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Pure Discrete Spectrum for One-dimensional Substitution Systems of Pisot Type

Dedicated to Robert V. Moody on his 60-th birthday

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Abstract. We consider two dynamical systems associated with a substitution of Pisot type: the usual $\mathbb{Z}$-action on a sequence space, and the $\mathbb{R}$-action, which can be defined as a tiling dynamical system or as a suspension flow. We describe procedures for checking when these systems have pure discrete spectrum (the "balanced pairs algorithm" and the "overlap algorithm") and study the relation between them. In particular, we show that pure discrete spectrum for the $\mathbb{R}$-action implies pure discrete spectrum for the $\mathbb{Z}$-action, and obtain a partial result in the other direction. As a corollary, we prove pure discrete spectrum for every $\mathbb{R}$-action associated with a two-symbol substitution of Pisot type (this is conjectured for an arbitrary number of symbols).

1 Introduction

Substitution dynamical systems and their spectral properties have been much studied (see [18], [22]), but there remain many open problems, especially in the *non-constant length* case (see Section 2 for definitions). The standard substitution dynamical system is defined as the shift on the associated sequence space, resulting in a minimal uniquely ergodic $\mathbb{Z}$-action in the primitive case. There is a conjecture that, for every substitution of Pisot type, the resulting measure-preserving transformation has pure discrete spectrum. There are many partial results in this direction; this conjecture has been confirmed in the two-symbol case, due to the recent progress made by Barge and Diamond [2] on the Strong Coincidence Conjecture, see [9], [10]. There is a combinatorial "balanced pair algorithm," introduced by Livshits [15, 16], for checking pure discrete spectrum of a substitution $\mathbb{Z}$-action. It was used in [9], [10] in the two-symbol case, and remains, perhaps, the most efficient method for practical checking of pure discrete spectrum for any number of symbols. In Section 3 we present a version of this algorithm convenient for us, and sketch the proof of why it works. We implemented this algorithm in *Mathematica*; some examples that we worked out are presented in the Appendix.

Another one-dimensional system associated to a substitution, is the $\mathbb{R}$-action, which can be defined as a suspension flow for the $\mathbb{Z}$-action, with the height function constant on cylinder sets corresponding to symbols of the alphabet. It can also be viewed as a tiling dynamical system. The most natural choice for heights/tile

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lengths is given by the Perron-Frobenius eigenvector of the substitution matrix; then the corresponding tilings of the line are self-similar. Tiling dynamical systems and their spectral properties have been studied in [27] and references therein. In particular, [27] showed that the Pisot eigenvalue condition is necessary for the $\mathbb{R}$-action to have non-trivial discrete component in its spectrum, and developed a combinatorial-geometric "overlap algorithm" for checking when the spectrum is pure discrete. In Section 4, we describe the overlap algorithm and point out some of its special features in the one-dimensional case ([27] was mostly concerned with the planar case). A natural question arises: what is the relation between the spectral properties of the $\mathbb{Z}$-action and the $\mathbb{R}$-action associated to the substitution? More specifically: what is the relation between the balanced pair algorithm and the overlap algorithm? This is investigated in Section 5. As a corollary, we obtain the following results:

1. *For any substitution of Pisot type, if the corresponding $\mathbb{R}$-action has pure discrete spectrum, then the $\mathbb{Z}$-action has pure discrete spectrum.*
2. *For any substitution of Pisot type on two symbols, the corresponding $\mathbb{R}$-action has pure discrete spectrum.*

A strong motivation for questions on pure discrete spectrum comes from the physics of quasicrystals. Aperiodic tilings are used as models of atomic configurations, and pure discrete spectrum of the dynamical system is equivalent to pure point diffractivity [5], [8], [13]. It follows, in particular, that every one-dimensional system of Pisot type with two kinds of "atoms" is pure point diffractive. (We note that the terms "pure discrete" and "pure point" are synonymous; the latter is used in the literature on quasicrystals, but the former is standard in Ergodic Theory.)

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2 Preliminaries

Here we briefly recall some basic facts about substitutions and associated dynamical systems; proofs and additional information can be found in [18], [22].

Let $\mathcal{A} = \{1, \dots, d\}$, for some $d \geq 2$, and denote by $\mathcal{A}^* = \bigcup_{i=0}^{\infty} \mathcal{A}^i$ the set of finite words in the alphabet $\mathcal{A}$. A *substitution* is a map $\zeta \colon \mathcal{A} \to \mathcal{A}^*$. The substitution ζ is extended to maps (also denoted by ζ) $\mathcal{A}^* \to \mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ by concatenation. We write $l_i(w)$ for the number of occurrences of the letter i in the word w and denote $\mathbf{l}(w) = [l_1(w), \dots, l_d(w)]^T$. The matrix associated with the substitution ζ is defined by

$$M_\zeta = (m_{i,j})_{d\times d}, \quad \text{where} \quad m_{i,j} = l_i\big(\zeta(j)\big).$$

Note that $M_\zeta\big(\mathbf{l}(w)\big) = \mathbf{l}\big(\zeta(w)\big)$ for all $w \in \mathcal{A}^*$. The substitution is *primitive* if there exists k such that all entries of M_ζ^k are strictly positive; equivalently, if for every $i, j \in \mathcal{A}$, the symbol j occurs in $\zeta^k(i)$. Throughout the paper we assume that $\zeta(1)$ begins with 1 and $|\zeta(1)| > 1$. The sequence $\mathbf{u} = u_1u_2u_3\cdots = \lim_{n\to\infty} \zeta^n(1) \in \mathcal{A}^{\mathbb{N}}$ is a fixed point for ζ; we call it the *substitution sequence* for ζ.

Consider the sequence space $\mathcal{A}^{\mathbb{Z}}$ with the product topology, and let σ be the shift transformation $(\sigma v)_n = v_{n+1}$ on $\mathcal{A}^{\mathbb{Z}}$. For a sequence $\mathbf{x}$ denote by $\mathcal{L}(\mathbf{x})$ the *language* of $\mathbf{x}$, that is, the set of all finite words that occur in $\mathbf{x}$. Define

$$\Omega_\zeta := \{\mathbf{x} \in \mathcal{A}^{\mathbb{Z}} : \mathcal{L}(\mathbf{x}) \subset \mathcal{L}(\mathbf{u})\}.$$

Then (Ω_ζ, σ) is a topological dynamical system, called the *substitution dynamical system (for ζ)*. For a primitive substitution, this dynamical system is *minimal*, *i.e.* every orbit is dense; equivalently, $\mathcal{L}(\mathbf{x}) = \mathcal{L}(\mathbf{u})$ for all $\mathbf{x} \in \Omega_\zeta$. Minimality implies that the substitution sequence $\mathbf{u}$ (as well as any element of Ω_ζ) is *uniformly recurrent*: for every $W \in \mathcal{L}(\mathbf{u})$, there exists L such that W occurs in any block of $\mathbf{u}$ of length L. For a primitive ζ, the system (Ω_ζ, σ) is *uniquely ergodic*, *i.e.* there is a unique σ-invariant invariant Borel probability measure μ. Unique ergodicity implies that every $W \in \mathcal{L}(\mathbf{u})$ occurs in $\mathbf{u}$ with a well-defined positive frequency.

Now we have a measure-preserving transformation $(\Omega_\zeta, \sigma, \mu)$; its spectral type is, by definition, the spectral type of the unitary operator $U_\zeta\colon f(\cdot) \mapsto f(\sigma\cdot)$ on $L^2(\Omega_\zeta, \mu)$. The dynamical system $(\Omega_\zeta, \sigma, \mu)$ has *pure discrete spectrum* if and only if there is a basis for $L^2(\Omega_\zeta, \mu)$ consisting of eigenfunctions for U_ζ. By the Halmos-Von Neumann Theorem, a measure-preserving transformation has pure discrete spectrum if and only if it is measure-theoretically isomorphic to a translation on a compact Abelian group, see [28]. Abusing the terminology a little, we will say that ζ has pure discrete spectrum when the measure-preserving transformation $(\Omega_\zeta, \sigma, \mu)$ does.

The substitution ζ is said to have *constant length* if $i \mapsto |\zeta(i)|$ is constant on $\mathcal{A}$; otherwise, it has *non-constant length*.

Definition A substitution ζ satisfies the *strong coincidence condition* (on prefixes) if for any two letters $i, j \in \mathcal{A}$ there exist $n \in \mathbb{N}$, $\alpha \in \mathcal{A}$, and $p, q, r, t \in \mathcal{A}^*$, such that

$$\zeta^n(i) = p\alpha t \text{ and } \zeta^n(j) = q\alpha r, \quad \text{with } \mathbf{l}(p) = \mathbf{l}(q).$$

Dekking [4] proved that a primitive substitution of constant length has pure discrete spectrum if and only if it satisfies the strong coincidence condition.

The substitution ζ is said to be of *Pisot type* if the Perron-Frobenius eigenvalue of the matrix M_ζ is a Pisot number and the characteristic polynomial is irreducible; equivalently, if all non-Perron-Frobenius eigenvalues are strictly between zero and one in modulus. A substitution of Pisot type is necessarily primitive, has non-constant length, and the corresponding substitution sequences are non-periodic, see [3]. It is conjectured that every substitution of Pisot type satisfies the strong coincidence condition. This has recently been proven by Barge and Diamond in the two-symbol case [2].

3 The Balanced Pair Algorithm

In this section we describe a version of the algorithm, first introduced by Livshits [15, 16], for checking pure discrete spectrum for a substitution of non-constant length.

There is another approach, pioneered by Rauzy [21], which proceeds by constructing a geometric representation of a substitution dynamical system on a torus. It was further extended by [12], [1], [3], [25], [23]. We do not discuss it here.

We say that a pair of words U, V is a *balanced pair* if they have the same length and the same occurrence of symbols, *i.e.* $\mathbf{l}(U) = \mathbf{l}(V)$. We will denote the balanced pair U, V as $[U, V]$ or $\begin{vmatrix} U \\ V \end{vmatrix}$. It *splits* at m if $[U_{[1,m-1]}, V_{[1,m-1]}]$ is a balanced pair, with $2 \le m \le |U|$. A balanced pair is called *irreducible* if it does not split nontrivially. The process of splitting a balanced pair into irreducible balanced subpairs is called *reduction*. Clearly, if $[U, V]$ is a balanced pair, then $[\zeta(U), \zeta(V)]$ is a balanced pair as well. We may also consider a reduction for a pair of infinite words $[\mathbf{x}, \mathbf{y}]$ in the alphabet $\mathcal{A}$ (although it may happen that it never splits).

Let $\mathbf{u}$ be the substitution sequence and let W be a nonempty prefix of $\mathbf{u}$. The set of irreducible balanced pairs arising in the reduction of $[\mathbf{u}, \sigma^{|W|}\mathbf{u}]$ is denoted by $I_1(W)$ (they are called *initial* irreducible balanced pairs). We claim that $I_1(W)$ is finite, and every element of $I_1(W)$ occurs with a well-defined positive frequency in the reduction of $[\mathbf{u}, \sigma^{|W|}\mathbf{u}]$. Indeed, writing $\mathbf{u} = WU_1WU_2W\cdots$, where $U_i \in \mathcal{A}^*$ does not contain W, we see that $[\mathbf{u}, \sigma^{|W|}\mathbf{u}]$ splits into balanced pairs as follows:

$$\begin{vmatrix} \mathbf{u} \\ \sigma^{|W|}\mathbf{u} \end{vmatrix} = \begin{vmatrix} WU_1 \\ U_1W \end{vmatrix} \begin{vmatrix} WU_2 \\ U_2W \end{vmatrix} \cdots .$$

There are finitely many balanced pairs arising this way since W occurs in $\mathbf{u}$ with bounded gaps. They need not be irreducible; performing further reduction yields $I_1(W)$, which is therefore also finite. The existence of frequencies follows from the fact that W occurs in $\mathbf{u}$ with a well-defined positive frequency.

Next, we define inductively, for $n > 1$,

$$\begin{aligned} I_n(W) = \{[X, Y] : [X, Y] &\text{ is an irreducible balanced subpair of} \\ &[\zeta(V), \zeta(Z)], \text{ for some } [V, Z] \in I_{n-1}(W)\}. \end{aligned} \tag{3.1}$$

Equivalently, $I_n(W)$ is the set of all irreducible balanced pairs that arise from the reduction of $[\mathbf{u}, \sigma^{|\zeta^{n-1}(W)|}\mathbf{u}]$. We define the set of balanced pairs as $I = I(W) = \bigcup_{n=1}^{\infty} I_n(W)$.

We call this process the *balanced pair algorithm* associated to the prefix W, or bpa-W for short. We say that the bpa-W *terminates* if $I(W)$ is finite. In this case we obtain a new substitution $\hat{\zeta}$, with the alphabet $I(W)$, and the substitution map that takes $[U, V] \in I(W)$ into the reduction of $[\zeta(U), \zeta(V)]$.

A balanced pair $[i, i]$, for $i \in \mathcal{A}$, is called a *coincidence*. Coincidences need not be in $I(W)$, but if they are, then the new substitution $\hat{\zeta}$ is not primitive, since $\hat{\zeta}([i, i])$ is the reduction of $[\zeta(i), \zeta(i)]$, and thus contains only coincidences.

We say that the balanced pair $[U, V]$ *leads to a coincidence* if there exists m such that the reduction of $[\zeta^m(U), \zeta^m(V)]$ contains a coincidence.

***Theorem 3.1* (Livshits [15, 16])** *Let ζ be a primitive substitution such that $\zeta(1)$ starts with* 1.

(a) *If for some prefix W the* bpa-W *terminates and every balanced pair in $I(W)$ leads to a coincidence, then ζ has pure discrete spectrum.*

(b) *If the* bpa-W *terminates for some prefix $W = u_0 \cdots u_m$, such that $u_{m+1} = u_0$, and ζ has pure discrete spectrum, then every balanced pair in $I(W)$ leads to a coincidence.*

We include a sketch of the proof of Theorem 3.1 below, for the reader's convenience, but first we make some remarks and then give an example of how the algorithm works.

Remarks 1. This theorem was proved by Livshits in [16] (in a slightly different form), but similar ideas appeared before. As already mentioned, the idea of coincidences was introduced by Dekking [4] in the constant length case. For substitutions of non-constant length, the balanced pair algorithm essentially goes back to the paper of Michel [17]. The main steps of the proof of part (a) (in a special case) can be found in Queffélec's book [18, VI.5]. Hollander [9] has worked out the details of this proof, and our sketch below largely follows his argument.

2. Note that the Pisot condition is not assumed in the theorem. However, it seems that if the characteristic polynomial of the substitution matrix is irreducible and the Perron-Frobenius eigenvalue is not Pisot, then ζ does not have pure discrete spectrum and the bpa does not terminate. We do not know if these claims have been proved in full generality; partial results in this direction were obtained in [26], [6].

3. It is easy to see that if the balanced pair algorithm terminates and ζ satisfies the strong coincidence condition, then every balanced pair leads to a coincidence. Hollander [9] (see also [10]) proved that for every substitution of Pisot type on two symbols, the balanced pair algorithm terminates. Together with the recent result of Barge and Diamond [2] and Theorem 3.1, this implies that all such substitutions have pure discrete spectrum.

Example Let ζ be the substitution:

$$1 \to 12, \quad 2 \to 13, \quad 3 \to 1.$$

So $\mathbf{u} = 12131211213\cdots$. We take the shortest prefix $W = 1$. First we need to find all the words in $\mathbf{u}$ of the form $1U1$ where U does not contain 1's. This is easy to do, and we get the following set of initial balanced pairs:

$$I_1(W) = \left\{ \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \begin{vmatrix} 2 \\ 2 \end{vmatrix}, \begin{vmatrix} 3 \\ 3 \end{vmatrix}, \begin{vmatrix} 12 \\ 21 \end{vmatrix}, \begin{vmatrix} 13 \\ 31 \end{vmatrix} \right\}$$

The substitution and reduction of the pairs which are not coincidences, is as follows:

$$\begin{vmatrix} 12 \\ 21 \end{vmatrix} \to \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{vmatrix} 213 \\ 312 \end{vmatrix}$$

$$\begin{vmatrix} 13 \\ 31 \end{vmatrix} \to \begin{vmatrix} 1 \\ 1 \end{vmatrix} \begin{vmatrix} 21 \\ 12 \end{vmatrix}$$

We got a new irreducible pair, which we break down:

$$\begin{vmatrix}213\\312\end{vmatrix} \to \begin{vmatrix}1\\1\end{vmatrix}\ \begin{vmatrix}3121\\1213\end{vmatrix}$$

Continuing the reduction process:

$$\begin{vmatrix}3121\\1213\end{vmatrix} \to \begin{vmatrix}1\\1\end{vmatrix}\ \begin{vmatrix}12\\21\end{vmatrix}\ \begin{vmatrix}13\\31\end{vmatrix}\ \begin{vmatrix}12\\21\end{vmatrix}$$

So

$$I(W) = \left\{\begin{vmatrix}1\\1\end{vmatrix}, \begin{vmatrix}2\\2\end{vmatrix}, \begin{vmatrix}3\\3\end{vmatrix}, \begin{vmatrix}12\\21\end{vmatrix}, \begin{vmatrix}13\\31\end{vmatrix}, \begin{vmatrix}213\\312\end{vmatrix}, \begin{vmatrix}3121\\1213\end{vmatrix}\right\}$$

Since every balanced pair leads to a coincidence, by Theorem 3.1(a), the $\mathbb{Z}$-action associated to this substitution has pure discrete spectrum.

Sketch of the Proof of Theorem 3.1 Let W be a prefix of the substitution sequence $\mathbf{u}$. Denote $D_m = \{n : u_{n+m} \neq u_n\}$ and $p_l = |\zeta^l(W)|$. Recall that for $R \subset \mathbb{N}$ the *density* is defined by

$$\operatorname{dens}(R) = \lim_{k\to\infty} \frac{\#(R \cap [1,k])}{k},$$

if the limit exists. The existence of $\operatorname{dens}(D_m)$ follows from unique ergodicity of the substitution dynamical system. We are interested in the behavior of $\operatorname{dens}(D_{p_l})$ as $l \to \infty$. Let

$$\mathbf{a}^{(l)} = \alpha_1^{(l)}\alpha_2^{(l)}\cdots \tag{3.2}$$

be the reduction of $[\mathbf{u}, \sigma^{p_l}\mathbf{u}]$ into irreducible balanced pairs. For a balanced pair $\beta = [U,V]$ let $|\beta| = |U| = |V|$ and $\Delta(\beta) = \#\{i \le |\beta| : U_i \neq V_i\}$. Then

$$\operatorname{dens}(D_{p_l}) = \lim_{N\to\infty} \frac{\sum_{j=1}^N \Delta(\alpha_j^{(l)})}{\sum_{j=1}^N |\alpha_j^{(l)}|}. \tag{3.3}$$

Consider the the substitution $\hat{\zeta}$ on the set of irreducible balanced pairs $I = I(W)$. According to (3.2), we have $\mathbf{a}^{(l)} \in I^{\mathbb{N}}$ and $\mathbf{a}^{(l)} = (\hat{\zeta})^l(\mathbf{a}^{(0)})$. Here $\mathbf{a}^{(0)}$ is the reduction of $[\mathbf{u}, \sigma^{|W|}\mathbf{u}]$ into irreducible balanced pairs.

There is a directed graph $\mathcal{G}(\hat{\zeta})$ associated with the substitution $\hat{\zeta}$. Its vertices are labelled by $I = I(W)$, and for every vertex β there are directed edges from β into the letters of $\hat{\zeta}(\beta)$ (with multiplicities).

(a) Let $\beta \in I(W)$ be an irreducible balanced pair which is not a coincidence. By assumption, there is a path in the graph $\mathcal{G}(\hat{\zeta})$ leading from β to a coincidence. But all the edges from coincidences lead to coincidences, so β is a "non-essential" vertex. It is not hard to deduce (and it is a standard argument) that the frequency of the symbol β in $\mathbf{a}^{(l)} = (\hat{\zeta})^l(\mathbf{a}^{(0)})$ goes to zero geometrically fast, as $l \to \infty$. Since $I(W)$ is finite, and $\Delta(\beta) > 0$ if and only if β is a non-coincidence, it follows from (3.3) that

$$\operatorname{dens}(D_{p_l}) \le \text{const} \cdot \gamma^l \tag{3.4}$$

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for some $\gamma \in (0, 1)$. Now we can conclude as in [18, VI.4]: the sequence $\mathbf{u}$ is mean-almost periodic so ζ has pure discrete spectrum.

(b) Let $W = u_0 \cdots u_m$ be such that $u_{m+1} = u_0$, and denote $p_l = |\zeta^l(W)|$. It follows from [11] that $\lim_{l\to\infty} \lambda^{p_l} = 1$ for any eigenvalue λ of the dynamical system $(\Omega_\zeta, \sigma, \mu)$. If the spectrum is pure discrete, we have that the eigenfunctions span a dense subset of $L^2(\Omega_\zeta)$, and hence $\lim_{l\to\infty} \|U_\zeta^{p_l} f - f\|_2 = 0$ for every $f \in L^2(\Omega_\zeta)$. Taking f to be the characteristic function of the cylinder set corresponding to $i \in \mathcal{A}$, we obtain, as in [18, VI.26], that $\lim_{l\to\infty} \mathrm{dens}(D_{p_l}) = 0$. Suppose that there is an irreducible balanced pair in $I(W)$ which does not lead to a coincidence. This implies that there exists an irreducible (strongly connected) component $\mathcal{G}_0$ of the graph $\mathcal{G}(\hat{\zeta})$ which contains no coincidences. There exists l_0 such that for every $l \geq l_0$ elements of the component $\mathcal{G}_0$ occur in $\mathbf{a}^{(l)}$ with a positive frequency. (Note that different elements of $\mathcal{G}_0$ may occur for different l). A standard argument then shows that this frequency is bounded away from zero, as $l \to \infty$. Since $\Delta(\beta) > 0$ for all $\beta \in \mathcal{G}_0$, in view of (3.3), it follows that $\mathrm{dens}(D_{p_l}) \not\to 0$, which is a contradiction. ■

4 Tiling Dynamical Systems

The reader is referred to [19], [27] for preliminaries on tiling dynamical systems. Here the situation is much simplified since we consider the one-dimensional case only.

Let M_ζ be the incidence matrix of the substitution ζ and let θ be its Perron-Frobenius eigenvalue. We assume θ to be a Pisot number and the characteristic polynomial of M_ζ to be irreducible. Then there exists a left Perron-Frobenius eigenvector $(t_1, \ldots, t_d)$ such that $t_i \in \mathbb{Z}[\theta]$, $i \leq d$ (first we find an eigenvector with entries in $\mathbb{Q}(\theta)$ and then multiply it by an appropriate scalar polynomial in θ). Furthermore, it is not hard to see that the numbers $\{t_1, \ldots, t_d\}$ are independent over the rationals (see, *e.g.* [3, Proposition 3.1] where independence is proved for a right eigenvector of M_ζ; replacing M_ζ by its transpose yields the desired result).

To the substitution ζ we can associate a tiling $\mathcal{T}$ of the half-line $\mathbb{R}_+$ by intervals. The "prototiles" are closed intervals of length $t_1, \ldots, t_d$. We say that any closed interval of length t_i is a *tile of type i*. The tiling $\mathcal{T}$ is obtained by taking the substitution sequence $\mathbf{u} = u_1 u_2 \cdots$ and putting the tiles of types $u_1, u_2, \ldots$ adjacent to each other, starting from the origin. (Tiles are sometimes equipped with "labels," when there is a need to distinguish congruent tiles. However, in our case all t_i's are distinct by rational independence, so the type of a tile is uniquely determined by its length.) The tiling space $X_\mathcal{T}$ is defined as the set of tilings $\mathcal{S}$ of the line $\mathbb{R}$ such that every "patch" of $\mathcal{S}$ is a translate of a $\mathcal{T}$-patch. The usual topology is introduced on $X_\mathcal{T}$, making it a compact space. The *tiling dynamical system* is the $\mathbb{R}$-action $(X_\mathcal{T}, \Gamma_x)$ where $\Gamma_x(\mathcal{S}) = \mathcal{S} - x$ for $x \in \mathbb{R}$. Observe that this $\mathbb{R}$-action is topologically conjugate to the suspension flow over the $\mathbb{Z}$-action (Ω_ζ, σ), with the height function equal to t_i on the cylinder corresponding to the symbol i. The tiling $\mathcal{T}$ is *self-similar*, where the similarity map is the multiplication by θ and the tile-substitution is the geometric version of ζ. (Here the situation is slightly different from that considered in [27] since $\mathcal{T}$ is a tiling of the half-line rather than of the whole line; however, everything extends readily to this case. It is also possible to find another tiling $\mathcal{T}'$, of the whole line, which is self-similar with respect to a power of the substitution, and which generates the same tiling space.) Since the substitution ζ is primitive, the tiling dynamical system $(X_{\mathcal{T}}, \Gamma_x)$ is uniquely ergodic, that is, there is a unique invariant Borel probability measure ν on $X_{\mathcal{T}}$, see [20], [27]. The system $(X_{\mathcal{T}}, \Gamma_x, \nu)$ is said to have *pure discrete spectrum* if there is a basis of $L^2(X_{\mathcal{T}}, \nu)$ consisting of eigenfunctions for the $\mathbb{R}$-action.

4.1 Overlap Algorithm

This algorithm was introduced in [27] for tilings in the plane. It extends to the case of tilings in the line; in fact, the arguments become easier, but there are also some features which are exclusively one-dimensional.

Let $\mathcal{T}$ be the self-similar tiling of the half-line as above, with a Pisot expansion coefficient θ. Let $T, S \in \mathcal{T}$ and $y \in \mathbb{R}_+$. The triple (T, S, y) is called an *overlap* if the interiors of T and $S - y$ intersect; this overlap is denoted by $\mathcal{O}_{T,S,y}$. The overlaps $\mathcal{O}_{T,S,y}$ and $\mathcal{O}_{T',S',y'}$ are *equivalent* if $T = T' + z$, $S - y = S' - y' + z$ for some real number z. We will denote by $[\mathcal{O}_{T,S,y}]$ the equivalence class of an overlap. There are finitely many equivalence classes for a fixed y, since the tiling has a finite number of patches up to translation. We will denote by $\mathcal{O}_y$ the set of all overlaps $\mathcal{O}_{T,S,y}$ with $T, S \in \mathcal{T}$, and let $[\mathcal{O}_y]$ be the corresponding set of equivalence classes. The set of overlaps $\mathcal{O}_y$ has the following geometric interpretation: Consider the tiling $\mathcal{T}$ "on top" of the tiling $\mathcal{T} - y$. The pairs of tiles from these tilings, whose interiors intersect, give rise to the overlaps in $\mathcal{O}_y$. We say that an overlap $\mathcal{O}_{T,S,y}$ is an *overlap-coincidence* if $T = S - y$. We call $\mathcal{O}_{T,S,y}$ a *half-coincidence* if one of the endpoints of T coincides with an endpoint of $S - y$. (Thus, every overlap-coincidence is a half-coincidence too). Let

$$\Xi(\mathcal{T}) = \{x \in \mathbb{R} : \exists T, T' \in \mathcal{T} \text{ with } T' = T + x\} \tag{4.1}$$

be the set of translation vectors between tiles of the same type. Since the lengths of tiles are in $\mathbb{Z}[\theta]$, we have $\Xi(\mathcal{T}) \subset \mathbb{Z}[\theta]$. For a fixed $x \in \mathbb{Z}[\theta] \cap \mathbb{R}_+$ we will construct a *subdivision graph of overlaps* $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$. It is a directed graph whose vertices are the elements of $\bigcup_{n \geq 0} [\mathcal{O}_{\theta^n x}]$. Applying the expansion we have

$$\theta\big(T \cap (S - \theta^n x)\big) = \bigcup_{T' \subset \theta T, S' \subset \theta S} \big(T' \cap (S' - \theta^{n+1} x)\big). \tag{4.2}$$

There is an edge in the graph from $[\mathcal{O}_{T,S,\theta^n x}]$ to $[\mathcal{O}_{T',S',\theta^{n+1} x}]$ for every intersection with non-empty interior in the right-hand side of (4.2). *A priori*, this graph may have infinitely many vertices; however, since the expansion coefficient θ is a Pisot number, the graph is finite. This is proved in [27, Proposition 6.4] (although it was assumed in [27] that $x \in \Xi(\mathcal{T})$, only $x \in \mathbb{Z}[\theta]$ was used). The graph $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$ has two special kinds of vertices: the overlap-coincidences and the half-coincidences. It is clear that every edge from an overlap-coincidence leads to an overlap-coincidence. On the other hand, for a half-coincidence, there is at least one edge leading to some (possibly the same) half-coincidence.

The overlap algorithm associated to a translation vector $x \in \mathbb{Z}[\theta] \cap \mathbb{R}_+$ (abbreviated as oa-x) runs as follows: We construct the subdivision graph of overlaps for

x, *i.e.* $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$, starting with $[\mathcal{O}_x]$. Then expansions and subdivisions in (4.2) are applied. New vertices and edges are added as described above. The process is repeated until $[\mathcal{O}_{\theta^{n+1}x}] \subset [\mathcal{O}_x] \cup \cdots \cup [\mathcal{O}_{\theta^n x}]$. As we pointed out, the graph is finite when θ is Pisot, so the overlap algorithm always terminates in this case. We say that the overlap algorithm associated to a translation vector $x \in \mathbb{Z}[\theta] \cap \mathbb{R}_+$ *terminates with coincidences* (resp. *half-coincidences*) if from any vertex of the graph $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$, there is a path leading to an overlap-coincidence (resp. half-coincidence). This construction is valid in higher dimensions, except that the concept of half-coincidence is one-dimensional.

Theorem 4.1 **(Solomyak [27])**

(a) *The dynamical system $(X_{\mathcal{T}}, \Gamma_x)$ has pure discrete spectrum if and only if for all $x \in \Xi(\mathcal{T}) \cap \mathbb{R}_+$, the corresponding overlap algorithm terminates with coincidences.*
(b) *If for some $x \in \mathbb{Z}[\theta] \cap \mathbb{R}_+$ the corresponding overlap algorithm terminates with coincidences, then $(X_{\mathcal{T}}, \Gamma_x)$ has pure discrete spectrum.*

The proof is essentially contained in [27]. There are two distinctions. First, [27] is mostly concerned with tilings in the plane, but the arguments only get simpler for tilings of the line. Second, here $\mathcal{T}$ is a tiling of the half-line, so, strictly speaking, it is not self-similar in the sense of [27]. However, $\mathcal{T}$ is repetitive and has uniform patch frequencies in $\mathbb{R}_+$, so this change does not cause any difficulties. Part (b) holds by [27, Theorem 6.1]. The key step is showing that $\operatorname{dens}\big(\mathcal{T} \setminus (\mathcal{T} + \theta^n x)\big) \to 0$, as $n \to \infty$, geometrically fast, which is analogous to (3.4).

5 Relation Between the $\mathbb{R}$-Action and the $\mathbb{Z}$-Action

There are similarities between the balanced pair algorithm for symbolic systems ($\mathbb{Z}$-action) and the overlap algorithm for the corresponding tiling systems ($\mathbb{R}$-action). In this section we explore the relation between them and the consequences for the spectrum of the systems.

Theorem 5.1 *The following are equivalent:*

(i) *the balanced pair algorithm associated to some prefix W of $\mathbf{u}$ terminates and every balanced pair in $I(W)$ leads to a coincidence;*
(ii) *there exists $x \in \mathbb{Z}[\theta] \cap \mathbb{R}_+$ such that the corresponding overlap algorithm terminates with coincidences.*

The proof requires some preparation; it is given at the end of the section. First we deduce several corollaries.

Corollary 5.2 *Let ζ be a substitution of Pisot type. If the $\mathbb{R}$-action $(X_{\mathcal{T}}, \Gamma_x)$ has pure discrete spectrum, then the $\mathbb{Z}$-action (Ω_ζ, σ) has pure discrete spectrum.*

Proof If the $\mathbb{R}$-action $(X_{\mathcal{T}}, \Gamma_x)$ has pure discrete spectrum, then oa-x terminates with coincidences for all $x \in \Xi(\mathcal{T})$, by Theorem 4.1(a). By Theorem 5.1 and Theorem 3.1, the $\mathbb{Z}$-action (Ω_ζ, σ) has pure discrete spectrum. ■

Corollary 5.3 *The $\mathbb{R}$-action $(X_{\mathcal{T}}, \Gamma_x)$ has pure discrete spectrum if and only if there exists a prefix W of $\mathbf{u}$ such that the corresponding balanced pair algorithm terminates and every balanced pair in $I(W)$ leads to a coincidence.*

Proof This is immediate from Theorem 5.1 and Theorem 4.1. For the "only if" direction we use that $\Xi(\mathcal{T}) \subset \mathbb{Z}[\theta]$. ■

This "if" part of the last corollary is important since it gives us a checkable criterion for pure discrete spectrum of the $\mathbb{R}$-action. The balanced pair algorithm is much easier to deal with, from the computational point of view, than the overlap algorithm.

Corollary 5.4 *For any substitution of Pisot type on two symbols, the corresponding $\mathbb{R}$-action $(X_{\mathcal{T}}, \Gamma_x)$ has pure discrete spectrum.*

Proof Combining [9] and [2] (see also [10]), we know that there exists a prefix W such that the bpa-W terminates and every balanced pair leads to a coincidence. Now the claim follows from Corollary 5.3. ■

Next we start preparation for the proof of Theorem 5.1.

A *geometric balanced pair* is a pair of patches $[P_1, P_2]$ where P_1 is a collection of consecutive tiles $\{T_{i_1}, \dots, T_{i_m}\} \subset \mathcal{T}$ and P_2 is a collection of consecutive tiles $\{S_{j_1} - y, \dots, S_{j_n} - y\} \subset \mathcal{T} - y$, for some $y \in \mathbb{R}_+$, such that the left endpoints of T_{i_1}, $S_{j_1} - y$, and the right endpoints of T_{i_m}, $S_{j_n} - y$, coincide.

Lemma 5.5 *If $[P_1, P_2]$ is a geometric balanced pair, then $m = n$ and $[i_1 \cdots i_m, j_1 \cdots j_n]$ is a balanced word pair.*

Proof Clearly, $t_{i_1} + \cdots + t_{i_m} = t_{j_1} + \cdots + t_{j_n}$. Now the statement is immediate from the rational independence of $t_1, \dots, t_k$. ■

Notice that a geometric balanced pair for $(\mathcal{T}, \mathcal{T} - y)$ gives rise to a number of overlaps, among which the first one, (T_{i_1}, S_{j_1}, y), and the last one, (T_{i_m}, S_{j_n}, y), are half-coincidences. (In the special case, when $m = n = 1$, there is just one overlap-coincidence corresponding to the geometric balanced pair.)

Let $W = u_1 \cdots u_r$ be a prefix of $\mathbf{u}$. To this prefix we associate a positive real number $x = x(W)$, the sum of the lengths of tiles given by the symbols in W, that is,

$$x = x(W) = t_{u_1} + \cdots + t_{u_r}.$$

Thus, $x \in \mathbb{Z}[\theta] \cap \mathbb{R}_+$ and $\mathcal{T} - x$ has a tile $[0, t_{u_{r+1}}]$. Moreover, the restriction of the tiling $\mathcal{T} - x$ to $\mathbb{R}_+$ corresponds precisely to the infinite word $\sigma^{|W|}\mathbf{u}$, in the sense that the former is the sequence of tile types for the latter, written in the order they appear. Recall that the set of initial balanced word pairs $I_1(W)$ arises from the reduction of $[\mathbf{u}, \sigma^{|W|}\mathbf{u}]$. Every balanced word pair corresponds to a geometric balanced pair for the tilings $(\mathcal{T}, \mathcal{T} - x)$. Further, the set of balanced word pairs $I_n(W)$ arises from the reduction of $[\mathbf{u}, \sigma^{|\zeta^{n-1}(W)|}\mathbf{u}]$. By the choice of t_i as components of the left eigenvector

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for θ, the infinite word $\sigma^{|\zeta^{n-1}(W)|}\mathbf{u}$ is the sequence of tile types for the tiling $\mathcal{T}-\theta^{n-1}x$ restricted to $\mathbb{R}_+$. Recall that the pair of tilings $(\mathcal{T}, \mathcal{T}-\theta^{n-1}x)$ yields the overlaps $\mathcal{O}_{\theta^{n-1}x}$ which come up in the overlap algorithm. Thus, there is a clear link between the bpa-W and the oa-x, for $x = x(W)$.

The following theorem is of independent interest, since it allows one to check when the balanced pair algorithm terminates.

Theorem 5.6 *The following are equivalent:*

(i′) *the balanced pair algorithm associated with a prefix W terminates;*
(ii′) *the overlap algorithm corresponding to $x = x(W)$ terminates with half-coincidences;*
(iii′) *the distance between consecutive half-coincidences arising from $(\mathcal{T}, \mathcal{T} - \theta^n x)$ is bounded, with a bound independent of n.*

Proof (i′) ⇔ (iii′) We employ the correspondence between the bpa-W and oa-x indicated above. The bpa-W terminates if and only if there are finitely many balanced word pairs in $\bigcup_{n\geq 0} I_n(W)$. This happens if and only if there are finitely many geometric balanced pairs arising from $(\mathcal{T}, \mathcal{T} - \theta^n x)$ (for all $n \geq 0$), up to translation. Since every geometric balanced pair starts and ends with a half-coincidence, (i′) ⇒ (iii′) follows. Conversely, whenever we have consecutive half-coincidences arising from $(\mathcal{T}, \mathcal{T} - y)$, there is a geometric balanced pair between them, see Figure 1. This proves (iii′) ⇒ (i′).

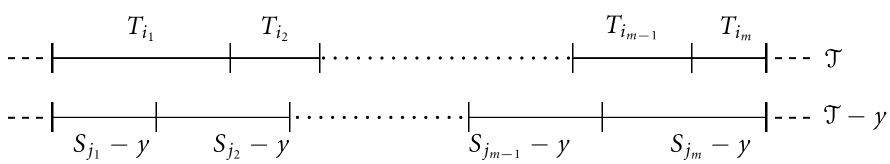

Figure 1: A geometric balanced pair in $\mathcal{T}$ and $\mathcal{T} - y$.

(ii′) ⇔ (iii′) All the overlaps in oa-x arise from the pairs of tilings $(\mathcal{T}, \mathcal{T}-\theta^n x)$, $n \geq 0$. Fix any of these overlaps, say, $\mathcal{O}_{T,S,y}$. The (equivalence classes of) overlaps that it leads to, are those which arise from $\left(\theta^n T, \theta^n(S - y)\right)$, $n \geq 1$. Since $T \cap (S - y)$ has positive length, the length of $\theta^n T \cap \theta^n(S - y)$ tends to infinity. If the oa-x terminates with half-coincidences, then there exists $N \in \mathbb{N}$ independent of T, S, y, such that $\left(\theta^N T, \theta^N(S - y)\right)$ contains a half-coincidence. This implies (iii′). Conversely, if (iii′) holds, then for n large, $\left(\theta^n T, \theta^n(S - y)\right)$ contains a half-coincidence, which means that (ii′) holds. ■

Proof of Theorem 5.1 (i) ⇒ (ii) We let $x = x(W)$ as above. By Theorem 5.6, we already know that oa-x terminates with half-coincidences. Every balanced word pair

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leads to a letter coincidence, hence the distance between consecutive such coincidences in the reduction of $[\mathbf{u}, \sigma^{|\zeta^n(W)|}\mathbf{u}]$ is bounded by a constant independent of n. Every letter coincidence yields a geometric overlap-coincidence. Then arguing as in the proof of Theorem 5.6, we obtain that oa-x terminates with coincidences.

(ii) $\Rightarrow$ (i) By Theorem 4.1(b), the tiling dynamical system $(X_{\mathcal{T}}, \Gamma_x)$ has pure discrete spectrum, and hence, by Theorem 4.1(a), oa-y terminates with coincidences for any $y \in \Xi(\mathcal{T})$. Choose a prefix $W = u_1 \cdots u_r$ of $\mathbf{u}$ so that $u_{r+1} = u_1$. Then $x = x(W) \in \Xi(\mathcal{T})$, since this is a translation vector between two congruent tiles, those that correspond to u_1 and u_{r+1}. Thus, oa-x terminates with coincidences. By Theorem 5.6, bpa-W terminates, and it only remains to check that every balanced word pair leads to a letter coincidence. But, as we saw already, any geometric balanced pair corresponds to a balanced word pair, and any overlap-coincidence corresponds to a letter coincidence, so the theorem is proved. ■

6 Appendix: Examples

We implemented the balanced pair algorithm in *Mathematica*; the code of the program can be found at the URL: http://www.ma.usb.ve/˜vsirvent/software/bpa.html

The algorithm was tested on a number of examples which we considered "difficult," for some vague heuristic reasons, in the hope of finding a counterexample to "Pisot implies pure discrete spectrum" conjecture. However, in all examples the bpa-W terminated with coincidences, so the associated dynamical systems (the $\mathbb{Z}$-action and the $\mathbb{R}$-action) have pure discrete spectrum. Four of the examples are presented below; in all of them we considered the prefix $W = 1$. Recall that a substitution ζ is unimodular if $\det(M_\zeta) = \pm 1$.

1. Consider the following non-unimodular Pisot substitution:

$$1 \to 12$$

$$2 \to 223$$

$$3 \to 11$$

The balanced pair algorithm produces 559 different irreducible balanced pairs (we count $[U, V]$ and $[V, U]$ as the same pair) and the maximal length of an irreducible balanced pair is 1673. This is, perhaps, due to the fact that the conjugates of the Pisot number are $\approx .89$ in modulus, which is rather close to 1.

2. Consider the following unimodular Pisot substitution:

$$1 \to 1111112223$$

$$2 \to 2231111$$

$$3 \to 311122$$

The balanced pair algorithm produces 260 different irreducible balanced pairs, and the maximal length of an irreducible balanced pair is 194. This substitution has high complexity and its Rauzy fractal is very irregular, see [24, 25] for details.

3. Consider the following unimodular Pisot substitution:

$$1 \to 112$$
$$2 \to 3$$
$$3 \to 14$$
$$4 \to 1$$

The balanced pair algorithm produces 37 different irreducible balanced pairs, and the maximal length of an irreducible balanced pair is 41. The Pisot number θ of this substitution has the property that there are integers whose "greedy" expansions in base θ is not finite, see [7] for details.

4. Consider the following unimodular Pisot substitution:

$$1 \to 1111223$$
$$2 \to 41$$
$$3 \to 2111$$
$$4 \to 121$$

The balanced pair algorithm produces 628 different irreducible balanced pairs, and the maximal length of an irreducible balanced pair is 2306. The associated Pisot number of this substitution is the cube of the previous example.

Added In Proof After this paper was accepted, we became aware of the preprint math.DS/0201152, available at http://xxx.lanl.gov/, by A. Clark and L. Sadun, entitled "When size matters: subshifts and their related tiling spaces". Their Theorem 3.1 implies that the $\mathbb{Z}$-action, corresponding to the substitution of Pisot type, is pure discrete if and only if the corresponding $\mathbb{R}$-action is pure discrete.

References

[1] P. Arnoux and S. Ito, *Pisot substitutions and Rauzy fractals*. Bull. Belg. Math. Soc. Simon Stevin **8**(2001), 181–207.

[2] M. Barge and B. Diamond, *Coincidence for substitutions of Pisot type*. 2001, preprint.

[3] V. Canterini and A. Siegel, *Geometric representation of substitutions of Pisot type*. Trans. Amer. Math. Soc. **353**(2001), 5121–5144.

[4] F. M. Dekking, *The spectrum of dynamical systems arising from substitutions of constant length*. Z. Wahrsch. Verw. Gebiete **41**(1978), 221–239.

[5] S. Dworkin, *Spectral theory and X-ray diffraction*. J. Math. Phys. **34**(1993), 2965–2967.

[6] S. Ferenczi, C. Mauduit and A. Nogueira, *Substitution dynamical systems: algebraic characterization of eigenvalues*. Ann. Sci. École Norm. Sup. **29**(1996), 519–533.

[7] C. Frougny and B. Solomyak, *Finite beta-expansions*. Ergodic Theory Dynamical Systems **12**(1992), 713–723.

[8] A. Hof, *Diffraction by aperiodic structures*. In: The Mathematics of Long-Range Aperiodic Order, (ed. R. V. Moody), Kluwer, 1997, 239–268.

[9] M. Hollander, *Linear numeration systems, finite β-expansions, and discrete spectrum for substitution dynamical systems*. Ph.D. Thesis, University of Washington, 1996.

[10] M. Hollander and B. Solomyak, *Two-symbol Pisot substitutions have pure discrete spectrum*. 2001, preprint.

[11] B. Host, *Valeurs propres de systèmes dynamiques définis par de substitutions de longueur variable*. Ergodic Theory Dynamical Systems **6**(1986), 529–540.
[12] ———, *Représentation géométrique des substitutions sur 2 lettres*. Unpublished manuscript, 1992.
[13] J.-Y. Lee, R. V. Moody and B. Solomyak, *Diffraction and Multi-dimensional Substitution Systems*. 2001, preprint.
[14] D. Lind and B. Marcus, *Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.
[15] A. N. Livshits, *On the spectra of adic transformations of Markov compacta*. Russian Math. Surveys **42**(1987), 222–223.
[16] ———, *Some examples of adic transformations and substitutions*. Selecta Math. Soviet. **11**(1992), 83–104.
[17] P. Michel, *Coincidence values and spectra of substitutions*. Z. Wahrsch. Verw. Gebiete **42**(1978), 205–227.
[18] M. Queffélec, *Substitution Dynamical Systems—Spectral Analysis*. Lecture Notes in Math. **1294**, Springer-Verlag, Berlin, 1987.
[19] C. Radin, *Space tilings and substitutions*. Geom. Dedicata **55**(1995), 257–264.
[20] C. Radin and M. Wolff, *Space tilings and local isomorphism*. Geom. Dedicata **42** (1992), 355–360.
[21] G. Rauzy, *Nombres algébriques et substitutions*. Bull. Soc. Math. France **110**(1982), 147–178.
[22] A. Siegel, *Chapter 7: Spectral theory and geometric representation of substitutions*. In: Substitutions in Dynamics, Arithmetics and Combinatorics, 199–252, (eds. V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel), Lecture Notes in Math., Springer, to appear.
[23] ———, *Répresentation des systèmes dynamiques substitutifs non unimodulaires*. Ergodic Theory Dynamical Systems, to appear.
[24] V. F. Sirvent, *Modelos geométricos asociados a substituciones*. Trabajo de ascenso, Universidad Simón Bolívar, 1998.
[25] V.F. Sirvent and Y. Wang, *Self-affine tiling via substitution dynamical systems and Rauzy fractals*. Pacific J. Math., to appear.
[26] B. Solomyak, *On the spectral theory of adic transformations*. Adv. Soviet Math. **9**(1992), 217–230.
[27] ———, *Dynamics of self-similar tilings*. Ergodic Theory Dynamical Systems **17**(1997), 695–738. Corrections, ibid. **19**(1999), 1685.
[28] P. Walters, *An Introduction to Ergodic Theory*. Springer Graduates Texts in Math., Springer-Verlag, New York, 1982.

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