

## LETTERS TO THE EDITOR

### RANDOM SET AND COVERAGE MEASURE

GUILLERMO AYALA,  
JUAN FERRANDIZ AND  
FRANCISCO MONTES\*, *Universitat de València*

#### Abstract

It is well known that a random set determines its random coverage measure. The paper gives a necessary and sufficient condition for the reverse implication. An equivalent formulation of the condition constitutes a first step in the search for a way to recognize a random measure as being the random coverage measure of a random set.

RANDOM MEASURE; RANDOM COVERAGE MEASURE

#### 1. Introduction

The interest in random closed sets (Matheron (1975)) has increased during recent years along with the increasing development of their applications. These sets are a cornerstone in the model approach to stereology, as can be seen in Stoyan (1990). Given a random set  $\Phi$ , its associated random coverage measure is a partial description of it. Recent results in stereology are focused on the estimation of random set characteristics related to this associated coverage measure, in particular its second-moment measure (Cruz-Orive (1989), Jensen et al. (1990)).

Obviously, a random set always determines its coverage measure. But under what conditions does the coverage measure determine the distribution of a random set? The answer to this question can be found as a corollary to another more general question: how can one recognize a random measure  $\mu$  as being the random coverage measure of a random set with distribution determined by  $\mu$ ?

#### 2. Results

In  $\mathbb{R}^k$  with Borel  $\sigma$ -field  $\beta^k$ , a random  $(d, k)$ -set  $\Phi$  is defined as a measurable mapping from a probability space into the measurable space of  $\nu$ -rectifiable closed sets in  $\mathbb{R}^k$ , where  $\nu$  stands for the corresponding  $d$ -dimensional Hausdorff measure in  $\mathbb{R}^k$  (see Jensen et al. (1990) and Zähle (1982) for more details). This random set determines a unique random coverage measure defined as  $\mu_\Phi(B) = \nu(\Phi \cap B)$ ,  $B \in \beta^k$ . The following theorem establishes which condition the random closed set must satisfy in order to recover its distribution from the associated coverage measure.

---

Received 19 March 1991; revision received 16 July 1991.

\* Postal address: Departamento de Estadística e Investigación Operativa, Universitat de València, 46100-Burjassot (València), Spain.

This work was supported in part by DGICYT grant no. PB 87-0992.

*Theorem 1.* Let  $\Phi$  be a random  $(d, k)$ -set and  $\mu_\Phi$  its associated coverage measure. The distribution of  $\Phi$  is recoverable from  $\mu_\Phi$  if and only if

$$(2.1) \quad P\{\Phi \cap K \neq \emptyset, \mu_\Phi(K \oplus \varepsilon B) = 0 \text{ for some } \varepsilon\} = 0$$

for all compact sets  $K$ , where  $\oplus$  denotes Minkowski addition and  $B$  is the open unit ball in  $\mathbb{R}^k$ .

*Proof.* The distribution of  $\Phi$  is determined by the probabilities  $T(K) = P\{\Phi \cap K \neq \emptyset\}$  for every compact  $K$  (see Matheron (1975)).

Notice that

$$(2.2) \quad \mu_\Phi(K \oplus \varepsilon B) > 0, \quad \forall \varepsilon > 0 \Rightarrow \Phi \cap K \neq \emptyset.$$

On the other hand,  $\{\Phi \cap K \neq \emptyset\}$  can be written as the disjoint decomposition

$$(2.3) \quad \begin{aligned} \{\Phi \cap K \neq \emptyset\} &= \{\mu_\Phi(K \oplus \varepsilon B) > 0, \forall \varepsilon > 0\} \\ &\cup \{\Phi \cap K \neq \emptyset, \mu_\Phi(K \oplus \varepsilon B) = 0 \text{ for some } \varepsilon\}, \end{aligned}$$

and  $\{\mu_\Phi(K \oplus \varepsilon B) > 0, \forall \varepsilon > 0\}$  being an event with its probability determined by the distribution of  $\mu_\Phi$ , the sufficiency of (2.1) is established.

To prove necessity, suppose that for some compact  $K$  and  $\varepsilon > 0$ ,  $P(\Phi \cap K \neq \emptyset, \mu_\Phi(K \oplus \varepsilon B) = 0) > 0$ . Set

$$\Phi' = \begin{cases} \Phi - (K \oplus \varepsilon B), & \text{if } \mu_\Phi(K \oplus \varepsilon B) = 0 \\ \Phi, & \text{otherwise.} \end{cases}$$

It follows immediately that  $\mu_\Phi = \mu_{\Phi'}$ , but the disjoint decomposition

$$\{\Phi' \cap K = \emptyset\} = \{\Phi \cap K = \emptyset\} \cup \{\Phi \cap K \neq \emptyset, \mu_\Phi(K \oplus \varepsilon B) = 0\}$$

implies  $P\{\Phi \cap K \neq \emptyset\} > P\{\Phi' \cap K \neq \emptyset\}$ .

It follows from the theorem that  $P(\Phi \cap K \neq \emptyset) = \lim_{i \rightarrow \infty} P\{\mu_\Phi(K \oplus \varepsilon_i B) > 0\}$  for any sequence of  $\varepsilon_i$  decreasing to 0. In fact, we can associate with each random measure  $\mu$  a random closed set  $\Phi_\mu$ , satisfying the condition (2.1), as follows.

*Definition.* Given a random measure  $\mu$  let  $\Phi_\mu$  be the random closed support of  $\mu$ , defined for any sequence of  $\varepsilon_i$  decreasing to 0 and for any  $\{x_1, x_2, \dots\}$  dense in  $\mathbb{R}^k$  by

$$\Phi_\mu = \bigcap_{i=1}^{\infty} \text{closure } \{x_j, \mu(x_j \oplus \varepsilon_i B) > 0\}.$$

Note that this definition is independent of the choice of sequences  $\{\varepsilon_i\}$  and  $\{x_1, x_2, \dots\}$  and it allows us the following alternative formulation of Theorem 1.

*Theorem 2.* The distribution of the random closed set  $\Phi$  is recoverable from  $\mu_\Phi$  if and only if  $\Phi$  is distributed as the random closed support of  $\mu_\Phi$ .

Theorem 2 gives a natural answer to the second question in the introduction: a random measure  $\mu$  will be the random coverage of a random closed set when its closed support  $\Phi_\mu$  has  $\mu$  as its random coverage measure. But the fundamental question remains: what natural and verifiable conditions must be imposed on  $\mu$  in order for it to have this property?

**Acknowledgement**

We are grateful to Professor D. Stoyan for his helpful comments. For the final version of this paper we are much indebted to the interesting and detailed comments of the referee.

**References**

- CRUZ-ORIVE, L. M. (1989) Second-order stereology: estimation of reduced second-moment volume measures. *Acta Stereol.* **8**, 641–646.
- JENSEN, E. B., KIEU, K. AND GUNDERSEN, H. J. G. (1990) On the stereological estimation of reduced moment measures. *Ann. Inst. Statist. Math.* To appear.
- MATHERON, G. (1975) *Random Sets and Integral Geometry*. Wiley, New York.
- STOYAN, D. (1990) Stereology and stochastic geometry. *Internat. Statist. Rev.* **58**, 227–242.
- ZÄHLE, M. (1982) Random processes of Hausdorff rectifiable closed sets. *Math. Nachr.* **108**, 49–72.