# THE MAXIMUM NUMBER OF STRONGLY CONNECTED SUBTOURNAMENTS*

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In the ranking of a collection of p objects by the method of paired comparisons, a measure of consistency is provided by the relative number of transitive (or consistent) triples and cyclic (or inconsistent) triples. This point of view was introduced by Kendall and Babington Smith [4]. They found a formula for the maximum number of cyclic triples, thereby determining the greatest inconsistency possible. The purpose of this note is to extend the result to obtain the maximum number of "strongly connected" collections of n objects among the given p objects.

For definitions we refer to the review article [2] by Harary and Moser; see also Chapter 11 of the book [3]. We also use the following terminology for brevity: an n-tournament, n-subtournament, and n-cycle have n points each. Because we will refer to them in the text, we show in Figures 1 and 2 all the 3- and 4-tournaments.

Figure 1

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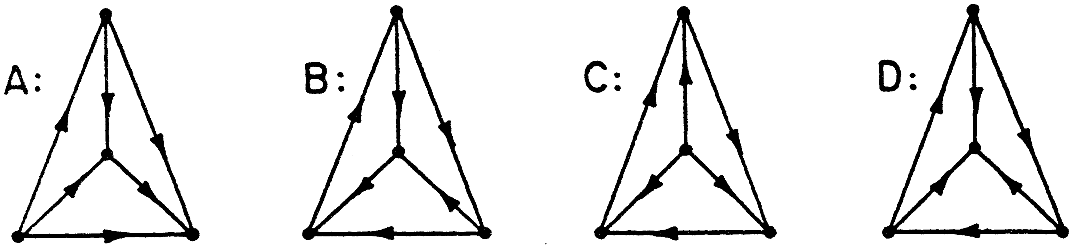


Figure 2

In these terms, the theorem of [4] gives the maximum number of 3-cycles, i.e., strong 3-tournaments, in any p-tournament. It has often been rediscovered by scholars in various fields (cf. Berge [1] p. 129). Because it is not difficult and gives some insight into the general problem, we review the proof of the result here.

THEOREM. The maximum number of 3-cycles in any p-tournament is

$$\begin{cases} \frac{1}{24}p(p+1)(p-1) & \text{if } p \text{ is odd,} \\ \frac{1}{24}p(p+2)(p-2) & \text{if } p \text{ is even.} \end{cases}$$

Proof: The total number of triples of points in a p-tournament is $\binom{p}{3}$, and each triple is either cyclic or transitive as shown in Figure 1. If the point $v_i$ is adjacent to two other points, the triple of these three points is necessarily transitive. Since each transitive triple has just one point adjacent to the other two, $v_i$ is such a point in $\binom{s_i}{2}$ transitive triples, where $s_i$ is the score or "outdegree" of point $v_i$. Hence the number of transitive triples in a tournament is $\sum_{i=1}^{p} \binom{s_i}{2}$.

The maximum number of cyclic triples in a p-tournament occurs when the number of transitive triples is minimum. It is a well-known combinatorial result that the above sum is minimum when the $s_i$ are as nearly equal as possible. That is, if $p$ is odd, take each $s_i = \frac{p-1}{2}$, if $p$ is even, take half the $s_i$ as $\frac{p}{2}$ and the others as $\frac{p}{2} - 1$. Routine computations yield the theorem.

We remark, as a corollary to the proof, that the number of strong 3-tournaments (being 3-cycles) in a tournament with a given score sequence is independent of the structure of the tournament and is $\sum_{i=1}^{p} \binom{s_i}{2}$. This is not true for strong 4-tournaments. The two 5-tournaments in Figure 3 both have score sequence (3, 3, 2, 1, 1), but the first has two 4-cycles while the second has three. Referring to Figure 2, we see that B is the only strong 4-tournament and also that B has exactly one 4-cycle. Hence in any tournament, the number of 4-cycles equals the number of its strong 4-tournaments.

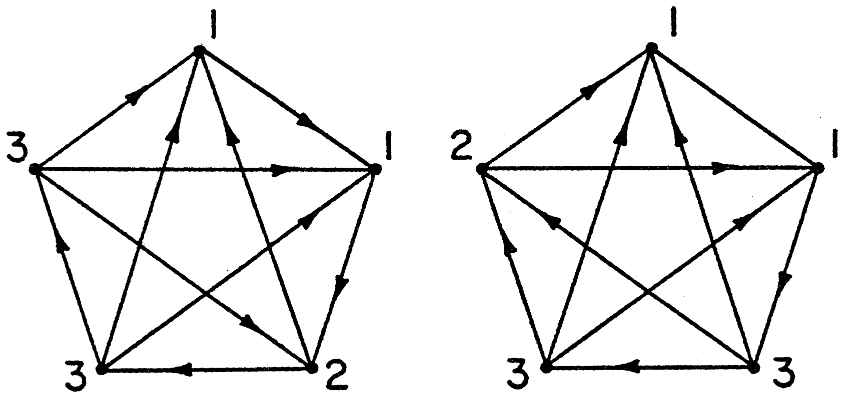


Figure 3

We now prove the principal result.



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THEOREM. The maximum number of strong $n$-tournaments in any $p$-tournament is

$$t(p,n) = \begin{cases} \binom{p}{n} - p\binom{(p-1)/2}{n-1} & p \text{ odd} , \\ \binom{p}{n} - \frac{p}{2}\left[\binom{p/2}{n-1} + \binom{(p-2)/2}{n-1}\right] & p \text{ even} . \end{cases}$$

Proof. Since every set of $n$ points in a tournament induces an $n$-tournament, a $p$-tournament contains $\binom{p}{n}$ $n$-tournaments. A transitive $n$-tournament contains a point with score $n-1$, and any $n$-tournament with a point of score $n-1$ is not strong. Thus, a $p$-tournament $T$ with score sequence $(s_1, s_2, \ldots, s_p)$ has at least $\sum_{i=1}^{p} \binom{s_i}{n-1}$ $n$-tournaments which are not strong. Therefore, $T$ has at most $\binom{p}{n} - \sum_{i=1}^{p} \binom{s_i}{n-1}$ strong $n$-tournaments. A well-known combinatorial result states that for a given sum $\sum_{i=1}^{p} s_i$, the number $\sum_{i=1}^{p} \binom{s_i}{n-1}$ is minimum when the integers $s_i$ are as nearly equal as possible. That is, if $p$ is odd, each $s_i = \frac{p-1}{2}$; if $p$ is even, half the scores are $\frac{p}{2} - 1$ and the others are $\frac{p}{2}$. We will give a construction of $p$-tournaments with such scores in which every subtournament is either strong or transitive.

First, let $p$ be odd. Let $T_o(p)$ be the special tournament constructed by taking every point $v_i$ adjacent to the points $v_{i+1}, v_{i+2}, \ldots, v_{i+(p-1)/2}$, where the subscripts are reduced modulo $p$. The result is indeed a tournament since for any two distinct integers $i$ and $j$, with $1 \le i, j \le p$, either $j < i + \frac{p-1}{2}$ or $i < j + \frac{p-1}{2}$. Each point in $T_o$ has



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score $\frac{p-1}{2}$.

We now show that every 4-tournament in $T_o(p)$ is transitive or strong, that is, is isomorphic to A or B in Figure 2. Suppose that $T_o(p)$ has a 4-tournament T isomorphic to C. In T, let $v_1 v_i v_j v_1$ denote the cycle of length 3 and let $v_k$ be the point with score 3. Then by the construction of $T_o(p)$, $i \leq 1 + \frac{p-1}{2}$, $j \leq i + \frac{p-1}{2}$, and $j > 1 + \frac{p-1}{2}$, since the lines $v_1 v_i$, $v_i v_j$, and $v_j v_1$ are in T. Also, because of the presence of lines $v_k v_1$ and $v_k v_i$, $p \geq k > 1 + \frac{p-1}{2}$ and $k > i + \frac{p-1}{2}$. Therefore $\frac{p+1}{2} < j < k \leq p$, and line $v_j v_k$ must appear in $T_o(p)$. Because $v_k v_j$ appears by our assumption, this is a contradiction, proving that C is not a subtournament of $T_o(p)$. That D is not contained in $T_o(p)$ can be shown analogously or by directional duality. Hence every 4-tournament of $T_o(p)$ is of type A and transitive or is of type B and strong.

For the case when p is even, we first construct the (p+1)-tournament $T_o(p+1)$ as in the odd case above. Now for p even, let $T_o(p)$ be any p-subtournament (all of which are isomorphic) of $T_o(p+1)$. Then every 4-tournament of $T_o(p)$ is again of type A or B.

Now let $n \geq 4$. If T is an n-tournament which is not transitive, it has a cycle. An elementary result of the theory of tournaments implies that if there is any cycle in a tournament, there is a 3-cycle in it.

Let $v_1 v_i v_j v_1$ be such a cycle in T. If T is not strong, there is a point $v_k$ which is either adjacent to or adjacent from all three points in the cycle. Therefore, if T is neither strong nor transitive, it has a 4-tournament isomorphic to C or D in

Figure 2. Since the p-tournaments $T_o(p)$ constructed earlier in the proof have no such subtournaments, every n-tournament in $T_o(p)$ is either strong or transitive. Since $T_o(p)$ has scores which minimize $\sum_{i=1}^{p} \binom{s_i}{n-1}$ for all n, the theorem is proved.

By the construction of the special p-tournament $T_o(p)$, there are as many strong n-tournaments in $T_o(p)$ as possible, for every n. Hence the following corollary is immediate.

COROLLARY 1. The maximum number of strong sub-tournaments in any p-tournament is $t(p) = \sum_{n=3}^{p} t(p,n)$.

It is easy to form the sum stated in Corollary 1 (separately for p odd and p even) of the numbers $t(p,n)$ over all pertinent values of n by using standard combinatorial identities. The result is as follows.

COROLLARY 2. The maximum number of strong sub-tournaments in any p-tournament is

$$t(p) = \begin{cases} 2^p - p\,2^{(p-1)/2} - 1 & p \text{ odd}, \\ 2^p - 3p\,2^{(p-4)/2} - 1 & p \text{ even}. \end{cases}$$

Formulae for $t(p,n)$ are given below for $n = 3, 4,$ and $5$. Note that $t(p,4) = \frac{1}{2}(p-3)\,t(p,3)$ for all values of p.

$$t(p,3) = \begin{cases} \frac{1}{24}p(p+1)(p-1) & p \text{ odd}, \\ \frac{1}{24}p(p+2)(p-2) & p \text{ even}; \end{cases}$$



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$$t(p,4) = \begin{cases} \frac{1}{48} p(p+1)(p-1)(p-3) & p \text{ odd}, \\ \frac{1}{48} p(p+2)(p-2)(p-3) & p \text{ even}; \end{cases}$$

$$t(p,5) = \begin{cases} \frac{1}{1920} p(p+1)(p-1)(p-3)(11p-47) & p \text{ odd}, \\ \frac{1}{1920} p(p+2)(p-2)(p-4)(11p-36) & p \text{ even}. \end{cases}$$

By an earlier remark, the maximum number of 4-cycles in any p-tournament is $t(p,4)$. For, a 4-tournament can have at most one such cycle, and has one if and only if it is of type B. Still unsolved are the problems of finding the maximum number of n-cycles, $n > 4$, in a p-tournament and the maximum total number of cycles.

As noted earlier, the number of transitive triples in any p-tournament is $\sum_{i=1}^{p} \binom{s_i}{2}$ and the number of cyclic triples is $\binom{p}{3} - \sum_{i=1}^{p} \binom{s_i}{2}$. We close by presenting a simple algorithm for finding the numbers of transitive quadruples and 4-cycles in an arbitrary p-tournament T. Let $r_i = p-1-s_i$ so that $r_i$ is the "indegree" of $v_i$. The number of 4-tournaments in T of type A or C is clearly $\Sigma\binom{s_i}{3}$ and of those of type A or D, by duality, $\Sigma\binom{r_i}{3}$ since each has a unique point of score 0 in that subtournament. Now let $s_{ij}$ be the score of point $v_j$ in the subtournament of T containing all the points adjacent from point $v_i$. Then $n_i = \sum_{j=1}^{p} \binom{s_{ij}}{2}$ is the number of transitive triples in the i' th such subtournament. The number of transitive quadruples in T is then $\sum_{i=1}^{p} n_i$. Therefore, the number of 4-cycles in T must be

$$\binom{p}{4} - \Sigma\binom{s_i}{3} - \Sigma\binom{r_i}{3} + \Sigma n_i .$$

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