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Asymptotic centres of filters on uniformly rotund spaces

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The notion of asymptotic centre of a bounded sequence of points in a uniformly convex Banach space was introduced by Edelstein in order to prove, in a quasi-constructive way, fixed point theorems for nonexpansive and similar maps.

Similar theorems have also been proved by, for example, adding a compactness hypothesis to the restrictions on the domain of the maps. In such proofs, which are generally less constructive, it may be possible to weaken the uniform convexity hypothesis.

In this paper Edelstein's technique is extended by defining a notion of asymptotic centre for an arbitrary set of nonempty bounded subsets of a metric space. It is shown that when the metric space is uniformly rotund and complete, and when the set of bounded subsets is a filter base, this filter base has a unique asymptotic centre. This fact is used to derive, in a uniform way, several fixed point theorems for nonexpansive and similar maps, both single-valued and many-valued.

Though related to known results, each of the fixed point theorems proved is either stronger than the corresponding known result, or has a compactness hypothesis replaced by the assumption of uniform convexity.

1. Introduction

1.1. In this paper we use the notion of uniform rotundity for metric

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spaces which was introduced in Staples [4]. The definition is as follows, where we write $B(m, r)$ for the following subset of a metric space (M, d) :

$$\{x \in M : d(x, m) \leq r\} \ , \quad m \in M \ , \quad r \ \text{real.}$$

A metric space (M, d) is called uniformly rotund if for all nonempty bounded $S \subseteq M$, all real positive ε, p, q with $p \leq q$, there is real positive δ such that for all $x, y \in S$ there is $z \in M$ such that for all real $D \in [p, q]$,

$$B(x, D) \cap B(y, D) \subseteq B(z, D-\delta) \ .$$

The supremum of such numbers $\delta \leq q$ may be denoted $\delta(S, \varepsilon, z, p, q)$.

1.2. Recall that a filter base F of subsets of a set M is a non-empty set of nonempty subsets of M such that for all $A, B \in F$ there is $C \in F$ such that $C \subseteq A \cap B$.

We define the asymptotic radius ρ of a set F of bounded subsets of a metric space M to be:

$$\inf\{r : \text{for some } S \in F \text{ and } x \in M, S \subseteq B(x, r)\} \ .$$

If in particular there is $c \in M$ such that for all $\varepsilon > 0$ there is $S_\varepsilon \in F$ such that

$$S_\varepsilon \subseteq B(c, \rho+\varepsilon) \ ,$$

then c is called an asymptotic centre of F . Our basic result is as follows.

1.3. *If M is a nonempty, complete and uniformly rotund metric space, and if F is a filter base of bounded subsets of M , then F has a unique asymptotic centre.*

Proof of existence. Write ρ for the asymptotic radius of F . If $\rho = 0$ then it is straightforward to check that F is convergent, to a point of M which is an asymptotic centre of F . Hence suppose that $\rho > 0$.

Choose, for each positive integer n , $x_n \in M$ such that for some $S_n \in F$,

$$S_n \subseteq B\left(x_n, \rho+1/n\right) \ .$$

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We show that (x_n) is a Cauchy sequence; its limit is then trivially an asymptotic centre of F.

We may as well suppose that $S_i \supseteq S_{i+1}$, $i \geq 1$. Suppose that there is $\varepsilon > 0$ such that for infinitely many n, and for each such n some $m \geq n$, so that

$$d(x_m, x_n) \geq \varepsilon .$$

Then for all such m and n,

$$S_m \subseteq S_n \subseteq B(x_n, \rho+1/n)$$

and

$$S_m \subseteq B(x_m, \rho+1/m) \subseteq B(x_m, \rho+1/n) .$$

As $x_n \in B(X_1, \rho+1)$ for all n, then from the definition of uniform convexity there is real positive δ, and for infinitely many n there is $z_n \in M$, such that for some $m \geq n$,

$$S_m \subseteq B(x_n, \rho+1/n) \cap B(x_m, \rho+1/n) \subseteq B(z_n, \rho+1/n-\delta) ;$$

a contradiction when we take $n > 1/\delta$.

Proof of uniqueness. If F has two asymptotic centres c and c', with $c \neq c'$, then $\delta > 0$ and we obtain a contradiction in the same way.

Both these arguments are closely similar to the corresponding arguments in Staples [4]; in particular the uniqueness result can again be obtained from the slightly weaker assumption on M of regular local uniform rotundity.

1.4. A map $f : M \to M$ on a metric space M is said to preserve a set F of subsets of M, if for all $S \in F$ there is $T \in F$ such that $T \subseteq f(S)$.

Our basic fixed point theorem is as follows, where for $S \subseteq M$ and $x \in M$ we write $d(S, x)$ for $\sup\{d(s, x) : s \in S\}$.

1.5. *If M is a complete, uniformly rotund metric space, if F is a filter base of bounded subsets of M, if $f : M \to M$ preserves F and if for all $\varepsilon > 0$ there is $T_\varepsilon \in F$ such that for all $S \subseteq T_\varepsilon$,*

$S \in F$ implies $f(S)$ is bounded and $d(f(S), f(c)) \le d(S, c) + \varepsilon$, where c is the asymptotic centre of F, then c is a fixed point of f.

Proof. For arbitrary $\varepsilon > 0$ we find $U \in F$ such that

$$U \subseteq B(f(c), \rho+\varepsilon) ;$$

it then follows that $f(c)$ is the asymptotic centre of F; that is that $f(c) = c$.

Choose $S \in F$ such that

$$d(S, c) \le \rho + \varepsilon/2 ;$$

we suppose, as we may, that $S \subseteq T_{\varepsilon/2}$. Then

$$d(f(S), f(c)) \le d(S, c) \le \rho + \varepsilon .$$

Since f preserves F there is $U \in F$ such that $U \subseteq f(S)$. Hence

$$d(U, f(c)) \le \rho + \varepsilon ,$$

as required.

1.6. Clearly the inequality in the statement of 1.5 can be apparently weakened so as to require merely that for some positive integer $k(\varepsilon)$,

$$d(f^{k(\varepsilon)}(S), f(c)) \le d(S, c) + \varepsilon .$$

However it is also clear that this weakening is only apparent.

1.7. In order to compare 1.5 with the basic result of Edelstein [2, 3], extended to metric spaces as in Staples [4], we now state a correspondingly general form of the latter result:

If M is a complete, uniformly rotund metric space, if $x \in M$, if $f : M \to M$ is such that $(f^n(x))$ is a bounded sequence, and if for all $\varepsilon > 0$ there is $N(\varepsilon)$ and $k(\varepsilon) \ge 1$ such that for all $n \ge N(\varepsilon)$,

$$d(f^{n+k(\varepsilon)}(x), f(c)) \le d(f^n(x), c) + \varepsilon ,$$

where c is the asymptotic centre of $(f^n(x))$, then c is a fixed point of f.

1.8. In a trivial sense there is no point in comparing 1.5 and 1.7, since each gives a necessary and sufficient condition for the existence of a fixed point in M. That is, if f has a fixed point c, then we can choose $F = \{\{c\}\}$ in 1.5, and $x = c$ in 1.7, and satisfy the hypotheses with $T_\varepsilon = \{c\}$ and $N(\varepsilon) = k(\varepsilon) = 1$ respectively.

However it is also natural to compare the strengths of 1.5 and 1.7 as sufficient conditions for the existence of fixed points of f, when these trivial cases are excluded. When that is done, 1.5 is seen to be stronger, as is shown by the following example of a function $f : [0, 1] \to [0, 1]$.

1.9. Define f as follows:

(i) on $[1/2, 1]$, $f(1/2) = 3/8$,

$$f\left(1-2^{-n}+x\right) = 1 - 2^{-n} + 2x ,$$
$$0 < x \le 2^{-n-1} , \quad n \ge 1 ,$$

$$f(1) = 3/4 ;$$

(ii) on $\left[2^{-n-1}, 2^{-n}\right)$, $n \ge 1$,

$$f(x) = (1/2).f(2x) ;$$

(iii) $f(0) = 0$.

It is straightforward to check that the only fixed point is 0, and that for $x \neq 0$, the sequence of iterates $\left(f^n(x)\right)$ either converges to 2^{-k} for some $k \ge 0$, or else eventually alternates between 2^{-k} and $\frac{3}{4}.2^{-k}$ for some $k \ge 0$. In none of these cases is 0 the asymptotic centre of $\left(f^n(x)\right)$. Hence the fixed point is not within the scope of 1.7, started from any $x \neq 0$.

On the other hand, consider the filter base of sets

$$\left[0, 2^{-n}\right] , \quad n = 0, 1, 2, \ldots .$$

It is preserved by f, and

$$f\left(\left[0, 2^{-n}\right]\right) = \left[0, 2^{-n}\right] , \quad n = 0, 1, 2, \ldots ,$$

and the asymptotic centre of this filter base is 0, so the inequality

$$d\left(f(S), f(c)\right) \le d(S, c)$$

is satisfied by all sets S in the filter. Thus the fixed point is within the scope of 1.5, started from a suitable $F \neq \{\{0\}\}$.

2. Applications

2.1. The following result is a strengthening of Edelstein [3, Theorem 4]; *cf.* also Cooper and Michael [1, Theorem 4.1] and Staples [4, 5.2 (ii)]. The proof is also much simpler.

If M is a complete, uniformly rotund metric space, if $C \subseteq M$ is nonempty, if H is a semigroup of maps $M \to M$ such that for all $T, U \in H$,

(i) $T(C)$ is bounded,

(ii) there is $V \in H$ such that $V(C) \subseteq T(C) \cap U(C)$,

(iii) for all $\varepsilon > 0$ there is $V_\varepsilon \in H$ such that

$$d\big(T(V_\varepsilon(C)),\ T(c)\big) \le d\big(V_\varepsilon(C),\ c\big) + \varepsilon ,$$

where c is the asymptotic centre of the filter base $F = \{T(C) : T \in H\}$, then c is a fixed point of every $T \in H$.

This result is an obvious corollary of 1.5. It strengthens Edelstein's result because the hypotheses are satisfied if M is bounded and the elements of H commute and satisfy Edelstein's nonexpansive condition. Indeed it is enough to require, as, for example, Cooper and Michael [1] do, that the semigroup H should be left reversible (and its elements nonexpansive), for the following reason.

For H to be left reversible means that for all $T, U \in H$ there are $T', U' \in H$ such that

$$TT' = UU' .$$

But then, choosing $V = TT'$,

$$V(C) = TT'(C) = UU'(C) \subseteq T(C) \cap U(C) \text{ , as required.}$$

2.2. The next result is a straightforward extension of Staples [4, 3.4 (ii)]. There are several related results (see, for example, Wong [5] and the further references there) which apparently are not simplified or extended by the method of this paper. Note also that Wong's Theorem 5 extends straightforwardly by the method of Staples [4] to give a corresponding result for uniformly rotund metric spaces, while Wong's proof of his Theorem 6 appears to make essential use of the linear structure which he assumes.

2.3. In order to state the result we first define a set F of subsets of a set M to be invariant under a map $f : M \to M$ if $f(S) \subseteq S$ for all $S \in F$. If f also preserves F, we say that F is invariantly preserved.

2.4. *If M is a complete, nonempty, uniformly rotund metric space, if F is a filter base of bounded subsets of M which is invariantly preserved under $f : M \to M$, and if for all $\varepsilon > 0$ there is $T_\varepsilon \in F$ such that for all $S \in F$ such that $S \in T_\varepsilon$,*

$$d\big(f(S),\ f(c)\big) \le \varepsilon + \sum_{n=0}^{\infty} \alpha_n(S,\ \varepsilon).d\big(f^n(S),\ c\big) ,$$

where c is the asymptotic centre of F and for all such S, ε,

$$\sum_{n=0}^{\infty} \alpha_n(S,\ \varepsilon) \le 1 + \varepsilon ,$$

then c is a fixed point of f.

Proof. In order to apply 1.5, consider arbitrary $\varepsilon > 0$ and arbitrary $S \in F$ such that $S \subseteq T_{\delta(\varepsilon)} \cap T_\varepsilon$, where

$$\delta(\varepsilon) = \frac{1}{2\big(1+d\big(T_\varepsilon, c\big)\big)} .$$

Since F is invariant under f,

$$d\big(f^n(S),\ c\big) \le d(S,\ c) ,$$

so

$$\begin{aligned} d\big(f(S),\ f(c)\big) &\le \varepsilon/2 + \sum_{n=0}^{\infty} \alpha_n\big(S,\ (\varepsilon)\big).d(S,\ c) \\ &\le \varepsilon/2 + d(S,\ c) + \varepsilon/2 = d(S,\ c) + \varepsilon , \end{aligned}$$

as required to apply 1.5.

3. Related results

3.1. The following elementary result can be proved by the method of this paper, but it is a degenerate case which has a much simpler proof.

If M *is a complete, uniformly rotund metric space, if* $D \subseteq M$ *is bounded and nonempty, and if* $f : M \to M$ *maps* D *onto a dense subset of* D *so that*

$$d\big(f(D),\ f(c)\big) \le d(D,\ c)\ ,$$

where c *is the Chebychev centre of* D *in* M (in the sense of Staples [4]), *then* c *is a fixed point of* f .

Indeed, since $f(D)$ is dense in D ,

$$d\big(D,\ f(c)\big) = d\big(f(D),\ f(c)\big) \le d(D,\ c)\ ,$$

so $f(c) = c$ by definition of Chebychev centre and the uniform rotundity of M .

3.2. The method of this paper extends in natural ways to many-valued mappings on a metric space. A many-valued mapping f of a metric space M to M is just a (single-valued) map

$$M \to P(M)$$

from M into the set $P(M)$ of subsets of M . For $S \subseteq M$ we write

$$f(S) = \bigcup_{x \in S} f(x)\ .$$

With this notation, the definition of a many-valued map f preserving a filter F of subsets of M is exactly as in 1.4.

3.3. There are two natural definitions of a fixed point of a many-valued map f on M , as follows:

(i) a *strong* fixed point $x \in M$ of f is a point such that $f(x) = \{x\}$;

(ii) a *weak*, or Kakutani, fixed point $x \in M$ of f is a point such that

$$x \in f(x)\ .$$

3.4. For each of the notions of fixed point just mentioned there is a corresponding natural notion of distance between nonempty bounded subsets of M , as follows:

(i) extending a notation which was introduced in 1.4, we define

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a function d on nonempty bounded subsets S and T of M by:

$$d(S, T) = \sup\{d(s, t) : s \in S, t \in T\} ,$$

(ii) the Hausdorff pseudometric d_H on nonempty bounded subsets S and T of M is defined by:

$d_H(S, T)$ is the larger of $\sup_{s \in S} \inf_{t \in T} \{d(s, t)\}$ and $\sup_{t \in T} \inf_{s \in S} \{d(s, t)\}$.

3.5. Consider first strong fixed points. Our basic result 1.5 extends with only minor changes in the wording, as follows:

If M *is a complete, nonempty, uniformly rotund metric space, if* F *is a filter base of bounded subsets of* M *, if* $f : M \to P(M)$ *preserves* F *, and if for all* $\varepsilon > 0$ *there is* $T_\varepsilon \in F$ *such that for all* $S \in F$ *,* $S \subseteq T_\varepsilon$ *implies* $f(S)$ *is bounded and*

$$d(f(S), f(c)) \leq d(S, c) + \varepsilon ,$$

where c *denotes the asymptotic centre of* F *and* $f(c)$ *is assumed bounded, then* c *is a strong fixed point of* f *.*

Proof. The definition of $d(f(S), f(c))$ ensures that for all $x \in f(c)$, all $\varepsilon > 0$, and all $S \in F$ such that $S \subseteq T_\varepsilon$,

$$d(f(S), x) \leq d(S, c) + \varepsilon .$$

Hence, as in the proof of 1.5, $x = c$.

3.6. Similarly the results of 2.1 and 3.1 extend to strong fixed point theorems for many-valued maps. Provided that the trivial extension to many-valued maps is made of the notion of invariance (2.3), the result 2.4 also extends in the same way to a strong fixed point theorem for many-valued maps.

3.7. Consider now weak fixed points of many-valued maps. The theorem corresponding to 1.5 is as follows.

If M *is a complete, uniformly rotund metric space, if* F *is a filter base of bounded subsets of* M *, if* $f : M \to P(M)$ *preserves* F *, and if for all* $\varepsilon > 0$ *there is* $T_\varepsilon \in F$ *such that for all* $S \in F$ *,* $S \subseteq T_\varepsilon$ *implies*

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$$d_H\big(f(S),\ f(c)\big) \le d_H(S,\ c) + \varepsilon\ ,$$

where c *is the asymptotic centre of* F *and* $f(c)$ *is assumed bounded and closed, then* c *is a weak fixed point of* f .

Proof. The hypotheses ensure that for all positive integers n there is $c_n \in f(c)$ such that for some $S_n \in F$,

$$d\big(S_n,\ c_n\big) \le \rho + 1/n\ ,$$

where ρ is the asymptotic radius of F . As in 1.3, the uniform rotundity of M ensures that $\big(c_n\big)$ is a Cauchy sequence convergent to c , so that as $f(c)$ is closed, $c \in f(c)$ as required.

3.8. It is easy to check that there are also corresponding extensions of 2.1, 2.4, and 3.1.

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