BULL. AUSTRAL. MATH. SOC. 26B25, 58C20
VOL. 50 (1994) [123–134]

SECOND-ORDER NORMAL VECTORS TO A CONVEX EPIGRAPH

ALBERTO SEEGER

The second–order behaviour of a nonsmooth convex function f is reflected by the so–called second–order subdifferential mapping $\partial^2 f$. This mathematical object has been intensively studied in recent years. Here we study $\partial^2 f$ in connection with the geometric concept of "second–order normal vector" to the epigraph of f.

1. MATHEMATICAL BACKGROUND

Throughout this note $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is assumed to be a lower–semicontinuous proper convex function. As usual, the class of such functions is denoted by $\Gamma_0(\mathbb{R}^n)$. The purpose of this work is to provide the reader with some additional mathematical tools for a better understanding of the second–order behaviour of f around a reference point $x \in \mathbb{R}^n$. Recall that the first–order behaviour of f around x is reflected by the set

$$\partial f(x) := \{y \in \mathbb{R}^n : f(x') \geqslant f(x) + \langle y, x' - x\rangle \text{ for all } x' \in \mathbb{R}^n\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ stands for the usual Euclidean product in the space $\mathbb{R}^n$. The set (1.1) is known as the subdifferential of f at x, and each of its elements is called a subgradient of f at x (see [**6**]).

Second–order information on f is captured by a family of sets

$$\{\partial^2 f[x, y] : y \in \partial f(x)\}.$$

The precise definition of $\partial^2 f[x, y]$, and some new results concerning this set, will be given in Section 3.

Twice epi–differentiability is a fundamental concept in the definition of $\partial^2 f[x, y]$. A new characterisation of this notion will be given in Section 2.

For convenience in our exposition, we recall below the concept of epigraphical convergence.

Received 27th October, 1993

DEFINITION 1.1: (see, for instance, Attouch [1]). A sequence $\{\varphi_k\}_{k\in\mathrm{N}}$ of functions $\varphi_k : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be epi–convergent to $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ if for every $h \in \mathbb{R}^n$, the following properties are satisfied:

$$\exists\ \{h_k\} \to h \text{ such that } \varphi(h) \geqslant \limsup \varphi_k(h_k); \tag{1.2}$$

$$\forall\ \{h_k\} \to h \text{ one has } \varphi(h) \leqslant \liminf \varphi_k(h_k). \tag{1.3}$$

A family $\{\varphi_t\}_{t>0}$ of functions $\varphi_t : \mathbb{R}^n \to \overline{\mathbb{R}}$ epiconverges to $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ (as t goes to 0^+), if for all $\{t_k\} \to 0^+$, the sequence $\{\varphi_{t_k}\}$ epi–converges to φ. In such a case one says that φ is the epigraphical limit of the family $\{\varphi_t\}_{t>0}$, and one writes $\varphi = \text{epi-}\lim_{t\to 0^+} \varphi_t$.

2. On Twice Epi–differentiability

In connection with the second–order analysis of nonsmooth functions, Rockafellar's concept of twice epi–differentiability has drawn the attention of many authors. In the case of nonsmoooth convex functions, this notion can be introduced in the following terms:

DEFINITION 2.1: Let $f \in \Gamma_0(\mathbb{R}^n)$ be finite at x, and let $y \in \partial f(x)$. The function f is said to be twice epi–differentiable at x relative to y if the epigraphical limit

$$f''[x, y; \cdot] := \text{epi-}\lim_{t\to 0^+} \delta_t^2 f[x, y; \cdot] \tag{2.1}$$

exists, where

$$\delta_t^2 f[x, y; h] := \frac{2}{t}\left[\frac{f(x+th) - f(x)}{t} - \langle y, h\rangle\right] \quad \text{for all } h \in \mathbb{R}^n.$$

The function $f''[x, y; \cdot]$ is called the second–order epi–derivative of f at x relative to y.

Important classes of convex functions enjoying the above twice epi–differentiability property have been singled out by Rockafellar [9] (see also [**2**, **8**]). The existence of the second–order epi–derivative $f''[x, y; \cdot]$ has been characterised in several equivalent ways by Moussaoui and Seeger [**5**]. These authors have shown that ε–subdifferentials, distance functions, and projections, are useful tools for studying this question. Here we follow another approach which consists in emphasising the role of the epigraph

$$\text{epi } f := \{(x, \beta) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leqslant \beta\},$$

or more precisely, of its indicator function

$$\psi_{\text{epi } f}(x, \beta) := \begin{cases} 0 & \text{if } (x, \beta) \in \text{ epi } f, \\ +\infty & \text{otherwise} \end{cases}.$$

https://doi.org/10.1017/S0004972700009631 Published online by Cambridge University Press

A well-known fact in convex analysis is that

$$y \in \partial f(x) \text{ if and only if } (y,-1) \in N[f;x], \tag{2.2}$$

where

$$N[f;x] := \partial\psi_{\text{epi } f}(x, f(x))$$

corresponds to the normal cone to epi f at the point $(x, f(x))$. The equivalence (2.2) is sometimes expressed in the form

$$\partial f(x) = \{y \in \mathbb{R}^n : (y,-1) \in N[f;x]\}. \tag{2.3}$$

One of the main goals of this paper is to show that a somewhat similar formula also holds at a second-order level. This leads us to study the relationship between $f''[x,y;\cdot]$ and $\psi''_{\text{epi } f}[(x,f(x)),(y,-1);(\cdot,\cdot)]$, the latter term being of course the second-order epi-derivative of $\psi_{\text{epi } f}$ at $(x, f(x))$ relative to $(y,-1)$. As a first step in our study, we look at the second-order differential quotients

$$\varphi_t(h) := \delta_t^2 f[x,y;h]$$

and

$$\psi_t(h,\alpha) := \delta_t^2\psi_{\text{epi } f}[(x,f(x)),\ (y,-1);\ (h,\alpha)].$$

A simple matter of calculation yields:

LEMMA 2.1. *Let $f \in \Gamma_0(\mathbb{R}^n)$ be finite at x, and let $y \in \partial f(x)$. Then, for all $t > 0$ and $h \in \mathbb{R}^n$, one has*

$$\varphi_t(h) = \inf_{\alpha\in\mathbb{R}} \psi_t(h,\alpha). \tag{2.4}$$

Moreover, if the function f is finite at $x + th$, then the infimum in (2.4) is attained at $\alpha = t^{-1}[f(x+th) - f(x)]$.

PROOF: By definition one has

$$\psi_t(h,\alpha) := \frac{2}{t}\Big[\frac{\psi_{\text{epi } f}((x,f(x)) + t(h,\alpha)) - \psi_{\text{epi } f}(x,f(x))}{t} - \langle (y,-1),\ (h,\alpha)\rangle\Big].$$

After a short calculation one gets

$$\psi_t(h,\alpha) = \frac{2}{t}\Big[\psi_{\text{epi } f}(x+th, f(x)+t\alpha) - \langle y,h\rangle + \alpha\Big],$$

that is to say,

$$\psi_t(h,\alpha) = \begin{cases} \frac{2}{t}[\alpha - \langle y,h\rangle] & \text{if } (f(x+th)-f(x))/t \leqslant \alpha, \\ +\infty & \text{otherwise} \end{cases}. \tag{2.5}$$

https://doi.org/10.1017/S0004972700009631 Published online by Cambridge University Press

If f is not finite at $x+th$, then both terms in (2.4) are equal to $+\infty$. Otherwise, the function $\psi_t(h,\cdot)$ is minimised at $\alpha = t^{-1}[f(x+th)-f(x)]$, and its infimum is just $\varphi_t(h)$. ∎

Next we would like to pass to the limit as $t\to 0^+$ in formula (2.4). An epigraphical limit is however a subtle concept, and needs to be handled with care. To avoid some undesirable technicalities, suppose that x is a point at which the function $f\in\Gamma_0(\mathbb{R}^n)$ is continuous. This requirement is not too stringent and helps to keep our presentation clear. Under this continuity assumption, the directional derivative

$$h\in\mathbb{R}^n \mapsto f'(x;h) := \lim_{t\to 0^+} t^{-1}[f(x+th)-f(x)]$$

is finite everywhere, and one has

$$\lim t_k^{-1}[f(x+t_kh_k)-f(x)] = f'(x;h) \text{ for all } \{(t_k,h_k)\}\to(0^+,h).$$

Now one can state the main result of this section.

Theorem 2.1. *Let $f\in\Gamma_0(\mathbb{R}^n)$ be continuous at x, and let $y\in\partial f(x)$. Then the following statements are equivalent:*

(a) *f is twice epi-differentiable at x relative to y;*

(b) *$\psi_{\mathrm{epi}\ f}$ is twice epi-differentiable at $(x,f(x))$ relative to $(y,-1)$.*

For convenience, we split the proof of the above theorem into two lemmas.

Lemma 2.2. *Let $f\in\Gamma_0(\mathbb{R}^n)$ be continuous at x, and let $y\in\partial f(x)$. Suppose $\psi_{\mathrm{epi}\ f}$ is twice epi-differentiable at $(x,f(x))$ relative to $(y,-1)$. Then, f is twice epi-differentiable at x relative to y. Moreover, for all $h\in\mathbb{R}^n$, one can write*

$$\begin{aligned} f''[x,y;h] &= \inf_{\alpha\in\mathbb{R}} \psi''_{\mathrm{epi}\ f}[(x,f(x)),(y,-1);(h,\alpha)] \\ &= \psi''_{\mathrm{epi}\ f}[(x,f(x)),(y,-1);(h,f'(x;h))]. \end{aligned} \tag{2.6}$$

Proof: Take any $(h,\alpha)\in\mathbb{R}^n\times\mathbb{R}$, and write

$$\psi(h,\alpha) := \psi''_{\mathrm{epi}\ f}[(x,f(x)),(y,-1);(h,\alpha)] = \left[\text{epi-}\lim_{t\to 0^+}\psi_t\right](h,\alpha).$$

If $\alpha < f'(x;h)$, then

$$t^{-1}[f(x+th')-f(x)] > \alpha'$$

for all (t,h',α') close to $(0^+,h,\alpha)$. This fact, together with expression (2.5), implies that

$$\liminf_{(t,h',\alpha')\to(0^+,h,\alpha)} \psi_t(h',\alpha') = +\infty. \tag{2.7}$$

Thus, $\psi(h,\alpha) = +\infty$. Consider now the case $\alpha > f'(x;h)$. Since $y \in \partial f(x)$, one has necessarily $f'(x;h) \geqslant \langle y,h\rangle$. Hence, $\alpha - \langle y,h\rangle$ is strictly positive, and the term

$$\frac{2}{t}[\alpha' - \langle y,h'\rangle]$$

goes to $+\infty$ as (t,h',α') goes to $(0^+,h,\alpha)$. Thus, we are again in the situation described by (2.7). Summarising, $\psi(h,\alpha) = +\infty$ whenever $\alpha \neq f'(x;h)$. This implies of course that

$$\inf_{\alpha\in\mathbb{R}} \psi(h,\alpha) = \psi(h, f'(x;h)).$$

It remains now to show that the function

$$h \in \mathbb{R}^n \mapsto \psi(h, f'(x;h))$$

is the epigraphical limit of the family $\{\varphi_t\}_{t>0}$ as $t\to 0^+$. Take any sequence $\{t_k\}\to 0^+$ and any $h \in \mathbb{R}^n$. One needs to prove the conditions

$$\exists\ \{h_k\}\to h \text{ such that } \psi(h,f'(x;h)) \geqslant \limsup\ \varphi_{t_k}(h_k), \tag{2.8}$$

and

$$\forall\ \{h_k\}\to h \text{ one has } \psi(h,f'(x;h)) \leqslant \liminf\ \varphi_{t_k}(h_k). \tag{2.9}$$

Since the epigraphical limit ψ exists, one has

$$\psi(h,f'(x;h)) \leqslant \liminf \psi_{t_k}(h_k,\alpha_k)$$

for all $\{(h_k,\alpha_k)\}\to(h,f'(x;h))$. But, for the particular choice

$$\alpha_k = t_k^{-1}[f(x+t_kh_k) - f(x)], \tag{3.10}$$

one gets

$$\psi_{t_k}(h_k,\alpha_k) = \varphi_{t_k}(h_k).$$

[See Lemma 2.1.]

Condition (2.9) is proven in this way. To prove (2.8) we use again the existence of the epigraphical limit ψ. One knows that

$$\psi(h,f'(x;h)) \geqslant \limsup \psi_{t_k}(h_k,\alpha_k) \tag{2.11}$$

for some $\{(h_k,\alpha_k)\}\to(h,f'(x;h))$. If $\psi(h,f'(x;h)) = +\infty$, then condition (2.8) holds trivially. So, one can suppose that $\psi(h,f'(x;h)) < +\infty$, and

$$\alpha_k \geqslant \beta_k := t_k^{-1}[f(x+t_kh_k) - f(x)].$$

https://doi.org/10.1017/S0004972700009631 Published online by Cambridge University Press

Now, notice that

$$\psi_{t_k}(h_k,\alpha_k) \geqslant \psi_{t_k}(h_k,\beta_k) = \varphi_{t_k}(h_k),$$

and hence

$$\limsup \psi_{t_k}(h_k,\alpha_k) \geqslant \limsup \varphi_{t_k}(h_k).$$

Condition (2.8) follows by combining (2.11) and the above inequality. ∎

Next we state the converse of Lemma 2.2.

Lemma 2.3. *Let $f \in \Gamma_0(\mathbb{R}^n)$ be continuous at x, and let $y \in \partial f(x)$. Assume that f is twice epi–differentiable at x relative to y. Then, $\psi_{\text{epi } f}$ is twice epi–differentiable at $(x, f(x))$ relative to $(y,-1)$. Moreover, for all $(h,\alpha) \in \mathbb{R}^n \times \mathbb{R}$, one can write*

$$\psi''_{\text{epi } f}[(x,f(x)),(y,-1);(h,\alpha)] = \begin{cases} f''[x,y;h] & \text{if } \alpha = f'(x;h) \\ +\infty & \text{otherwise} \end{cases}. \tag{2.12}$$

Proof: We keep the same notation as in the proof of the previous lemma. Take any sequence $\{t_k\} \to 0^+$ and any $(h,\alpha) \in \mathbb{R}^n \times \mathbb{R}$. One has seen already that if $\alpha \neq f'(x;h)$, then the second–order epi–derivative ψ is well defined at (h,α), and it is equal to $+\infty$. So, we just need to consider the case $\alpha = f'(x;h)$, and prove the conditions

$$\exists \, \{(h_k,\alpha_k)\} \to (h, f'(x;h)) \text{ such that } f''[x,y;h] \geqslant \limsup \psi_{t_k}(h_k,\alpha_k), \tag{2.13}$$

and

$$\forall \, \{(h_k,\alpha_k)\} \to (h, f'(x;h)) \text{ one has } f''[x,y;h] \leqslant \liminf \psi_{t_k}(h_k,\alpha_k). \tag{2.14}$$

But the first one follows from the existence of $f''[x,y;h]$ and the possibility of choosing $\{\alpha_k\}$ as in (2.10). To prove the second condition, observe that

$$\psi_{t_k}(h_k,\alpha_k) \geqslant \varphi_{t_k}(h_k)$$

[see Lemma 2.1] no matter how one chooses the sequence $\{\alpha_k\}$. Thus

$$f''[x,y;h] \leqslant \liminf \varphi_{t_k}(h_k) \leqslant \liminf \psi_{t_k}(h_k,\alpha_k).$$

This completes the proof of the lemma. ∎

Lemmas 2.2 and 2.3 not only serve to prove Theorem 2.1, but also provide some formulae linking the second–order epi–derivatives $f''[x,y;\cdot]$ and $\psi''_{\text{epi } f}[(x, f(x)),(y,-1); (\cdot,\cdot)]$ in a simple way. These formulae have many interesting consequences, some of which will be explored in the next section.

3. On Second–Order Normal Directions

As explained by the author in [**11, 12, 5**], to each second–order epi–derivative $f''[x,y;\cdot]$ one can associate a unique nonempty closed convex set $\partial^2 f[x,y]$ in such a way that

$$f''[x,y;h] = [\sup\{\langle z,h\rangle : z \in \partial^2 f[x,y]\}]^2 \text{ for all } h \in \mathbb{R}^n.$$

More precisely:

DEFINITION 3.1: Let f, x, and y, be as in Definition 2.1. The second–order subdifferential of f at x relative to y is the set given by

$$\partial^2 f[x,y] := \{z \in \mathbb{R}^n : \langle z,h\rangle \leqslant \{f''[x,y;h]\}^{1/2} \text{ for all } h \in \mathbb{R}^n\}. \tag{3.1}$$

Each vector z in $\partial^2 f[x,y]$ is called a second–order subgradient of f at x relative to y.

REMARK 3.1. A variant of the set (3.1) is obtained by using pointwise convergence instead of epigraphical convergence (see [**3, 4, 10**]). However, such a variant is of less interest, at least in the context of this note.

Second–order normal directions to a given convex set are obtained by applying the concept of second–order subdifferentiability to its corresponding indicator function. In the specific case of a convex epigraph, one has:

DEFINITION 3.2: Let $f \in \Gamma_0(\mathbb{R}^n)$ be finite at x, and let $y \in \partial f(x)$. If $\psi_{\text{epi } f}$ is twice epi–differentiable at $(x, f(x))$ relative to $(y,-1)$, then each vector in the set

$$N^2[f;x,y] := \partial^2 \psi_{\text{epi } f}[(x,f(x)),(y,-1)] \tag{3.2}$$

is called a second–order normal vector to epi f at $(x, f(x))$ relative to $(y,-1)$.

An equivalent definition of the set $N^2[f;x,y]$ can be found in our previous work [**12**]. The superscript 2 over the capital letter N reminds us that we are working at a second–order level. $N^2[f;x,y]$ is a closed convex set in $\mathbb{R}^n \times \mathbb{R}$ which contains the origin. However, this set is not always a cone.

The purpose of this section is to explore the connection existing between the second–order subgradients of a convex function, and the second–order normal vectors to its epigraph. As an extension of formula (2.3), one gets the following nice result:

THEOREM 3.1. *Let $f \in \Gamma_0(\mathbb{R}^n)$ be continuous at x, and let $y \in \partial f(x)$. Assume any of the equivalent conditions in Theorem 2.1. Then,*

$$\partial^2 f[x,y] = \{z \in \mathbb{R}^n : (z,0) \in N^2[f;x,y]\}. \tag{3.3}$$

PROOF: By definition, $\partial^2 f[x,y]$ is the subdifferential at $0 \in \mathbb{R}^n$ of the sublinear function $q := \{f''[x,y,\cdot]\}^{1/2}$. Similarly, $N^2[f;x,y]$ is the subdifferential at

$(0,0) \in \mathbb{R}^n \times \mathbb{R}$ of the sublinear function $\ell := \{\psi''_{\text{epi } f}[(x, f(x)), (y, -1); (\cdot, \cdot)]\}^{1/2}$. Now, according to Lemma 2.2, one can write

$$q(h) = \inf_{\alpha \in \mathbb{R}} \ell(h, \alpha) \quad \text{for all } h \in \mathbb{R}^n.$$

Moreover, for $h = 0$ the above infimum is attained at $\alpha = f'(x; 0) = 0$. By applying Rockafellar's rule [7, Theorem 24] on the subdifferential of a marginal function, one gets

$$\partial q(0) = \{z \in \mathbb{R}^n : (z, 0) \in \partial \ell(0,0)\}.$$

But this is just another way of writing formula (3.3). □

Theorem 3.1 says that $\partial^2 f[x,y]$ can be identified with the section of $N^2[f; x, y]$ corresponding to the height $\gamma = 0$. Recall that for computing first-order subgradients one has to cut the normal cone $N[f; x]$ at the level $\gamma = -1$. Below we illustrate this situation with the help of an example.

EXAMPLE 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \max\left\{\frac{1}{2}(x-1)^2, \frac{1}{2}(x+1)^2\right\}.$$

For $x = 0$, one has $N[f; x] = \{(y, \gamma) \in \mathbb{R} \times \mathbb{R} : |y| + \gamma \leqslant 0\}$. By cutting this normal cone at the level $\gamma = -1$, one gets $\partial f(x) = \{y \in \mathbb{R} : |y| - 1 \leqslant 0\} = [-1, 1]$. Take, for instance, the subgradient $y = 1$. As a matter of computation one gets $N^2[f; x, y] = \{(z, \gamma) \in \mathbb{R} \times \mathbb{R} : z + \gamma \leqslant 1\}$. The set $\partial^2 f[x, y]$ is obtained by setting $\gamma = 0$ in the inequality $z + \gamma \leqslant 1$. Thus, $\partial^2 f[x, y] = \{z \in \mathbb{R} : z \leqslant 1\} = (-\infty, 1]$.

The next result is somehow the converse of Theorem 3.1. It tells us how to compute $N^2[f; x, y]$ in terms of $\partial^2 f[x, y]$.

THEOREM 3.2. *Under the same assumptions as in Theorem 3.1, one can write*

$$N^2[f; x, y] = \{(z, \gamma) \in \mathbb{R}^n \times \mathbb{R} : z + \gamma y \in \partial^2 f[x, y]\}. \tag{3.4}$$

PROOF: By definition, $(z, \gamma) \in N^2[f; x, y]$ if and only if

$$\langle (z, \gamma), (h, \alpha) \rangle \leqslant \left\{\psi''_{\text{epi } f}[(x, f(x)), (y, -1); (h, \alpha)]\right\}^{1/2} \text{ for all } (h, \alpha) \in \mathbb{R}^n \times \mathbb{R}.$$

According to Lemma 2.3, the above condition reduces to

$$\langle (z, \gamma), (h, f'(x; h)) \rangle \leqslant \{f''[x, y; h]\}^{1/2} \text{ for all } h \in \mathbb{R}^n.$$

This is clearly equivalent to

$$\langle z, h \rangle + \gamma f'(x; h) \leqslant \{f''[x, y; h]\}^{1/2} \text{ for all } h \in D \tag{3.5}$$

where

$$D := \{h \in \mathbb{R}^n : f''[x,y;h] < +\infty\}$$

denotes the effective domain of $f''[x,y;\cdot]$. But on the set D, the directional derivative $f'(x;\cdot)$ coincides with the linear function $\langle y,\cdot\rangle$. Thus, condition (3.5) can be written in the form

$$\langle z+\gamma y, h\rangle \leqslant \{f''[x,y;h]\}^{1/2} \text{ for all } h \in D.$$

The latter inequality amounts to saying that $z+\gamma y \in \partial^2 f[x,y]$. □

We mention that Theorem 3.2 yields a simple expression for the polar set of $N^2[f;x,y]$ in terms of the polar set of $\partial^2 f[x,y]$. Polarity is an interesting tool in the analysis of closed convex sets containing the origin. By definition, the polar set of $C \subset \mathbb{R}^n$ is given by

$$C^0 := \{h \in \mathbb{R}^n : \langle z,h\rangle \leqslant 1 \text{ for all } z \in C\}.$$

COROLLARY 3.1. *Under the same assumptions as in Theorem 3.1, the polar set of $N^2[f;x,y]$ is given by*

$$\begin{aligned}\{N^2[f;x,y]\}^0 &= \{(h,\langle y,h\rangle) : h \in (\partial^2 f[x,y])^0\}\\ &= \{(h,\langle y,h\rangle) : f''[x,y;h] \leqslant 1\}.\end{aligned}$$

PROOF: Let $L : \mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$ be the linear mapping given by

$$L(z,\gamma) = z+\gamma y.$$

By applying Theorem 3.2 and a standard calculus rule on polar sets (see [**6**, Corollary 16.3.2]), one obtains

$$\{N^2[f;x,y]\}^0 = \{L^{-1}(\partial^2 f[x,y])\}^0 = L^*(\{\partial^2 f[x,y]\}^0),$$

where $L^* : \mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}$ stands for the adjoint mapping of L. It suffices now to observe that L^* is given by $L^*h = (h,\langle y,h\rangle)$. □

We end this section by mentioning another by-product of the formulae established in Lemmas 2.2 and 2.3. The next proposition deals with the second-order epi-derivative of the Legendre-Fenchel conjugate $f^* \in \Gamma_0(\mathbb{R}^n)$ of f. It has been proven by Rockafellar [**9**, Theorem 2.4] that the existence of $f''[x,y;\cdot]$ is equivalent to the existence of $(f^*)''[y,x;\cdot]$; moreover, both second-order epi-derivatives are related by the conjugacy relationship

$$\frac{1}{2}(f^*)''[y,x;z] = \left\{\frac{1}{2}f''[x,y;\cdot]\right\}^*(z) \text{ for all } z \in \mathbb{R}^n.$$

We show next that $(f^*)''[y,x;\cdot]$ can also be expressed in terms of the second–order epi–derivative

$$(z,\gamma)\in\mathbb{R}^n\times\mathbb{R}\mapsto \sigma''_{\text{epi } f}[(y,-1),(x,f(x));(z,\gamma)],$$

where

$$\sigma_{\text{epi } f} := \psi^*_{\text{epi } f}$$

stands for the support function of epi f.

Proposition 3.1. *Under the same assumptions as in Theorem 3.1, one can write*

$$\sigma''_{\text{epi } f}[(y,-1),(x,f(x));(z,\gamma)] = (f^*)''[y,x;z+\gamma y] \text{ for all } (z,\gamma)\in\mathbb{R}^n\times\mathbb{R}. \tag{3.6}$$

In particular,

$$(f^*)''[y,x;z] = \sigma''_{\text{epi } f}[(y,-1),(x,f(x));(z,0)] \text{ for all } z\in\mathbb{R}^n. \tag{3.7}$$

Proof: According to Lemma 2.3, one has

$$\frac{1}{2}\psi(h,\alpha) = \begin{cases} \frac{1}{2}f''[x,y;h] & \text{if } \alpha = f'(x;h), \\ +\infty & \text{otherwise} \end{cases}, \tag{3.8}$$

where $\psi(h,\alpha) := \psi''_{\text{epi } f}[(x,f(x)),(y,-1);(h,\alpha)]$. By taking the Legendre–Fenchel conjugate on both sides of (3.8), one gets

$$\left(\frac{1}{2}\psi\right)^*(z,\gamma) = \sup_{\substack{h\in\mathbb{R}^n\\ \alpha\in\mathbb{R}}}\left\{\langle z,h\rangle + \gamma\alpha - \frac{1}{2}f''[x,y;h] : \alpha = f'(x;h)\right\}.$$

In the above supremum, one can let h range over the effective domain D of $f''[x,y;\cdot]$. If h belongs to such a set D, then $f'(x;h) = \langle y,h\rangle$. Hence,

$$\begin{aligned}\left(\frac{1}{2}\psi\right)^*(z,\gamma) &= \sup_{h\in D} \ell\{\langle z+\gamma y,h\rangle - \frac{1}{2}f''[x,y;h]\}\\ &= \{\frac{1}{2}f''[x,y;\cdot]\}^*\,(z+\gamma y).\end{aligned}$$

By using Rockafellar's conjugacy relationship [**9**, Theorem 2.4], one obtains finally

$$\frac{1}{2}\sigma''_{\text{epi } f}[(y,-1),(x,f(x));(z,\gamma)] = \frac{1}{2}(f^*)''[y,x;z+\gamma y].$$

The proof of the proposition is now complete. ∎

https://doi.org/10.1017/S0004972700009631 Published online by Cambridge University Press

4. Conclusions

As one may expect, the epigraph epi f carries in a hidden way information on the second–order behaviour of the convex function f. To bring this information into light, it suffices to collect all second–order normal vectors to epi f. Of course, one can localise this search around a reference point x, and a reference subgradient y. Once we have evaluated the set $N^2[f;x,y]$, it is possible to get $\partial^2 f[x,y]$ by using the cutting procedure explained in Theorem 3.1. If one wishes to move in the opposite direction, one can invoke Theorem 3.2. Indeed, formula (3.4) tells us how to construct $N^2[f;x,y]$ starting from $\partial^2 f[x,y]$.

Up to some minor modifications, the results presented in this note can be extended to an infinite dimensional setting. For instance, on a reflexive Banach space, the symbol $\langle\cdot,\cdot\rangle$ has to be understood as a duality product, epigraphical convergence has to be changed by Mosco–convergence, and so on.

References

[1] H. Attouch, *Variational convergence for functions and operators*, Applied Math. Series (Pitman, 1984).

[2] R. Cominetti, 'On pseudo–differentiability', *Trans. Amer. Math. Soc.* **324** (1991), 843–865.

[3] J.-B. Hiriart–Urruty, 'A new set–valued second–order derivative for convex functions', in *Mathematics for optimization*, (J.-B. Hiriart–Urruty, Editor) (North–Holland, 1986).

[4] J.-B. Hiriart–Urruty and A. Seeger, 'Calculus rules on a new set–valued second–order derivative for convex functions', *Nonlinear Anal.* **13** (1989), 721–738.

[5] M. Moussaoui and A. Seeger, 'Second-order subgradients of convex integral functionals', preprint, Department of Mathematics, University of Avignon, July 1993.

[6] R.T. Rockafellar, *Convex analysis* (Princeton Univ. Press, Princeton, 1970).

[7] R.T. Rockafellar, *Conjugate duality and optimization*, Conference Board of Math. Sci. Series No. 16 (SIAM Publications, 1974).

[8] R.T. Rockafellar, 'First and second order epi-differentiability in nonlinear programming', *Trans. Amer. Math. Soc.* **307** (1988), 75–108.

[9] R.T. Rockafellar, 'Generalized second derivatives of convex functions and saddle functions', *Trans. Amer. Math. Soc.* **322** (1990), 810–822.

[10] A. Seeger, *Analyse du second ordre de problèmes non différentiables*, Ph.D. Thesis (Université Paul Sabatier, Toulouse, 1986).

[11] A. Seeger, 'Second derivatives of a convex function and of its Legendre–Fenchel transformate', *SIAM J. Optim.* **2** (1992), 405–424.

[12] A. Seeger, 'Limiting behavior of the approximate second–order subdifferential of a convex function', *J. Optim. Theory Appl.* **74** (1992), 527–544.

King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Dhahran 31261
Saudi Arabia