ON AN INTEGRAL INVOLVING THE H-FUNCTION

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Abstract. The aim of this note is to evaluate an integral involving the product of two H-functions.

1. Introduction

The H-function introduced by Fox [3], p. 408 will be defined and represented as follows:

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \frac{\prod_1^m \Gamma(b_j-\beta_j\xi)\prod_1^n\Gamma(1-a_j+\alpha_j\xi)}{\prod_{m+1}^q\Gamma(1-b_j+\beta_j\xi)\prod_{n+1}^p\Gamma(a_j-\alpha_j\xi)}x^\xi d\xi, \tag{1.1}$$

where x is not equal to zero and an empty product is interpreted as unity, p, q, m, n are integers satisfying $1 \leqq m \leqq q$, $0 \leqq n \leqq p$; α_j $(j = 1, \cdots, p)$, β_j $(j = 1, \cdots, q)$ are positive numbers; and a_j $(j = 1, \cdots, p)$, b_j $(j = 1, \cdots, q)$ are complex numbers such that no pole of $\Gamma(b_h-\beta_h\xi)$ $(h = 1, \cdots, m)$ coincides with any pole of $(1-a_i+\alpha_i\xi)$ $(i = 1, \cdots, n)$; i.e.,

$$\alpha_i(b_h+\nu) \neq \beta_h(a_i-\eta-1), \tag{1.2}$$

$(\nu, \eta = 0, 1, \cdots; h = 1, \cdots, m; i = 1, \cdots, n)$.

Further the contour L runs from $\sigma-i\infty$ to $\sigma+i\infty$ such that the points

$$\xi = (b_h+\nu)/\beta_h \quad (h = 1, \cdots, m;\ \nu = 0, 1, \cdots), \tag{1.3}$$

which are poles of $\Gamma(b_h-\beta_h\xi)$ $(h = 1, \cdots, m)$, lie to the right, and the points:

$$\xi = (a_i-\eta-1)/\alpha_i \quad (i = 1, \cdots, n;\ \eta = 0, 1, \cdots), \tag{1.4}$$

which are poles of $\Gamma\ (1-a_i+\alpha_i\xi)$ $(i = 1, \cdots, n)$, lie to the left of L.

Such a contour is possible on account of (1.2).

The conventional notation $\phi(s) \doteqdot h(x)$ will be used to denote the classical Laplace transform

$$\phi(s) = s\int_0^\infty e^{-sx}h(x)dx. \tag{1.5}$$

In what follows $\{(a_p, \alpha_p)\}$ stands for $(a_1, \alpha_1), \cdots, (a_p, \alpha_p)$.

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The following results [6; 5, p. 14; 5, p. 26] will be required:

(i)

$$(2.1)\quad \int_0^\infty x^{\eta-1} H_{p,q}^{m,n}\left[zx^\sigma \middle| \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] H_{\gamma,\delta}^{\alpha,\beta}\left[x \middle| \begin{matrix}\{(c_\gamma,d_\gamma)\}\\ \{(e_\delta,f_\delta)\}\end{matrix}\right] dx$$

$$= H_{p+\delta,q+\gamma}^{m+\beta,n+\alpha}\left[z \middle| \begin{matrix}\{(a_n,\alpha_n)\},\ \{(1-e_\delta-\eta f_\delta,\sigma f_\delta)\},\ (a_{n+1},\alpha_{n+1}),\cdots,(a_p,\alpha_p)\\ \{(b_m,\beta_m)\},\ \{(1-c_\gamma-\eta d_\gamma,\sigma d_\gamma)\},\ (b_{m+1},\beta_{m+1}),\cdots,(b_q,\beta_q)\end{matrix}\right]$$

where $\sigma > 0$, $\lambda > 0$, $\lambda' > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$,

$$R\left[\eta+\sigma\frac{b_h}{\beta_h}+\frac{e_i}{f_i}\right] > 0 \qquad (h = 1,\cdots,m;\ i = 1,\cdots,\alpha),$$

$$R\left[\eta+\sigma\frac{a_{h'}-1}{\alpha_{h'}}+\frac{c_j-1}{d_j}\right] < 0 \qquad (h = 1,\cdots,n;\ j = 1,\cdots,\beta)$$

and λ and λ' stands for the quantities

$$\sum_1^n(\alpha_j)-\sum_{n+1}^p(\alpha_j)+\sum_1^m(\beta_j)-\sum_{m+1}^q(\beta_j),$$

$$\sum_1^\beta(d_j)-\sum_{\beta+1}^\gamma(d_j)+\sum_1^\alpha(f_j)-\sum_{\alpha+1}^\delta(f_j),$$

respectively, throughout this paper.

(ii)

$$(2.2)\quad s^{1-l}H_{p+1,q}^{m,n+1}\left[zs^{-\sigma}\middle| \begin{matrix}(1-l,\sigma),\ \{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] \doteqdot x^{l-1}H_{p,q}^{m,n}\left[zx^\sigma\middle| \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right]$$

where $\sigma > 0$, $R(s) > 0$, $\lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$ and

$$R\left[l+\sigma\frac{b_h}{\beta_h}\right] > 0 \qquad (h = 1,\cdots,m).$$

(iii)

$$(2.3)\quad sH_{\gamma,\delta}^{\alpha,\beta}\left[(s+a)\middle| \begin{matrix}\{(c_\gamma,d_\gamma)\}\\ \{(e_\delta,f_\delta)\}\end{matrix}\right] \doteqdot \frac{1}{x}e^{-ax}H_{\gamma+1,\delta}^{\alpha,\beta}\left[\frac{1}{x}\middle| \begin{matrix}\{(c_\gamma,d_\gamma)\},\ (0,1)\\ \{(e_\delta,f_\delta)\}\end{matrix}\right]$$

where $R(s) > 0$, $R(a) > 0$, $\lambda' > 1$ and $R[(c_h-1)/d_h] < 0$ $(h = 1,\cdots,\beta)$.

3. The integral

The formula to be proved here is

(3.1)

$$\int_0^\infty x^{l-1} H_{p,q}^{m,n}\left[zx^\sigma \middle| \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] H_{\gamma,\delta}^{\alpha,\beta}\left[(x+a)\middle| \begin{matrix}\{(c_\gamma, d_\gamma)\}\\ \{(e_\delta, f_\delta)\}\end{matrix}\right] dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r a^r}{r!} H_{p+\delta+1, q+\gamma+1}^{m+\beta, n+\alpha+1}$$

$$\left[z \middle| \begin{matrix}(1-l, \sigma), \{(a_n, \alpha_n)\}, \{(1-e_\delta-(l-r)f_\delta, \sigma f_\delta)\}, (a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)\\ \{(b_m, \beta_m)\}, \{(1-c_\gamma-(l-r)d_\gamma, \sigma d_\gamma)\}, (1-l+r, \sigma), (b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)\end{matrix}\right]$$

where $\sigma > 0$, $R(a) > 0$, $\lambda > 0$, $\lambda' > 1$, $|\arg z| < \frac{1}{2}\lambda\pi$, $|\arg a| < \frac{1}{2}\lambda'\pi$,

$$R\left[l+\sigma \frac{a_i-1}{\alpha_i} + \frac{c_j-1}{d_j}\right] < 0 \qquad (i = 1, \cdots, n;\ j = 1, \cdots, \beta),$$

$$R\left[l+\sigma \frac{b_h}{\beta_h}\right] > 0 \qquad (h = 1, \cdots, m).$$

PROOF. If we use the operational pairs (2.2) and (2.3) in the Parseval-Goldstein theorem of operational Calculus [4, p. 105], we get after a little simplifications:

$$\int_0^\infty x^{l-1} H_{p,q}^{m,n}\left[zx^\sigma \middle| \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] H_{\gamma,\delta}^{\alpha,\beta}\left[(x+a)\middle| \begin{matrix}\{(c_\gamma, d_\gamma)\}\\ \{(e_\delta, f_\delta)\}\end{matrix}\right] dx$$

$$(3.2) \qquad = \int_0^\infty x^{-l-1} e^{-ax} H_{p+1,q}^{m,n+1}\left[zx^{-\sigma} \middle| \begin{matrix}(1-l, \sigma), \{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] \times$$

$$\times H_{\gamma+1,\delta}^{\alpha,\beta}\left[\frac{1}{x}\middle| \begin{matrix}\{(c_\gamma, d_\gamma)\}, (0, 1)\\ \{(e_\delta, f_\delta)\}\end{matrix}\right] dx$$

Now we expand e^{-ax} and then integrate the right hand side of (3.2) term by term to get (3.1).

The term by term integration is permissible [2, p. 500], since,

(i) $e^{-ax} = \sum_{r=0}^{\infty} ((-1)^r a^r x^r)/r!$ is uniformly convergent in any fixed interval $0 \leqq x \leqq b$

(ii) $x^{-l-1} H_{p+1,q}^{m,n+1}\left[zx^{-\sigma} \middle| \begin{matrix}(1-l, \sigma), \{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] H_{\gamma+1,\delta}^{\alpha,\beta}\left[\frac{1}{x}\middle| \begin{matrix}\{(c_\gamma, d_\gamma)\}, (0, 1)\\ \{(e_\delta, f_\delta)\}\end{matrix}\right]$

is continuous [1, p. 278] and

(iii) the integral on the right hand side of (3.2) is absolutely convergent under the conditions mentioned in (3.1).

For $\sigma = 1$, $\alpha_j = \beta_k = d_i = f_h = 1$ $(j = 1, \cdots, p;\ k = 1, \cdots, q;\ i = 1, \cdots, \gamma;\ h = 1, \cdots, \delta)$ (3.1) reduces to a result recently given by Saxena [7, p. 47].

Particular case. If we take $\alpha = 1, \beta = \gamma = 0, \delta = 2, e_1 = 0, f_1 = 1, e_2 = -\nu$ and $f_2 = \mu$ in (3.1), it reduces to the following integral involving Maitland's generalized Bessel function [8, p. 257], [1, p. 279]:

$$\int_0^\infty x^{l-1} H_{p,q}^{m,n}\left[zx^\sigma \middle| \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] J_\nu^\mu(x+a)\, dx$$
$$= \sum_{r=0}^\infty \frac{(-1)^r a^r}{r!} H_{p+2,q}^{m,n+1}\left[z \middle| \begin{matrix}(1-l, \sigma), \{(a_p, \alpha_p)\}, (1+\nu+r\mu-l\mu, \sigma\mu)\\ \{(b_q, \beta_q)\}\end{matrix}\right]$$

where $\sigma > 0$, $R(a) > 0$, $\lambda > 0$, $|\arg z| < \frac{1}{2}\lambda\pi$, $\mu < 1$, $|\arg a| < \frac{1}{2}(1-\mu)\pi$ and

$$R\left[l + \sigma \frac{b_h}{\beta_h}\right] > 0 \qquad (h = 1, \cdots, m).$$

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