex ξ, η plane (Figure 3 is a real projection) we have a Goursat problem. We can difference the equations using the mesh indicated in the figure and obtain

$$u(P) - u(Q) - \lambda_-(Q)\,(v(P) - v(Q)) = 0$$

$$u(P) - u(R) - \lambda_+(R)\,(v(P) - v(R)) = 0$$

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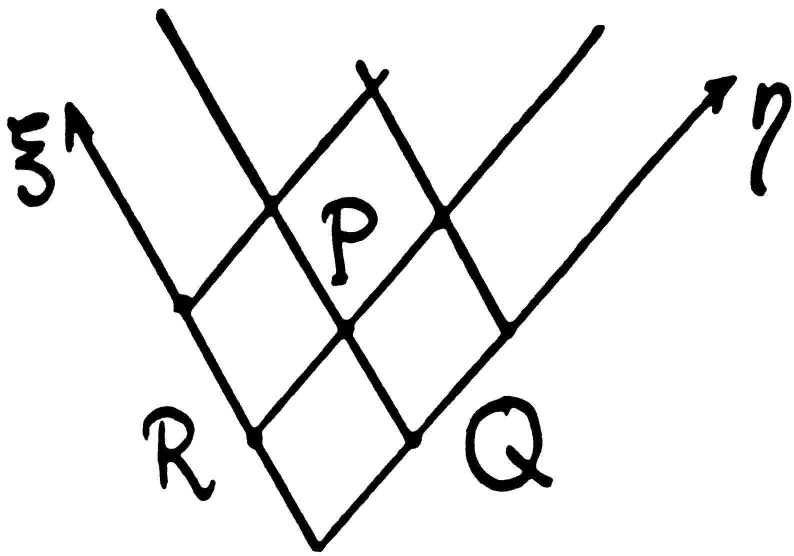

Figure 3. The Goursat Problem.

What data should we use? Long experience shows that by appropriate choice of data on the characteristic boundaries one can tune this scheme not only to avoid singularities in the real space and produce airfoils but to produce airfoils with maximal lift and many other important features.

We now have an airfoil. But long before it was designed it was clear that it would be special, that changes in its cruising speed would almost certainly lead to shocks.

The nicest thing about the corresponding theorem, [11], was that it really answered, mathematically, a puzzle for engineers.

Let us go back to our boundary value problem, in the symmetric case. ϕ satisfies a p.d.e., $\partial\phi/\partial n = 0$ on a given boundary $\partial\mathcal{D}$ and $\nabla\phi \to$ a given vector at ∞. The question we have to ask now is whether this problem is well posed and to study it, we must perturb the problem. We then discover:

THEOREM. The perturbation problem for the b.v.p. for ϕ is improperly posed for variations in $\partial\mathcal{D}$ if $\nabla\phi$ remains continuous.

First, what is the perturbation equation? A trick, due to Guderley, will get it easily. Suppose we have a system of conservation laws in two variables

$$u \in R^2,\ x \in R^2$$

$$(f(u))_x = 0$$

then

$$f_u u_x = 0 \quad \text{or} \quad h(u)x_u = 0$$

Next if $\nabla\phi = u$ reduces $f_x = 0$ we can introduce the Legendre potential, $\Phi(u) = u \cdot x - \phi(x)$. Then $\Phi_u = x$ and we obtain a second order equation for $\Phi(u)$.

On the other hand by perturbing the potential by $\delta\Phi$ we find $\delta\Phi + \Phi_u \cdot \delta u = \delta u \cdot x - \delta\phi$ and hence

$$\delta\Phi = -\delta\phi$$

Thus the perturbation $\delta\phi$ in ϕ satisfies the same equation as $\delta\Phi$ which is the same equation as for Φ in the "u" variables. In our case this is

$$K(\sigma)\Phi_{\theta\theta} + \Phi_{\sigma\sigma} = 0,$$

where θ is the flow angle, and σ depends only on speed.

This is an equation of mixed type because for supersonic flow $K < 0$ and for subsonic $K > 0$. Of course it makes no sense if you cannot map from the x, y plane into the (θ, σ) plane but in fact you can for a Garabedian airfoil.

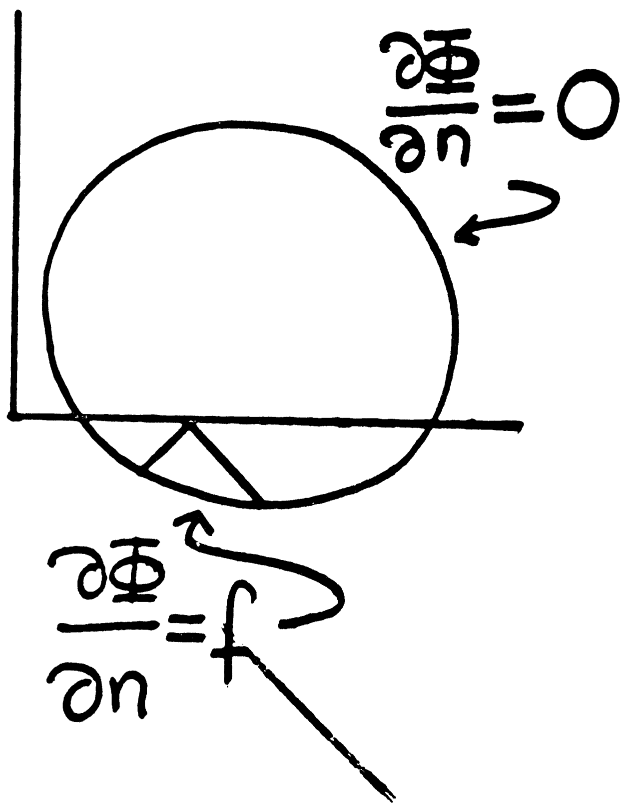

Figure 4. The Counting Problem.

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The best known of mixed type equations is the Tricomi equation

$$y\Phi_{xx} + \Phi_{yy} = 0.$$

However, it is easier to work with the Lavrentiev–Bitsadze equation

$$\pm\ \Phi_{xx} + \Phi_{yy} = 0, \qquad y \gtrless 0$$

$$\nabla\Phi \text{ continuous.}$$

We try a "counting" argument on this problem with the domain and data of Figure 4. One can write a homogeneous linear relation connecting Φ and $\partial\Phi/\partial y$ on $y = 0$ using the boundary data for $y > 0$ only and another from the boundary data for $y < 0$ omitting the arc between the characteristics where $\partial\Phi/\partial n = f$. Thus one would expect only the solution Φ = const. in the domain that excludes the region between the indicated characteristics. But continuing the Goursat problem we get $\partial\Phi/\partial n = f = 0$ in that domain too. So the problem appears overdetermined.

The reader can see how this counting argument extends to the profile problem. In fact, it can be completely proved that the null space for the linearized problem is finite dimensional, which shows that the problem is ill-posed.

Now we have roughly shown that we can construct a *supercritical* wing as these airfoils are now called but we have to stick to one cruising speed. The drag as a function of cruising speed is shown in Figure 2. A profile flying with Mach numbers .750 and .752 have very different distributions of speed in the neighborhood of shock formation, see Figure 5. Here CD is a function of velocity and M = .752 yields a distinct shock while M = .750 is smooth.

How do we find the flow away from design conditions and what theorems are appropriate? We must allow *weak* solutions: these are solutions satisfying, with $\chi \in C_0^\infty$,

$$\int \nabla\chi\cdot\rho\boldsymbol{q}|dx| = 0,$$

$$\int \nabla\chi\cdot\boldsymbol{q}|dx| = 0$$

If the solution is smooth then we get back our old equations.

To get boundary conditions we have to let χ have support that includes the boundary and require a third condition:

$$\int \chi\,\rho\,\boldsymbol{q}\cdot\boldsymbol{n}\,ds = 0, \qquad \text{on the boundary.}$$

We also need another condition that comes from the underlying thermodynamical condition that entropy increases. This becomes in our context: *shocks are compressible.*

Now we have a reasonable *proposition*:

$\exists\,!\ \boldsymbol{q}$ of bounded variation satisfying the boundary value problem and the entropy condition weakly.

This theorem which was weakly formulated, very weakly, back in the fifties is still an open problem.

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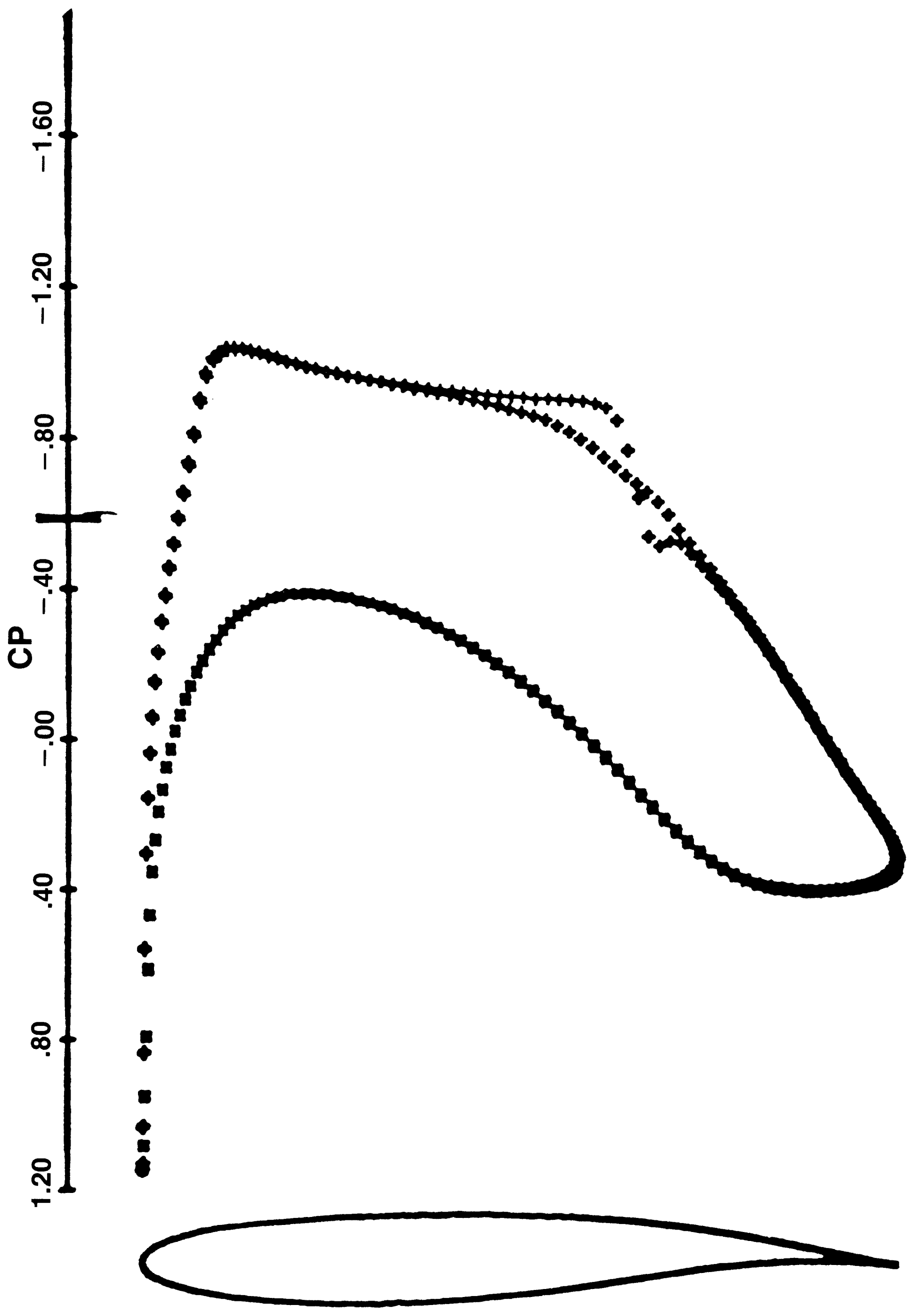

Figure 5. Velocity distribution at two very slightly different Mach numbers for a Korn–Garabedian airfoil. Computations by A. Jameson.

Let us think of purely supersonic flow past a thin airfoil. Goodman [5] has shown, using the techniques of Glimm and Lax [4] for nonlinear one dimensional gas dynamics, that there exists a weak solution to this problem. The technique is tricky and involves a rather extraordinary numerical method which has not been tried on the transonic problem.

What has been tried very successfully numerically is "up wind differencing" which introduces what is called "artificial viscosity". Let us look at a very simple equation due to V. Karmann:

$$\left(\frac{1}{2}\phi_x^2\right)_x + \phi_{yy} = 0$$

Murman and Cole, [13], cracked the problem of solving nonlinear equations of mixed type numerically by solving this one. We shall simply consider

$$(\phi_x^2)_x = 0$$

It has the solution $\phi_x^2 = u^2 = k$ and hence the weak solutions are $\phi_x = \pm\sqrt{k}$. The solution which is "compressive" i.e. satisfies the entropy condition is the one that takes ϕ_x from positive to negative values. So the desired solution u that is not a constant is a decreasing step function.

How do we get it numerically? Well, numerically we replace the derivative by a difference and we would ordinarily try to be as accurate as possible and center the difference but then we would not get the step function. Instead we step back a moment, alter our equation and add a viscous term in the "supersonic region" i.e. $(u^2)_x - \nu(u^2)_{xx} = 0$, $\nu > 0$, for $u > 0$ and $(u^2)_x = 0$ for $u < 0$. For $u > 0$ then $u^2 = k + Me^{x/\nu}$ and for $u < 0$, $u =$ constant. Then either u is identically a negative constant or $u = \sqrt{k}$ at $x = -\infty$. In the latter case $M < 0$ and u decreases smoothly until it vanishes. Then it jumps to a negative value and $u = -\sqrt{k}$ for larger values of x. This yields the right weak solution in the limit.

Now we can interpret this equation as a difference scheme of lower accuracy where ν is of the order of the mesh size and to first order accuracy in mesh size we still have

$$(\phi_x^2)_x = 0$$

and ϕ_x goes from positive to negative as required by the entropy condition. This was the idea of Murman and Cole. Switch to an inaccurate scheme when the equation is hyperbolic but get the entropy condition correctly. This idea was carried over by Jameson to the potential equation. Once we think of the problem in this way we can also switch around and use finite elements or variable mesh sizes.

The compressive condition has to affect a particle, and so upwind differencing means backwards on the particle path. This corresponds to the viscous equation

$$\text{div}\,\rho\nabla\phi + \nabla(\nu\,\text{div}\,\rho\nabla\phi)\cdot\nabla\phi = 0$$

but in practice there are many possibilities for introducing artificial viscosity.

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Now what does this tell us about approaching our theorem. We should try a limiting process with an artificial viscosity. One possibility is to "retard" Bernouilli's law:

$$\operatorname{div} \rho\nabla\phi = 0$$

$$\partial\phi/\partial n = 0 \qquad \text{on profile}$$

$$\nabla\phi \to \boldsymbol{q}_\infty \text{ at } \infty$$

$$\nu\, \nabla\rho\cdot\nabla\phi = |\nabla\phi|^2 - S^2(\rho), \qquad \nu > 0.$$

In the last equation if $\nu = 0$ Bernouilli's law holds by definition. Since we have increased the order of the system we have to add another boundary condition; say, $\rho \to$ its correct value upstream where $\phi \to -\infty$.

To make this problem manageable we consider the wing to be a small smooth bump on a wall and place everything in a box with some appropriate boundary conditions.

This appears solvable for every ν. Of course adding ν smooths out the solution and the solution is found as the fixed point of a mapping using say the Schauder theorem. But there does not appear to be any control on the max $|\nabla\phi|$. So we modify the problem. Things are being complicated by the fact that a gas can cavitate according to Bernouilli's law. We look at a free boundary problem, existence still unproven, which prevents the speed from exceeding the maximum cavitation speed of Bernouilli's law on the boundary of the domain.

Now we can control the speed in the interior and show that several estimates can be made.

$$|\nabla\phi| < \text{max Bernouilli speed}$$

$$\int |\log q - \log S|^2\, dx\, dy < K\nu$$

$$\nu \iint |\nabla|\nabla\phi||^2\, dx\, dy < K$$

This puts us close to the theory of compensated compactness due to Murat, [12]. This permits passage to the limit $\nu = 0$ for certain hyperbolic problems, see Tartar [14], Di Perna [3].

If we assume that at each point for all ν the speed is bounded away from stagnation and cavitation the estimates can be improved and the theory of compensated compactness can be extended to this case, see Morawetz [11].

What more can be done on the computational line? There are schemes today for solving Euler's equations. This was begun in the sixties by Magnus and Yoshihara [9] but failed because of its computer power needs. (Thousands of iterations were involved.) The idea was to solve the time dependent problem and wait for the steady state to emerge. This scheme in fact has been accelerated by a variety of techniques. Here we are carried into the whole area of hyperbolic nonlinear equations because the presence of time makes the equations everywhere hyperbolic. Hopefully bigger and better computers will open up such new three dimensional time dependent problems.

Perhaps the idea of proving the corresponding theorems will prove unfeasible but it will certainly take a lot of time.

I have been barely able to touch on the simpler problems. There are even some simple-to-state open problems. For example: what would be a *correct* perturbation problem? Perhaps in fact better evidence from the computer and the wind tunnel can help us formulate not only good problems but solvable ones. Already questions of uniqueness have been raised.

Let me end by saying that this is a field where the exchange of ideas between engineering or physics and mathematics is very good. Let it continue so.

REFERENCES

1. Bauer, F., Garabedian, P., and Korn, D., *A theory of supercritical wing sections with computer programs and examples*, Lecture Notes in Econom. and Math. Syst., vol. 66, Springer-Verlag, Berlin and New York, 1972. See also with A. Jameson, II, **108**, same series and III, **150**, same series.

2. Busemann, A., *Widerstand bei geschwindigkeiten naher der schallgeschwindigkeiten*, Proc. Third Internat. Congr. Appl. Mewch. **1** (1930), 282–285.

3. DiPerna, R. J., *Convergence of approximate solutions to conservation laws*, Arch. Rat. Mech. Anal. **82** (1983) 27–70.

4. Glimm, J., and Lax, P. D., *Decay of solutions of systems of nonlinear hyperbolic conservation laws*, Memoirs AMS, #101, 1979.

5. Goodman, J., Thesis, to appear as a paper.

6. Jameson, A., *Iterative solution of transonic flow over airfoils and wings including flows at Mach 1*, Comm. Pure Appl. Math. **27** (1974) 283–309.

7. Kacprzynski, J. J., Ohman, L. H., Garabedian, P. R., and Korn, D. G., National Research Council of Canada, Aeronautical Report LR-554.

8. Lighthill, M. J., *On the hodograph transformation for high-speed flow. II. A flow with circulation.* Quart. J. Mech. Appl. Math. **1** (1948), 442–450. *The hodograph transformation in transonic flow. III, Flow around a body*, Proc. Roy. Soc. London Ser. A **191** (1947), 341–369.

9. Magnus, R., and Yoshihara, H., *Inviscid transonic flow over airfoils, AIAA Jour.* **8** (1970), 2157–2162.

10. Morawetz, C. S., *On a weak solution for a transonic flow problem*, to appear in CPAM.

11. Morawetz, C. S., *On the nonexistence of continuous transonic flows past profiles.* I, Comm. Pure Appl. Math. **9** (1956), 45–68; II, **10** (1975), 107–132; III **11** (1958), 129–144. See also **17** (1964), 357–367.

12. Murat, F., *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa 5 (1978) 489–507.

13. Murman, E. M., and Cole, J. D., *Calculation of plane steady transonic flows*, A.I.A.A. Journal **9** (1971), 114–121.

14. Tartar, L. C., *Compensated compactness and applications to partial differential equations, Nonlinear Analysis and Mechanics*, Heriot–Watt Symp. IV, (1979) 136–192. Research Notes in Mathematics, Pitman.

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