ON PACKINGS OF UNEQUAL SPHERES IN R_n

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Suppose that a sequence of spheres $\{S_j\}_{j=1}^{\infty}$ is packed in order of decreasing diameters into the unit cube I_n of R_n. In recent work (**2**), I have shown that for $n = 2$, there exist positive constants K_2, s ($=0.97$) such that the area of $I_2 - \bigcup_{j=1}^{m} S_j$ has an asymptotic lower bound $K_2(d(S_m))^s$. Although the methods used were complicated and possibly only viable in two dimensions, it is intuitively clear that such a result should also be true in higher dimensions. The purpose of this note is to establish this result. During the past two years, considerable progress has been made on problems of this nature; the principal references of this work are (**1**; **3**; **4**; **5**; **6**).

THEOREM. *Suppose that a sequence of spheres $\{S_j\}_{j=1}^{\infty}$ is packed in order of decreasing diameters into the unit cube I_n of R_n. Then there exist positive constants K_n, s ($=0.97$) such that the volume of $I_n - \bigcup_{j=1}^{m} S_j$ is at least $K_n(d(S_m))^s$, $n \geqq 2$.*

LEMMA. *Suppose that A_1, A_2, A_3 are three disjoint disks of diameter r in R_2. Then every point p of R_2 is at least a distance $14r/25$ from at least one of the centres of A_1, A_2, A_3.*

Proof. If the lemma is not true, then the centres p_1, p_2, p_3 of A_1, A_2, A_3 must form a triangle whose sides have lengths lying between r and $28r/25$. The centroid of this triangle lies at a distance less than r from each of p_1, p_2, p_3 and therefore belongs to each of A_1, A_2, A_3, which is impossible.

Proof of the Theorem. We shall prove the theorem by induction on n ($n \geqq 2$) and, using the results of (**2**), we may assume that the theorem is true for $n = 2$. Therefore, we shall assume that $n \geqq 3$ and that the theorem is already true in R_{n-1}.

Let r be a positive number, and for a given packing $\{S_j\}_{j=1}^{\infty}$ of spheres into I_n, let $m = \sup\{j: d(S_j) \geqq r\}$. We write $I_n = [0, 1] \times I_{n-1}$ and $S_j(x) = S_j \cap (x \times I_{n-1})$ for $0 \leqq x \leqq 1$, $j = 1, 2, \ldots$. Our aim, of course, is to find a positive constant K_n such that the $(n-1)$-dimensional volume of $x \times I_{n-1} - \bigcup_{j=1}^{m} S_j(x)$ is at least $K_n r^s$. However, this is not immediately realizable from the inductive hypothesis since, in general, the collection $\{S_j(x)\}_{j=1}^{m}$ may contain $(n-1)$-spheres of arbitrarily small diameter.

Let $\mathfrak{D}_m$ denote the collection of all those $(n-1)$-spheres of $\{S_j(x)\}_{j=1}^{m}$ which have positive diameter less than $r/(100(n-1)^{\frac{1}{2}})$. If $S_\mu(x)$ is in $\mathfrak{D}$,

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let $C_\mu(x)$ denote the $(n-1)$-dimensional cube which circumscribes $S_\mu(x)$ and has faces parallel to those of I_{n-1}. Within the collection $\{S_j(x)\}_{j=1}^m$ we define a relationship $S_v(x) \sim S_{v'}(x)$ if the distance between $S_v(x)$ and $S_{v'}(x)$ is at most $r/100$. We assert that for $S_\mu(x)$ in $\mathfrak{D}$ there is at most one $S_{\mu'}(x)$, $\mu \neq \mu'$, such that $S_\mu(x) \sim S_{\mu'}(x)$. Having proved this, we can then say, in particular, that within $\mathfrak{D}$, the relation $\sim$ induces a partition of $\mathfrak{D}$ into subsets comprising at most two $(n-1)$-spheres.

Suppose the above assertion were false. That is to say, that there exist three distinct $(n-1)$-spheres $S_\mu(x)$, $S_{\mu'}(x)$, $S_{\mu''}(x)$ such that $S_\mu(x)$ is in $\mathfrak{D}$ and both $S_{\mu'}(x)$ and $S_{\mu''}(x)$ lie within a distance $r/100$ of $S_\mu(x)$. Then we can place n-spheres A_1, A_2, A_3 of diameter r into S_μ, $S_{\mu'}$, $S_{\mu''}$, respectively, so that their centres a_1, a_2, a_3 are at most a distance $13r/25$ from the centre y of $S_\mu(x)$. If P denotes the two-dimensional plane through a_1, a_2, a_3, then the projection y' of y onto P is at most a distance $13r/25$ from each of a_1, a_2, a_3, which contradicts the conclusions of the lemma.

Suppose, therefore, that $S_\mu(x)$ is in $\mathfrak{D}$ and that $S_{\mu'}(x)$, $\mu \neq \mu'$, is the other sphere, if any, such that $S_\mu(x) \sim S_{\mu'}(x)$. Let $R_{\mu,\mu'}$ be a hyperplane which strictly separates $S_\mu(x)$ from $S_{\mu'}(x)$. We divide $C_\mu(x)$ into 2^{n-1} equal half cubes by bisecting each side of $C_\mu(x)$ with a perpendicular hyperplane. Now $R_{\mu,\mu'}$ may meet one of these half cubes (U^*_μ, say), but in that case, $R_{\mu,\mu'}$ will not meet that half cube U_μ which lies diametrically opposite U^*_μ. For, otherwise, $R_{\mu,\mu'}$ would cut S_μ. If $R_{\mu,\mu'}$ does not meet one of the half cubes, we take any half cube to be U_μ. If $S_{\mu'}(x)$ is also in $\mathfrak{D}$ then $S_\mu(x)$, $S_{\mu'}(x)$ are uniquely determined relative to each other in the relationship $\sim$ and, therefore, there is no ambiguity in choosing $U_{\mu'}$ in $C_{\mu'}(x)$ as U_μ in $C_\mu(x)$ and with the same hyperplane $R_{\mu,\mu'}$. If $S_{\mu'}(x)$ and, hence, $R_{\mu,\mu'}$ are not defined, then we take any of the half cubes of $C_\mu(x)$ to be U_μ.

In the above manner, we may define a half cube U_μ for any $S_\mu(x)$ in $\mathfrak{D}$. We assert that the collection $\{U_\mu: S_\mu(x) \in \mathfrak{D}\}$ are pairwise disjoint. For, if there exist $S_\mu(x)$, $S_{\mu'}(x)$ in $\mathfrak{D}$ such that $U_\mu \cap U_{\mu'} \neq \emptyset$, then $S_\mu(x)$, $S_{\mu'}(x)$ are within a distance $r/100$ and $R_{\mu,\mu'}$ is defined. But then $R_{\mu,\mu'}$ strictly separates U_μ and $U_{\mu'}$, which is contradictory. We also note that $U_\mu - S_\mu$ is contained in $x \times I_{n-1} - \bigcup_{j=1}^m S_j$.

Suppose that the collection $\mathfrak{D}$ make a significant contribution to the $(n-1)$-dimensional volume of $\bigcup_{j=1}^m S_j(x)$, i.e., suppose

$$(1) \qquad \sum_{\mathfrak{D}} \nu_{n-1}(S_j(x)) \geqq \tfrac{1}{2}K_{n-1}(r/100(n-1)^{\frac{1}{2}}),$$

where $\nu_{n-1}(S)$ denotes the $(n-1)$-dimensional volume of S. Then, from above, it follows that

$$(2) \qquad \nu_{n-1}\left(x \times I_{n-1} - \bigcup_{j=1}^{m} S_j\right) \geqq \sum_{\mathfrak{D}} \nu_{n-1}(U_\mu - S_\mu) \geqq K'_n r^s,$$

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where $K'_n = 2^{-n}((W_{n-1})^{-1} - 1)K_{n-1}/(100(n-1)^{\frac{1}{2}})^s$, and W_{n-1} is the $(n-1)$-dimensional volume of the sphere of unit diameter in R_{n-1}.

If the spheres of $\mathfrak{D}$ satisfy the reverse inequality to (1), then those spheres $\mathfrak{E}$ of $\bigcup_{j=1}^{m} S_j(x)$ which do not lie in $\mathfrak{D}$ satisfy, using the inductive hypothesis,

$$\nu_{n-1}(x \times I_{n-1} - \bigcup_{\mathfrak{E}} S_j(x)) \geqq K_{n-1}(r/100(n-1)^{\frac{1}{2}})^s.$$

Hence, in this case,

$$\nu_{n-1}\left(x \times I_{n-1} - \bigcup_{j=1}^{m} S_j\right) \geqq K''_n r^s, \tag{3}$$

where $K''_n = \frac{1}{2}K_{n-1}/(100(n-1)^{\frac{1}{2}})^s$. If $K_n = \min(K'_n, K''_n)$, we conclude from (2) and (3) that

$$\nu_{n-1}\left(x \times I_{n-1} - \bigcup_{j=1}^{m} S_j\right) \geqq K_n r^s.$$

Hence

$$\nu_n\left(I_n - \bigcup_{j=1}^{m} S_j\right) = \int_0^1 \nu_{n-1}\left(x \times I_{n-1} - \bigcup_{j=1}^{m} S_j\right)dx \geqq K_n r^s,$$

which completes the proof of the theorem.

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