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EXCEPTIONAL ZEROES OF *P*-ADIC *L*-FUNCTIONS OVER NON-ABELIAN FIELD EXTENSIONS

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Abstract. Suppose E is an elliptic curve over $\mathbb{Q}$, and $p > 3$ is a split multiplicative prime for E. Let $q \neq p$ be an auxiliary prime, and fix an integer m coprime to pq. We prove the generalised Mazur–Tate–Teitelbaum conjecture for E at the prime p, over number fields $K \subset \mathbb{Q}\big(\mu_{q^\infty}, \sqrt[q^\infty]{m}\big)$ such that p remains inert in $K \cap \mathbb{Q}(\mu_{q^\infty})^+$. The proof makes use of an improved p-adic L-function, which can be associated to the Rankin convolution of two Hilbert modular forms of unequal parallel weight.

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1. Introduction. Let E denote a modular elliptic curve defined over the rationals of conductor N_E. The behaviour of its Hasse–Weil L-function $L(E, s)$ at $s = 1$ is a fundamental topic in modern number theory. Thanks to the efforts of Birch and Swinnerton-Dyer, there are some deep conjectures describing both the order of vanishing at $s = 1$ for these L-functions, and also a detailed formula predicting their leading terms. Despite much strong progress over the last thirty years, the original conjectures themselves remain unproven (except for curves whose analytic rank is ≤ 1).

Assume p is a prime number. We fix once and for all embeddings $\tau_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\tau_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, which enable us to view L-values both p-adically and over $\mathbb{C}$. In an attempt to understand these questions from a non-Archimedean standpoint, Mazur et al. [**13, 21**] constructed p-adic avatars of the classical complex L-series. For almost all primes, the order of vanishing of the p-adic L seems to agree with that of its complex cousin. However, in 1986, Mazur, Tate and Teitelbaum [**14**] discovered if p is a prime of split multiplicative reduction, the p-adic avatar vanishes at $s = 1$ regardless of how the classical L-function behaves there. Based on extensive calculation, they conjectured a *derivative formula* at $s = 1$, involving a mysterious $\mathcal{L}$-invariant term defined via Iwasawa's logarithm (normalised so that $\log_p(p) = 0$).

Throughout we suppose E has split multiplicative reduction at a prime $p \neq 2$. As a local $G_{\mathbb{Q}_p}$-module, the elliptic curve admits the rigid-analytic parametrisation

$$E\big(\overline{\mathbb{Q}}_p\big) \;\cong\; \overline{\mathbb{Q}}_p^\times \big/ \mathbf{q}_{E,p}^{\mathbb{Z}} \qquad \text{where } \mathbf{q}_{E,p} \in \mathbb{Q}_p^\times \text{denotes the Tate period of } E.$$

From the above discussion, the p-adic L-function for E over $\mathbb{Q}$ has a trivial zero caused by the vanishing of the p-Euler factor $\big(1 - a_p(E)^{-1}\big)$ whenever $a_p(E) = +1$. Greenberg and Stevens [**7**, Theorem 1.3] managed to prove the derivative formula at the central point $s = 1$ through an ingenious application of Hida theory [**8–10**].

In 2009, Mok [**15**, Theorem 1.1] extended the method to include totally real fields F, provided E has split multiplicative reduction at only a single place of F above p. Recently, Spieß [**19**, Theorem 5.10] has further developed work on the totally real case, and can now remove Mok's restriction that the elliptic curve be split multiplicative at only a single place.

In this paper, we treat a complementary scenario to those considered in [**7, 15, 19**]. The main goal is to establish a higher derivative formula for the p-adic L-function over number fields which are *not* totally real, yet lie in a false Tate curve extension. One now fixes an auxiliary prime number $q \neq p$, and a q-power free integer $m > 1$.

Here, we will prove the exceptional zero conjecture over $L_n = \mathbb{Q}\big(\mu_{q^n}, \sqrt[q^n]{m}\big)$.

In particular, these are non-abelian field extensions of $\mathbb{Q}$, whose Galois group is isomorphic to $(\mathbb{Z}/q^n\mathbb{Z}) \rtimes (\mathbb{Z}/q^n\mathbb{Z})^\times$. Let us also set $K_n = \mathbb{Q}(\mu_{q^n})$ and $F_n = K_n \cap \mathbb{R}$. One imposes the following four hypotheses on p, q, m, n and E:

(1.1.1) E is semistable over the extension $F_1 = \mathbb{Q}(\mu_q)^+$;
(1.1.2) $E_{/F_1}$ has good reduction at the prime above q;
(1.1.3) the positive integer m is coprime to $q \cdot N_E$;
(1.1.4) the prime number p remains inert in $\mathbb{Z}[\mu_{q^n}] \cap \mathbb{R}$.

We will first attach a p-adic L-function to E over the number field $\mathbb{Q}\big(\mu_{q^n}, \sqrt[q^n]{m}\big)$ by using the factorisation of $L(E/L_n, s)$ into a product of its various Artin twists. For each finite order character $\phi : \mathbb{Z}_p^\times \longrightarrow \mathbb{C}^\times$, one considers a multiplier term

$$\mathfrak{M}_p(L_n, \phi) \;:=\; \prod_{\rho} \big(\varepsilon\text{-factor of } \rho \otimes \phi\big)^{\mathfrak{m}(\rho)}, \qquad \text{(see Definition 6.7)}$$

where the product ranges over all the irreducible representations ρ of $\mathrm{Gal}(L_n/\mathbb{Q})$, and $\mathfrak{m}(\rho)$ counts the total number of copies of ρ inside the regular representation. (If ϕ is trivial then $\mathfrak{M}_p(L_n, \mathbf{1})$ is just the square root of the discriminant.)

THEOREM 1.1. *There exists a bounded measure $\mathrm{d}\mu^{(p)}_{E/L_n}$ defined on $\mathbb{Z}_p^\times$, interpolating*

$$\int_{x\in\mathbb{Z}_p^\times} \phi(x)\cdot \mathrm{d}\mu^{(p)}_{E/L_n}(x) \;=\; \mathfrak{M}_p(L_n, \phi) \times \frac{L\big(E/L_n, \phi^{-1}, 1\big)}{\big(\Omega_E^+\Omega_E^-\big)^{[L_n:\mathbb{Q}]/2}}$$

at almost all finite order characters $\phi \neq \mathbf{1}$, whilst $\int_{x\in\mathbb{Z}_p^\times} \mathrm{d}\mu^{(p)}_{E/L_n}(x) = 0$ when $\phi = \mathbf{1}$.

Here, the transcendental numbers $\Omega_E^\pm$ denote real/imaginary Néron periods for E. The analytic p-adic L-function is constructed via the Mazur–Mellin transform

$$\mathbf{L}_p\big(E/L_n, s\big) \;:=\; \int_{x\in\mathbb{Z}_p^\times} \exp\big((s-1)\log x\big)\cdot \mathrm{d}\mu^{(p)}_{E/L_n}(x)$$

and clearly vanishes at the point $s = 1$ because $\mu^{(p)}_{E/L_n}(\mathbb{Z}_p^\times) = 0$. The purpose of this article, therefore, is to recover as much of this missing L-value as possible.

Let $\mathfrak{e}_p = \mathfrak{e}_p(L_n)$ denote the number of places for $\mathbb{Q}\big(\mu_{q^n}, \sqrt[q^n]{m}\big)$ lying above p. Since the prime p was assumed to be inert in the totally real subfield $F_n = K_n \cap \mathbb{R}$, the positive integer $\mathfrak{e}_p(L_n)$ must divide into the index $[L_n : F_n] = 2 \times q^n$; henceforth, we shall denote by q^{n_0} the largest power of q dividing $\mathfrak{e}_p(L_n)$.

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THEOREM 1.2. *Under these hypotheses,* $\mathrm{order}_{s=1}\big(\mathbf{L}_p(E/L_n, s)\big) \ \geq \ \mathfrak{e}_p(L_n)$.

If $\Sigma_{L_n/\mathbb{Q}}$ is the regular representation for $\mathrm{Gal}(L_n/\mathbb{Q})$, it will be shown later that $\prod_{t=0}^{n_0}\big(1 - X^{[F_t:\mathbb{Q}]}\big)^{[K_t:\mathbb{Q}]}$ divides the characteristic polynomial of $\Sigma_{L_n/\mathbb{Q}}\big(\mathrm{Frob}_p^{-1}\big)$. Thus, one can replace the characteristic polynomial by

$$\mathcal{E}_p\big(\Sigma_{L_n/\mathbb{Q}}, X\big) \ := \ \det\Big(1 - X \cdot \Sigma_{L_n/\mathbb{Q}}\big(\mathrm{Frob}_p^{-1}\big)\Big) \times \prod_{t=0}^{n_0} \big(1 - X^{[F_t:\mathbb{Q}]}\big)^{-[K_t:\mathbb{Q}]}$$

which again has rational integer coefficients, yet is always non-zero at $X = 1$.

THEOREM 1.3. *If $p \geq 5$ remains inert in K_n, there is an exceptional zero formula*

$$\frac{1}{\mathfrak{e}_p!} \cdot \frac{\mathrm{d}^{\mathfrak{e}_p}\mathbf{L}_p\big(E/L_n, s\big)}{\mathrm{d}s^{\mathfrak{e}_p}}\Bigg|_{s=1} = \mathcal{L}_p(E/L_n) \times \mathcal{E}_p\big(\Sigma_{L_n/\mathbb{Q}}, 1\big) \times \frac{\sqrt{\mathrm{disc}(L_n)} \cdot L\big(E/L_n, 1\big)}{\big(\Omega_E^+\Omega_E^-\big)^{[L_n:\mathbb{Q}]/2}}$$

where the $\mathcal{L}$-invariant equals $\prod_{t=1}^{n_0}[F_t : \mathbb{Q}]^{(q-1)q^{t-1}} \times \big(\frac{\log_p(\mathbf{q}_{E,p})}{\mathrm{ord}_p(\mathbf{q}_{E,p})}\big)^{\mathfrak{e}_p}$.

This definition of $\mathcal{L}_p(E/L_n)$ above is compatible with those proposed in [**12, 15**]. Furthermore, the quantity $\log_p(\mathbf{q}_{E,p}) \neq 0$ by the main result of [**1**] hence the $\mathcal{L}$-invariant term is non-vanishing. We therefore directly obtain the following:

COROLLARY 1.4. *Assuming $p \geq 5$ remains inert in K_n, one deduces that*
(a) *If $L\big(E/L_n, 1\big) = 0$ then* $\mathrm{order}_{s=1}\big(\mathbf{L}_p(E/L_n, s)\big) > \mathfrak{e}_p(L_n)$*;*
(b) *If $L\big(E/L_n, 1\big) \neq 0$, then* $\mathrm{order}_{s=1}\big(\mathbf{L}_p(E/L_n, s)\big) = \mathfrak{e}_p(L_n)$.

We conjecture analogues of Theorem 1.3 and Corollary 1.4 hold if p splits in K_n/F_n, but are unable to prove it with the deformation techniques presented in this paper. (The author hopes to return to the situation where p splits in future work.)

REMARKS.
(a) It is natural to ask if these formulae over $L_n = \mathbb{Q}\big(\mu_{q^n}, \sqrt[q^n]{m}\big)$ can be derived from the existing results of Mok and Spieß? The answer is negative as the largest real subfield inside L_n is precisely $F_n = \mathbb{Q}(\mu_{q^n})^+$, and the number field $F_n(\sqrt[q^n]{m})$ will never be totally real (nor is it even Galois over $\mathbb{Q}$).
(b) While throughout one has assumed that $q \neq p$, the situation where $q = p$ has been addressed by Lei and the author in [**4**]. The p-adic L-functions constructed in [**4**, Theorems 1 and 4] also exhibit exceptional zeroes, and moreover satisfy various p-power congruences predicted by a non-commutative Iwasawa Main Conjecture.
(c) A worthwhile project would be to extend Theorem 1.3 to establish a derivative formula in the $q = p$ situation, then verify the p-power congruences numerically.

These derivative formulae are built out of a more technical result, which holds for the Artin twists of an elliptic curve E that is semistable over a totally real field F in which the prime p is inert. Let K/F denote a CM extension of number fields, and define $\rho := \mathrm{Ind}_K^F(\Phi)$ for some non-cyclotomic character $\Phi_{/K}$ such that $\rho^* \cong \rho$. One writes $\varepsilon_F(\rho)$ to indicate the ϵ-factor associated to the Artin representation ρ, which itself can be decomposed into a product of local terms.

The work of Hida and Panchiskin [**8, 16, 17**] on Rankin convolutions allows the construction of a p-adic L-function, $\mathbf{L}_p(E, \rho, s)$, interpolating cyclotomic ϕ-twists of the complex L-series associated to the motive $h^1(E) \otimes \mathrm{Ind}_K^{\mathbb{Q}}(\Phi)$.

THEOREM 1.5. *Assuming the p-adic L-function has an exceptional zero at $s=1$,*

$$\left.\frac{\mathrm{d}\mathbf{L}_p(E,\rho,s)}{\mathrm{d}s}\right|_{s=1} = -2\cdot\varepsilon_F(\rho)\cdot\left.\frac{\mathrm{d}\alpha_{\mathfrak{p}}(k)}{\mathrm{d}k}\right|_{k=2} \times \big(1-\beta_{\mathfrak{p}}(\rho)\big)\cdot\frac{L(E,\rho,1)}{\left(\Omega_E^+\Omega_E^-\right)^{[F:\mathbb{Q}]}},$$

where the ideal $\mathfrak{p}=p\cdot\mathcal{O}_F$, the local L-factor at $\mathfrak{p}$ is $L_{\mathfrak{p}}(\rho,X)=(1-X)\big(1-\beta_{\mathfrak{p}}(\rho)X\big)$, and $\alpha_{\mathfrak{p}}(k)$ denotes the $\mathfrak{p}$th eigenvalue (at weight k) of the Hida family lifting $E_{/F}$.

This formula gives a ρ-twisted generalisation of the results in **[7,15,19]** over real number fields (a more detailed version is stated as Theorem 6.2 later in the article).

EXAMPLE 1.6. We now study in detail the modular elliptic curve

$$E=525B1:\ y^2+xy = x^3+x^2+25x$$

with split multiplicative reduction at 7, non-split multiplicative reduction at 3, and potential good reduction at 5. Let $E^\dagger = E\otimes(\frac{-}{5})$ denote the twist of E by the quadratic character of conductor 5, which has a minimal Weierstrass equation

$$E^\dagger=21A4:\ y^2+xy = x^3+x.$$

Put $q=5$, $m=2$, $n=1$ and consider the non-abelian extension $L_1=\mathbb{Q}\big(\mu_5,\sqrt[5]{2}\big)$. The prime $p=7$ remains inert in $K=\mathbb{Q}(\mu_5)$, and over the real subfield $F=\mathbb{Q}(\mu_5)^+$ the twisted curve $E^\dagger$ also has split multiplicative reduction at the prime above 7. The Hasse–Weil L-function for E over L_1 decomposes into a product

$$L\big(E/L_1,s\big) = L(E,s)\times L(E^\dagger,s)\times L(E\otimes\theta_5,s)\times L(E^\dagger\otimes\theta_5,s)\times\Big(L\big(E,\rho,s\big)\Big)^4$$

where θ_5 denotes the Teichmüller character modulo 5, and ρ indicates the unique irreducible Artin representation of degree 4 factoring through $L_1/\mathbb{Q}$.

Since $E\cong E^\dagger$ over all three number fields F,K,L_1 clearly E satisfies each of the Hypotheses (1.1.1)–(1.1.4) mentioned at the beginning of the Introduction. Moreover, the ideal $7\cdot\mathcal{O}_K$ splits completely inside L_1/K, so applying Theorem 1.3

$$\frac{1}{5!}\cdot\left.\frac{\mathrm{d}^5\mathbf{L}_7\big(E/L_1,s\big)}{\mathrm{d}s^5}\right|_{s=1} = \mathcal{L}_7(E/L_1)\times\mathcal{E}_7\big(\Sigma_{L_1/\mathbb{Q}},1\big)\times\frac{\sqrt{\mathrm{disc}(L_1)}\cdot L\big(E/L_1,1\big)}{\left(\Omega_E^+\Omega_E^-\right)^{10}},$$

where $\mathrm{disc}(L_1)=2^{16}\cdot5^{23}$ and $\mathcal{L}_7(E)=2^4\left(\frac{\log_7(\mathbf{q}_{E,7})}{\mathrm{ord}_7(\mathbf{q}_{E,7})}\right)^5 = 16\big(4\times7+4\times7^2+\dots\big)^5$. Using the MAGMA package, the *LSeries* function determines numerically that

$$L(E/K,1) = 2.12709564136\dots \quad\text{and}\quad L(E,\rho,1) = 1.70167651313\dots$$

in which case $L\big(E/L_1,1\big)\neq0$. By Corollary 1.4(b), the $\mathrm{order}_{s=1}\mathbf{L}_7\big(E/L_1,s\big)=5$.

Here is a brief plan of the paper. We begin by recalling the theory of p-adic families of Hilbert modular forms, and their behaviour at classical points in weight space. In Section 3, we construct a Λ-adic HMF interpolating the product of a weight $(l,\dots,l)$ cusp form $\mathbf{g}_l$ with an Eisenstein series of variable weight. After working out its $\mathbf{f}_k$-isotypic projection, one can associate an improved p-adic L-function in Section 4 which is the natural generalisation of [**7**, Proposition 5.8] to Rankin convolution L-functions (the existence of such functions over general number fields was raised in [**3**, 4.17]).

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Finally, in Sections 5 and 6, we extend this construction to a two-variable deformation ring. The exceptional zero formula follows from a functional equation for the measure interpolating $L(\mathbf{f}_k, \mathbf{g}_1, s)$ at pairs (k, s), where $k \in \mathbb{Z}_{\geq 2}$ and $s \in \{1, \ldots, k-1\}$.

2. Preliminaries on Hilbert modular forms. We start by collecting together the various definitions needed from the theory of Hilbert modular forms (HMFs for short). Our approach requires us to construct measures on $\mathbb{Z}_p^\times \times \mathbb{Z}_p$, whose values lie inside deformations of these vector spaces. The main sources of reference are [**16–18**] and for the p-adic theory [**9, 10, 22**].

2.1. Modular forms over totally real fields. Let F be a totally real field of degree $n = [F : \mathbb{Q}]$, and write $\mathfrak{d}$ for it's different. One may then interpret $\mathrm{GL}_2(F)$ as a group $\mathfrak{G}_{\mathbb{Q}}$ of rational points for an associated $\mathbb{Q}$-subgroup inside $\mathrm{GL}_{2n}(\mathbb{Q})$. Its adèlisation $\mathfrak{G}_{\mathbb{A}}$ corresponds to the product

$$\mathrm{GL}_2(\mathbb{A}_F) \;=\; \mathrm{GL}_2(\mathbb{R})^n \times \mathrm{GL}_2(\hat{F}) \quad \text{where } \hat{F} := F \otimes \big(\varprojlim_m \mathbb{Z}/m\mathbb{Z}\big).$$

The subgroup $\mathrm{GL}_2^+(\mathbb{R})^n$ comprising vectors $\underline{v} = (v_1, \ldots, v_n)$ with $v_j = \left(\begin{smallmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{smallmatrix}\right)$ and $\alpha_j\delta_j > \beta_j\gamma_j$ for all $j \leq n$, acts naturally on n-copies of the upper half-plane $\mathfrak{H}$. If $\mathbf{i} = (i, \ldots, i)$, there is an isomorphism $\{\underline{v} \in \mathrm{GL}_2^+(\mathbb{R})^n \mid \underline{v}\cdot\mathbf{i} = \mathbf{i}\}/\mathbb{R}_+^n \cong \mathrm{SO}(2)^n$ and this quotient is maximally compact within $\mathrm{GL}_2(\mathbb{R})^n/\mathbb{R}_+^n$.

REMARKS.

(a) For any element $\underline{v} \in \mathrm{GL}_2^+(\mathbb{R})^n$ and function $f : \mathfrak{H}^n \longrightarrow \mathbb{C}$,

$$\big(f\big|_k \underline{v}\big)(z) \;:=\; \mathcal{N}\big(\gamma_j z_j + \delta_j\big)^{-k} \times f(\underline{v}\cdot z)\cdot \mathcal{N}\big(\det(\underline{v})\big)^{k/2} \quad \text{at integers } k > 0$$

where the norm of an n-tuple $z = (z_1, \ldots, z_n)$ is given by $\mathcal{N}(z) = z_1 \times \cdots \times z_n$.

(b) Let $\mathfrak{c}$ be an ideal of $\mathcal{O}_F$; one has localisations $\mathfrak{c}_\mathfrak{p} = \mathfrak{c}\cdot\mathcal{O}_{F,\mathfrak{p}}$ and $\mathfrak{d}_\mathfrak{p} = \mathfrak{d}\cdot\mathcal{O}_{F,\mathfrak{p}}$. We define open subgroups $W_\mathfrak{c} \subset \mathfrak{G}_{\mathbb{A}}$ by the product $W_\mathfrak{c} := \mathrm{GL}_2^+(\mathbb{R})^n \times \prod_\mathfrak{p} W(\mathfrak{p})$, with each local factor consisting of matrices

$$W(\mathfrak{p}) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(F_\mathfrak{p}) \;\middle|\; b \in \mathfrak{d}_\mathfrak{p}^{-1},\ c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p},\ \ a, d \in \mathcal{O}_{F,\mathfrak{p}}, ad - bc \in \mathcal{O}_{F,\mathfrak{p}}^\times\right\}.$$

(c) If $\hat{h}_F = \#\mathrm{Cl}^{\mathrm{nw}}(\mathcal{O}_F)$ denotes the narrow class number of F, one can always choose ideles $t_1, \ldots, t_{\hat{h}_F} \in \mathbb{A}_F^\times$ so that their associated $\mathcal{O}_F$-ideals $\widetilde{t}_1, \ldots, \widetilde{t}_{\hat{h}_F}$ form a complete set of representatives for $\mathrm{Cl}^{\mathrm{nw}}(\mathcal{O}_F)$. By the approximation theorem

$$\mathfrak{G}_{\mathbb{A}} \;=\; \bigcup_\lambda \mathfrak{G}_{\mathbb{Q}}\cdot x_\lambda \cdot W_\mathfrak{c} \;=\; \bigcup_\lambda \mathfrak{G}_{\mathbb{Q}}\cdot \big(x_\lambda^{-1}\big)^\iota \cdot W_\mathfrak{c},$$

where the elements $x_\lambda = \left(\begin{smallmatrix}1 & 0\\ 0 & t_\lambda\end{smallmatrix}\right)$, and the involution $\iota : \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix}d & -b\\ -c & a\end{smallmatrix}\right)$.

DEFINITION 2.1. Fix a weight $k > 0$, an ideal $\mathfrak{c}$, and a Hecke character ψ mod $\mathfrak{c}$. Then, a Hilbert automorphic form $\mathbf{f} : \mathfrak{G}_{\mathbb{A}} \longrightarrow \mathbb{C}$ of parallel weight $(k, \ldots, k)$, level $\mathfrak{c}$ and character ψ satisfies:

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(i) $\mathbf{f}(sgx) = \psi(s) \cdot \mathbf{f}(x)$ for all $x \in \mathfrak{G}_{\mathbb{A}}$, $s \in \mathbb{A}_F^\times$ and $g \in \mathfrak{G}_{\mathbb{Q}}$;

(ii) $\mathbf{f}(xw) = \psi(w^\iota) \cdot \mathbf{f}(x)$ for every $w \in W_{\mathfrak{c}}$ with $w_\infty = 1$;

(iii) $\mathbf{f}\big(xr(\theta)\big) = \mathbf{f}(x) \cdot \exp\big(ik\{\theta\}\big)$ where $r(\theta) = (\ldots, \left(\begin{smallmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{smallmatrix}\right), \ldots)$.

Definition 2.2. An automorphic form $\mathbf{f} : \mathfrak{G}_{\mathbb{A}} \longrightarrow \mathbb{C}$ is cuspidal provided

$$\int_{\mathbb{A}_F/F} \mathbf{f}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot g\right) \cdot \mathrm{d}t \;=\; 0 \quad \text{at each element } g \in \mathfrak{G}_{\mathbb{A}}.$$

If $\mathbf{f}$ satisfies the condition that for any $x \in \mathfrak{G}_{\mathbb{A}}$ with Archimedean component $x_\infty = 1$, there exists $h_x : \mathfrak{H}^n \longrightarrow \mathbb{C}$ such that $\mathbf{f}(xy) = \big(h_x\big|_k \underline{v}\big)(\mathbf{i})$ for all vectors $\underline{v} \in \mathrm{GL}_2^+(\mathbb{R})^n$ with each h_x holomorphic at the cusps, then $\mathbf{f}$ is called a Hilbert modular form of holomorphic type.

Remark. One writes $\mathcal{M}_k(\mathfrak{c}, \psi)$ to denote the space of HMFs of holomorphic type, and similarly the notation $\mathcal{S}_k(\mathfrak{c}, \psi)$ indicates the vector subspace of cusp forms. Setting $\mathbf{f}_\lambda = h_{(x_\lambda^{-1})^\iota}$, then $\mathbf{f}_\lambda(z) \in \mathcal{M}_k(\Gamma_\lambda(\mathfrak{c}), \psi)$, where at each $\lambda \in \{1, \ldots, \hat{h}_F\}$ one defines $\Gamma_\lambda(\mathfrak{c}) := x_\lambda \cdot W(\mathfrak{c}) \cdot (x_\lambda^{-1})^\iota \cap \mathfrak{G}_{\mathbb{Q}}$ to be the congruence modular subgroup of level $\mathfrak{c}$ (lying inside the $\mathbb{Q}$-group $\mathfrak{G}_{\mathbb{Q}}^+$ consisting of totally positive matrices).

The map $\mathbf{f} \mapsto \big(\mathbf{f}_1, \ldots, \mathbf{f}_{\hat{h}_F}\big)$ yields an isomorphism $\mathcal{M}_k(\mathfrak{c}, \psi) \cong \bigoplus_\lambda \mathcal{M}_k\big(\Gamma_\lambda(\mathfrak{c}), \psi\big)$. Consequently for all $\gamma \in \Gamma_\lambda(\mathfrak{c})$, one has $\mathbf{f}_\lambda\big|_k \gamma \;=\; \psi(\gamma)\mathbf{f}_\lambda$, and moreover

$$\mathbf{f}_\lambda(z) \;=\; \sum_\xi a_\lambda(\xi)\mathbf{e}_F(\xi z) \quad \text{with } \mathbf{e}_F(z) := \exp\big(2\pi i \mathrm{Tr}(z)\big),$$

the sum being taken over the totally positive elements $\xi \gg 0$ and also $\xi = 0$. Normalising the standard additive character $\chi_F : \mathbb{A}_F \to \mathbb{C}^\times$ by $\chi_F(x_\infty) = \mathbf{e}_F(x_\infty)$, then the adèlic expansion

$$\mathbf{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \;=\; |y|_{\mathbb{A}_F}^{k/2} \times \sum_{\xi \gg 0} C\big(\xi y \mathcal{O}_F, \mathbf{f}\big)\chi_F\big(\xi x\big)\mathbf{e}_F\big(\xi \mathbf{i} y_\infty\big) \;+\; |y|_{\mathbb{A}_F}^{k/2} C_0\big(y\mathcal{O}_F, \mathbf{f}\big)$$

has Fourier coefficients

$$C(\mathfrak{n}, \mathbf{f}) \;=\; \begin{cases} a_\lambda(\xi) \cdot \mathcal{N}_{F/\mathbb{Q}}(\widetilde{t}_\lambda)^{-k/2} & \text{if } \mathfrak{n} = \xi \widetilde{t}_\lambda^{-1} \text{ is integral,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ then the constant term $C_0\big(y\mathcal{O}_F, \mathbf{f}\big)$ is identically zero.

Fix a Hilbert modular form $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$, and also an auxiliary $\mathcal{O}_F$-ideal $\mathfrak{q}$. Throughout, ψ^* will denote the ideal character associated to ψ (although for reasons of space, we shall sometimes drop the * superscript, provided the context is clear).

Definition 2.3.

(i) The Hecke operator $U_{\mathfrak{q}} \in \mathrm{End}_{\mathbb{C}}\big(\mathcal{M}_k(\mathfrak{c}, \psi)\big)$ is defined to be

$$\mathbf{f}(x)\Big| U_{\mathfrak{q}} \;=\; \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})^{k/2-1} \sum_{v \in \mathcal{O}_F/\mathfrak{q}} \mathbf{f}\left(x \begin{pmatrix} 1 & v \\ 0 & q \end{pmatrix}\right) \quad \text{with } q \in \mathbb{A}_F^\times \text{ satisfying } \widetilde{q} = \mathfrak{q}.$$

(ii) The degeneracy map $V_{\mathfrak{q}} : \mathcal{M}_k(\mathfrak{c},\psi) \to \mathcal{M}_k(\mathfrak{cq},\psi)$ is given by

$$\mathbf{f}(x)\Big|V_{\mathfrak{q}} \;=\; \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})^{-k/2} \times \mathbf{f}\left(x\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}\right) \quad \text{where once again } \widetilde{q} = \mathfrak{q}.$$

The action of $U_{\mathfrak{q}}$ and $V_{\mathfrak{q}}$ on the Fourier coefficients of $\mathbf{f}$ is explicitly described by the twin formulae $C(\mathfrak{n},\mathbf{f}|U_{\mathfrak{q}}) = C(\mathfrak{qn},\mathbf{f})$ and $C(\mathfrak{n},\mathbf{f}|V_{\mathfrak{q}}) = C(\mathfrak{q}^{-1}\mathfrak{n},\mathbf{f})$, respectively. More generally, one has Hecke operators $T'_{\mathfrak{c}}(\mathfrak{q}) \in \mathrm{End}_{\mathbb{C}}(\mathcal{M}_k(\mathfrak{c},\psi))$ which act on each of the coefficients through $C(\mathfrak{n},\mathbf{f}|T'_{\mathfrak{c}}(\mathfrak{q})) = \sum_{\mathfrak{q}+\mathfrak{n}\subset\mathfrak{b}} \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{b})^{k-1}\cdot\psi^*(\mathfrak{b})\cdot C(\mathfrak{b}^{-2}\mathfrak{qn},\mathbf{f})$.

We adopt the convention that '$\mathfrak{n}(\mathbf{f})$' always refers to the exact conductor of $\mathbf{f}$. Assuming that $\mathfrak{c} \subset \mathfrak{n}(\mathbf{f})$, Shimura's J-operator [**18**, Section 2] is obtained via the rule

$$\left(\mathbf{f}|J_{\mathfrak{c}}\right)(x) \;=\; \psi\left(\det(x)^{-1}\right)\cdot\mathbf{f}(xb_0) \quad \text{such that } b_0 = \left(\begin{smallmatrix} 0 & 1 \\ c_0 & 0\end{smallmatrix}\right) \in \mathfrak{G}_{\widehat{\mathbb{Q}}} \text{ with } \widetilde{c}_0 = \mathfrak{c}\mathfrak{d}^2.$$

It has the property that $\mathbf{f}|J_{\mathfrak{c}} \in \mathcal{M}_k(\mathfrak{c},\psi^{-1})$. Furthermore, if $\mathbf{f}$ is a primitive Hecke eigenform of level $\mathfrak{c} = \mathfrak{n}(\mathbf{f})$, then $\mathbf{f}|J_{\mathfrak{n}(\mathbf{f})} = \varpi(\mathbf{f}) \times \mathbf{f}^{\#}$ where $\mathbf{f}^{\#}$ denotes the eigenform with conjugate coefficients, and the pseudo-eigenvalue $\varpi(\mathbf{f})$ has absolute value one.

EXAMPLE 2.4. Let η be a finite order character over F of conductor equal to $\mathfrak{m}_\eta$. Then, its associated Gauss sum is given by

$$\tau_F(\eta) \;=\; \sum_{x\in\mathfrak{m}_\eta^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \mathrm{sign}\left(\eta(x_\infty)\right)\cdot\eta^*\left(x\mathfrak{m}_\eta\mathfrak{d}\right)\cdot\mathbf{e}_F(x).$$

The twist of $\mathbf{f}$ by η is the unique Hilbert modular form $\mathbf{f}\otimes\eta \in \mathcal{M}_k(\mathfrak{cm}_\eta^2,\psi\eta^2)$ satisfying $C(\mathfrak{n},\mathbf{f}\otimes\eta) = \eta^*(\mathfrak{n})\cdot C(\mathfrak{n},\mathbf{f})$ at all ideals $\mathfrak{n}$ with $\mathfrak{n}+\mathfrak{c} = \mathcal{O}_F$.

Moreover, if $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c},\psi)$ is a primitive cusp form of conductor $\mathfrak{c}$ coprime to $\mathfrak{m}_\eta$ then $\mathbf{f}\otimes\eta$ will also be a primitive HMF of conductor $\mathfrak{cm}_\eta^2$, in which case

$$\mathbf{f}\otimes\eta\Big|J_{\mathfrak{cm}_\eta^2} \;=\; \varpi(\mathbf{f}\otimes\eta)\times\left(\mathbf{f}^{\#}\otimes\eta^{-1}\right) \;\in\; \mathcal{S}_k\left(\mathfrak{cm}_\eta^2,\psi^{-1}\eta^{-2}\right)$$

with twisted pseudo-eigenvalue $\varpi\left(\mathbf{f}\otimes\eta\right) = \psi^*(\mathfrak{m}_\eta)\,\eta^*(\mathfrak{c})\cdot\tau_F(\eta)^2\,\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{m}_\eta)^{-1}\times\varpi(\mathbf{f})$.

2.2. Klingen–Eisenstein series and their holomorphic projection. We shall now relax the condition that these Hilbert modular forms be holomorphic. One writes $\mathcal{M}_k^\infty(\mathfrak{c},\psi)$ (resp. $\mathcal{S}_k^\infty(\mathfrak{c},\psi)$) to denote the space of $\mathcal{C}^\infty$-modular forms (resp. the subspace of cusp forms) of weight $(k,\ldots,k)$, level $\mathfrak{c}$ and character ψ.

A plentiful supply of real-analytic modular forms is given by Eisenstein series. Fix a positive integer m, and also two fractional ideals $\mathfrak{a},\mathfrak{b}$. Let η denote any Hecke character modulo $\mathfrak{e}$ satisfying $\eta^*(x\cdot\mathcal{O}_F) = \mathrm{sign}\mathcal{N}(x)^m$ for all $x \equiv 1 \bmod^\times \mathfrak{e}$. Providing that $\mathrm{Re}(s) > 2-m$, one then defines

$$\begin{aligned}\mathbb{K}_m^q\left(z,s\,;\mathfrak{a},\mathfrak{b}\,;\eta\right) \;:=&\; (2\pi i)^{-\{q\}}\cdot(z-\overline{z})^{-q}\\ &\times\sum_{c,d}\mathrm{sign}\mathcal{N}(d)^m\eta^*(d\mathfrak{b}^{-1})\left(\frac{c\overline{z}+d}{cz+d}\right)^q\mathcal{N}(cz+d)^{-m}\Big|\mathcal{N}(cz+d)\Big|_\infty^{-2s},\end{aligned}$$

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where $q=(q_1,\ldots,q_n)\in\mathbb{Z}^n$, $\{q\}=q_1+\ldots q_n$, and the sum is over representatives $(c,d)\in\mathfrak{a}\times\mathfrak{b}$ under the equivalence $(c,d)\sim(uc,ud)$ with $u\in\mathcal{O}_F^\times$. In a similar way,

$$\mathbb{L}_m^q\big(z,s\,;\mathfrak{a},\mathfrak{b}\,;\eta\big):=(2\pi i)^{-\{q\}}\cdot(z-\overline{z})^{-q}$$
$$\times\sum_{c,d}\mathrm{sign}\mathcal{N}(c)^m\eta^*(c\mathfrak{a}^{-1})\left(\frac{c\overline{z}+d}{cz+d}\right)^q\mathcal{N}(cz+d)^{-m}\Big|\mathcal{N}(cz+d)\Big|_\infty^{-2s}.$$

Both $\mathbb{K}_m^q$ and $\mathbb{L}_m^q$ extend naturally to yield $\mathcal{C}^\infty$-functions on the adelisation of GL_2, first by putting $\mathbb{K}_m^q\big(s\,;\mathfrak{a},\mathfrak{b}\,;\eta\big)_\lambda(z)=\mathcal{N}(\widetilde{t}_\lambda)^{s+m/2}\cdot\mathcal{N}(y)^s\times\mathbb{K}_m^q\big(z,s\,;\widetilde{t}_\lambda\mathfrak{d}\mathfrak{a},\mathfrak{b}\,;\eta\big)$ and secondly $\mathbb{L}_m^q\big(s\,;\mathfrak{a},\mathfrak{b}\,;\eta\big)_\lambda(z)=\mathcal{N}(\widetilde{t}_\lambda)^{-s-m/2}\cdot\mathcal{N}(y)^s\times\mathbb{L}_m^q\big(z,s\,;\mathfrak{a},\widetilde{t}_\lambda^{-1}\mathfrak{d}^{-1}\mathfrak{b}\,;\eta\big)$.

LEMMA 2.5 ([**18**, p 672]). *If* $\Delta_m^q(z,s)=\pi^{-ns}y^{m+s}\prod_{\nu=1}^n\Gamma(s+m+q_\nu)$*, then*

$$\Delta_m^q(z,1-m-s)\cdot\mathbb{K}_m^q\big(1-m-s;\mathfrak{a},\mathfrak{b};\eta\big)$$
$$=\tau_F(\eta)\cdot\mathcal{N}\big(\mathfrak{d}\mathfrak{a}\mathfrak{b}\mathfrak{e}\big)^{m+2s-1}\times\Delta_m^q(z,s)\cdot\mathbb{L}_m^q\big(s;\mathfrak{a},\mathfrak{b}\mathfrak{e};\eta^{-1}\big).$$

As a corollary, the definition of the J-operator yields the functional equation

$$\mathbb{K}_m^q\big(s;\mathfrak{a},\mathfrak{b};\eta\big)\Big|J_\mathfrak{q}\quad=\quad(-1)^{m[F:\mathbb{Q}]}\cdot\mathcal{N}(\mathfrak{q}\mathfrak{d}^2)^{-s-m/2}\times\mathbb{L}_m^q\big(s;\mathfrak{b},\mathfrak{a}\mathfrak{q}^{-1};\eta\big).$$

Write $D_F=\mathcal{N}(\mathfrak{d})$ for the discriminant of the real field F, and suppose that $\mathfrak{e}\neq\mathcal{O}_F$.

PROPOSITION 2.6 ([**17**, Proposition 4.2]). *Assuming $s\in\mathbb{Z}$ satisfies $s\le q_\nu$ at every ν,*

$$\mathbb{L}_m^q\big(z,0;\ \mathcal{O}_F,\widetilde{t}_\lambda^{-1}\mathfrak{d}^{-1};\ \eta\big)$$
$$=\frac{(-2\pi i)^{[F:\mathbb{Q}](m+2s)}(-1)^{[F:\mathbb{Q}]s+\{q\}}}{\sqrt{D_F}\cdot\mathcal{N}(\widetilde{t}_\lambda)\prod_\nu\Gamma(s+m+q_\nu)}\times(4\pi y)^{-q}\sum_{0\ll\xi\in\widetilde{t}_\lambda}\mathcal{A}_\lambda(\xi,s,y,\eta)\,\mathbf{e}_F(\xi z),$$

where each real-analytic Fourier coefficient $\mathcal{A}_\lambda(\xi,s,y,\eta)$ equals

$$\sum_{\substack{\widetilde{\xi}=\widetilde{b}\times\widetilde{c},\\ b\in\widetilde{t}_\lambda,\ c\in\mathcal{O}_F}}\mathrm{sign}\mathcal{N}(\widetilde{b})^{m-1}\cdot\mathcal{N}(\widetilde{b})^{m+2s-1}\cdot\eta^*(\widetilde{c})\times\prod_{\nu=1}^{[F:\mathbb{Q}]}W\big(4\pi\xi_\nu y_\nu,m+s+q_\nu,s-q_\nu\big)$$

and the Whittaker function $W(y,\alpha,\beta):=\int_0^\infty(u+1)^{\alpha-1}u^{\beta-1}e^{-yu}\mathrm{d}u.$

The presence of this unpleasant quotient in the Fourier expansion above suggests some renormalisation is required. Let $\eta\neq\mathbf{1}$ denote a Hecke character modulo $\mathfrak{e}$. The Eisenstein series we will examine in detail is precisely

$$\mathbf{E}_m(s,\eta;\mathfrak{e})\quad:=\quad\frac{2^{-[F:\mathbb{Q}]}\cdot\sqrt{D_F}\cdot\Gamma(m+s)^{[F:\mathbb{Q}]}}{(-4\pi)^{-s[F:\mathbb{Q}]}(-2\pi i)^{[F:\mathbb{Q}](m+2s)}}\times\mathbb{L}_m^0\big(s;\ \mathcal{O}_F,\mathcal{O}_F;\ \eta\big).$$

NOTATION. *In the special case where $\mathfrak{p}$ is a prime of $\mathcal{O}_F$ and η' denotes the character associated to η modulo $\mathfrak{p}^n\mathfrak{e}$, one writes $\widetilde{\mathbf{E}}_m(s,\eta;\mathfrak{e}\mathfrak{p}^n)$ in the place of $\mathbf{E}_m(s,\eta';\mathfrak{e}\mathfrak{p}^n)$. In particular, the Fourier development of $\widetilde{\mathbf{E}}_m$ arising from Proposition 2.6 exhibits rather nice p-adic interpolation properties over weight-space, whilst $\mathbf{E}_m$ does not.*

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In the confines of this paper, a good reason to study $\mathbf{E}_m(s,\eta;\mathfrak{e})\in\mathcal{M}_k^{\infty}(\mathfrak{e},\eta)$ is that it naturally appears in an integral expression for the Rankin–Selberg L-function. Assume that $\mathbf{f}\in\mathcal{M}_k(\mathfrak{c},\psi)$ and $\mathbf{g}\in\mathcal{M}_l(\mathfrak{c},\theta)$ are two holomorphic modular forms. Their convolution L-function is defined by the summation

$$L(s,\mathbf{f},\mathbf{g}) \;=\; \sum_{\mathfrak{m}} C(\mathfrak{m},\mathbf{f})\,C(\mathfrak{m},\mathbf{g})\cdot\mathcal{N}(\mathfrak{m})^{-s}\qquad\text{for }\mathrm{Re}(s)\gg 0,$$

with a meromorphic continuation to the whole of $\mathbb{C}$.

DEFINITION 2.7. The completed Rankin–Selberg L-function is equal to

$$\mathfrak{D}^{(\mathfrak{c})}(s,\mathbf{f},\mathbf{g}) \;:=\; (2\pi)^{-2[F:\mathbb{Q}]s}\Gamma(s)^{[F:\mathbb{Q}]}\Gamma(s+1-l)^{[F:\mathbb{Q}]}\cdot\zeta_F^{(\mathfrak{c})}(2s+2-k-l,\psi\theta)\\ \times L(s,\mathbf{f},\mathbf{g}),$$

with $\zeta_F^{(\mathfrak{c})}(s,\psi\theta)=\sum_{\mathfrak{m}+\mathfrak{c}=\mathcal{O}_F}(\psi\theta)^*(\mathfrak{m})\cdot\mathcal{N}(\mathfrak{m})^{-s}$ denoting the twisted zeta-function. Whenever the ideal $\mathfrak{c}$ is the least common multiple of the conductors of $\mathbf{f}$ and $\mathbf{g}$, then we just drop the superscript $^{(\mathfrak{c})}$ from the notation altogether.

REMARKS.

(a) The Petersson inner product of $\mathbf{F}\in\mathcal{S}_k^{\infty}(\mathfrak{c},\psi)$ and $\mathbf{G}\in\mathcal{M}_k^{\infty}(\mathfrak{c},\psi)$ is given by the complex integral

$$\big\langle\mathbf{F},\mathbf{G}\big\rangle_{\mathfrak{c}} \;=\; \sum_{\lambda=1}^{\hat{h}_F}\int_{\Gamma_\lambda(\mathfrak{c})\backslash\mathfrak{H}^{[F:\mathbb{Q}]}}\overline{\mathbf{F}_\lambda(z)}\cdot\mathbf{G}_\lambda(z)\cdot\mathcal{N}(y)^k\,\mathrm{d}\mu_\infty(z)$$

upon selecting the hyperbolic metric $\mathrm{d}\mu_\infty(z)=\prod_{\nu=1}^{[F:\mathbb{Q}]}y_\nu^{-2}\cdot\mathrm{d}x_\nu\mathrm{d}y_\nu$.

(b) Provided $k>l$, one has the following integral representation

$$\mathfrak{D}^{(\mathfrak{c})}(s,\mathbf{f},\mathbf{g})=\sqrt{D_F}\,\left(\frac{\Gamma(s+1-l)}{\pi^s}\right)^{[F:\mathbb{Q}]}\\ \times\Big\langle\mathbf{f}^{\#},\;\mathbf{g}\cdot\mathbb{K}^0_{k-l}\big(s-k+1;\;\mathfrak{c},\mathcal{O}_F;\;\psi\theta^{-1}\big)\Big\rangle_{\mathfrak{c}}$$

which was established by Shimura in [**18**, equation (4.32)].

(c) In general, the product $\mathbf{g}\cdot\mathbb{K}^0_{k-l}\big(s-k+1;\;\mathfrak{c},\mathcal{O}_F;\;\psi\theta^{-1}\big)$ is only real-analytic, with moderate growth as a $\mathcal{C}^\infty$-modular form. It follows that the inner product above must then be equal to $\langle\mathbf{f}^{\#},\;\mathrm{Hol}(\mathbf{g}\cdot\mathbb{K}^0_{k-l}(s-k+1;\;\mathfrak{c},\mathcal{O}_F;\;\psi\theta^{-1}))\rangle_{\mathfrak{c}}$.

(d) Here, the holomorphic projection operator sends $\mathbf{G}=\big(\mathbf{G}_1,\ldots,\mathbf{G}_{\hat{h}_F}\big)\in\mathcal{M}_k^{\infty}(\mathfrak{c},\psi)$ with $\mathbf{G}_\lambda=\sum_\xi a_\lambda(\xi,y)\,\mathbf{e}_F(\xi x)$, to a holomorphic modular form $\mathrm{Hol}(\mathbf{G})\in\mathcal{M}_k(\mathfrak{c},\psi)$ under the condition that $\mathrm{Hol}(\mathbf{G})_\lambda=\sum_{0\ll\xi\in\widetilde{t}_\lambda}a_\lambda(\xi)\,\mathbf{e}_F(\xi z)$ where

$$a_\lambda(\xi)\;:=\;\frac{(4\pi)^{[F:\mathbb{Q}](k-1)}\mathcal{N}(\xi)^{k-1}}{\Gamma(k-1)^{[F:\mathbb{Q}]}}\;\times\;\int_{\mathbb{R}_+^{[F:\mathbb{Q}]}}a_\lambda(\xi,y)\,\mathbf{e}_F(\mathbf{i}\xi y)\cdot y^{k-2}\,\mathrm{d}y.$$

(This property that $\mathbf{G}$ has moderate growth is discussed at length in [**17**, Section 4.6].)

2.3. Controlling analytic families of HMFs. Before we can proceed further, let us recall some basic notions from measure theory. Fix a prime $p\neq 2$ and a tame level $\mathfrak{n}$ coprime to $p \cdot \mathcal{O}_F$. The narrow ray class group of conductor $\mathfrak{n}p^r$ is by definition $\mathrm{Cl}_F^{\mathrm{nw}}(\mathfrak{n}p^r) = F^\times \backslash \mathbb{A}_F^\times / F_\infty^+ \cdot (\widehat{\mathcal{O}}_F^\times \cap W_{1,\mathfrak{n}p^r})$. Taking the inverse limit over r, one obtains a decomposition

$$Z_F(\mathfrak{n}) := \varprojlim_r \mathrm{Cl}_F^{\mathrm{nw}}(\mathfrak{n}p^r) = W_F(\mathfrak{n}) \times Z_F(\mathfrak{n})_{\mathrm{tors}},$$

where $W_F(\mathfrak{n})$ is free over $\mathbb{Z}_p$ and $\# Z_F(\mathfrak{n})_{\mathrm{tors}} < \infty$. For each integer $r \geq 1$, define

$$Z_{F,r}(\mathfrak{n}) = \mathrm{Ker}\big(Z_F(\mathfrak{n}) \to \mathrm{Cl}_F^{\mathrm{nw}}(\mathfrak{n}p^r)\big) \quad \text{and} \quad W_{F,r}(\mathfrak{n}) = W_F(\mathfrak{n}) \cap Z_{F,r}(\mathfrak{n}).$$

We now explain how to make power series rings out of these various profinite groups.

NOTATIONS.

(a) *Let $\mathcal{O}$ be a finite extension of $\mathbb{Z}_p$; one considers Iwasawa algebras*

$$\mathcal{A}_F = \varprojlim_r \mathcal{O}\big[Z_F(\mathfrak{n})/Z_{F,r}(\mathfrak{n})\big] \quad \textit{and} \quad \Lambda_F = \varprojlim_r \mathcal{O}\big[W_F(\mathfrak{n})/W_{F,r}(\mathfrak{n})\big].$$

Clearly, $\mathcal{A}_F = \Lambda_F\big[Z_F(\mathfrak{n})_{\mathrm{tors}}\big]$, and Λ_F is a power series ring in $d = 1 + \delta_{F,p}$ variables where $\delta_{F,p}$ denotes the 'defect term' in Leopoldt's conjecture for the pair (F, p).

(b) *For an ideal $\mathfrak{l}$ coprime to $\mathfrak{n}p$, let $[\mathfrak{l}]$ be the corresponding group element of $\mathcal{A}_F$, and likewise $\langle[\mathfrak{l}]\rangle$ refers to the group element in Λ_F under the natural projection.*

(c) *Write $F^{\mathfrak{c}p^r}$ for the maximal abelian extension of F unramified outside $\mathfrak{n}p^r \cdot \infty$.*

Class field theory furnishes us with an isomorphism $\mathrm{Cl}_F^{\mathrm{nw}}(\mathfrak{n}p^r) \xrightarrow{\sim} \mathrm{Gal}(F^{\mathfrak{c}p^r}/F)$ by explicitly sending the class of $\mathfrak{a}$ to the Artin symbol $(\frac{\mathfrak{a}, F^{\mathfrak{c}p^r}}{F}) \in \mathrm{Gal}(F^{\mathfrak{c}p^r}/F)$. Passing to the inverse limit again,

$$\mathcal{A}_F \cong \mathcal{O}\big[\big[\mathrm{Gal}\big(F^{\mathfrak{c}p^\infty}/F\big)\big]\big] \quad \text{where the Lie extension } F^{\mathfrak{c}p^\infty} = \bigcup_{r \geq 1} F^{\mathfrak{c}p^r}.$$

Over the field of rational numbers $Z_{\mathbb{Q}}(1) \cong \mathbb{Z}_p^\times$ and $Z_{\mathbb{Q}}(1)_{\mathrm{tors}} \cong \mathbb{F}_p^\times$, so we denote by $\omega_{\mathbb{Q}} : Z_{\mathbb{Q}}(1) \twoheadrightarrow \mathbb{F}_p^\times$ and $\langle \cdot \rangle_{\mathbb{Q}} : Z_{\mathbb{Q}}(1) \twoheadrightarrow W_{\mathbb{Q}}(1) \cong 1 + p\mathbb{Z}_p$ the two projection maps. Analogously, we will write ω_F and $\langle \cdot \rangle_F$ for the composition of the above projections with the norm homomorphism $\mathcal{N} : Z_F(\mathfrak{n}p^r) \to Z_{\mathbb{Q}}(1) = \mathbb{Z}_p^\times$.

LEMMA 2.8.

(i) *There is an isomorphism $\mathrm{Meas}(Z_F(\mathfrak{n}), \mathcal{O}) \xrightarrow{\sim} \Lambda_F[Z_F(\mathfrak{n})_{\mathrm{tors}}]$ which at each character $\theta : Z_F(\mathfrak{n})_{\mathrm{tors}} \to \mathbb{C}_p^\times$, sends*

$$\mathrm{d}\mu \mapsto \int_{g \in Z_F(\mathfrak{n})} \theta(g)\, w_1^g \ldots w_d^g \cdot \mathrm{d}\mu(g) \in \mathcal{A}_F^{(\theta)}$$

where $W_F(\mathfrak{n}) \cong \mathbb{Z}_p \cdot \{w_1, \ldots, w_d\}$;

(ii) *if $\mathrm{Tw}_j : \mathcal{A}_F \to \mathcal{A}_F$ corresponds to the operation mapping $\mathrm{d}\mu \mapsto \mathcal{N}x_p^j \cdot \mathrm{d}\mu$ with $\mathcal{N}x_p$ the Tate character, then the twist operator $\mathrm{Tw}_j \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{A}_F)$ for all $j \in \mathbb{Z}$.*

Proof. See for instance **[21]** which contains an argument for h-admissible measures. □

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EXAMPLE 2.9. If ϵ is a finite order character of $W_F(\mathfrak{n})$ and $k \geq 2$ an integer weight, we write $P_{k,\epsilon} \in \mathrm{Hom}_{\mathcal{O}}\big(\Lambda_F, \overline{\mathbb{Q}}_p\big) = \mathrm{Spec}\big(\Lambda_F\big)(\overline{\mathbb{Q}}_p)$ for the specialisation extending the map $\langle[\mathfrak{l}]\rangle \mapsto \epsilon(\mathfrak{l})\langle\mathfrak{l}\rangle_F^{k-2}$ (such homomorphisms are designated 'classical points'). Assume the ideal $\mathfrak{c} \subset \mathfrak{n}$, and that ϕ is a Hecke character modulo $\mathfrak{c}p^r$ for some $r \geq 0$. The work of Deligne–Ribet [**6**] implies the existence of a p-adic zeta-function[1]

$$\zeta_{F,p\text{-adic}}^{(\mathfrak{c})}(\phi) \;\in\; \mathbb{Q} \otimes \Lambda_F\left\{\frac{1}{w_i - 1} \;\middle|\; i = 1, \ldots, d\right\} \;\subset\; \mathcal{Q}_{\Lambda_F} := \mathrm{Frac}(\Lambda_F)$$

interpolating $P_{k'+2,\epsilon}(\zeta_{F,p\text{-adic}}^{(\mathfrak{c})}(\phi)) = \tau_p(\zeta_F^{(\mathfrak{c}p)}(1-k', \epsilon\phi\omega_F^{2-k'}))$ at integers $k' > 0$. As the action of Tw_j extends to pseudo-measures, then Tate twisting j-times:

$$P_{k,\epsilon} \circ \mathrm{Tw}_j\Big(\zeta_{F,p\text{-adic}}^{(\mathfrak{c})}\big(\phi\omega_F^{j-2}\big)\Big) \;=\; \zeta_F^{(\mathfrak{c}p)}\big(3-k-j, \epsilon\phi\omega_F^{2-k}\big) \;\text{ for all weights } k > 2-j.$$

We conclude the background section with a discussion of Hida's deformation theory. First, endow $h_k(\mathfrak{n}p^\infty, \mathcal{O}) = \varprojlim_r h_k(\mathfrak{n}p^r, \mathcal{O})$ with its natural $\mathcal{A}_F$-algebra structure by sending each group element $[\mathfrak{l}]$ to the diamond operators $< \mathfrak{l} >_k \in h_k(\mathfrak{n}p^r, \mathcal{O})$. In fact, $h_k(\mathfrak{n}p^\infty, \mathcal{O})$ is independent of the weight $k \geq 2$, hence one writes $\mathbf{h}(\mathfrak{n}, \mathcal{O})$ for the universal p-adic Hecke algebra of tame level $\mathfrak{n}$.

If $\mathcal{S}_k(\mathfrak{n}p^\infty, \mathcal{O}) = \bigcup_{r\geq 1} \mathcal{S}_k(\mathfrak{n}p^r, \mathcal{O})$, then its p-adic completion $\overline{\mathcal{S}}_k(\mathfrak{n}p^\infty, \mathcal{O})$ is a module over $\mathcal{A}_F$ again, independent of $k \geq 2$ (which we drop from the notation). Moreover, there is a perfect pairing $[-,-]_{\mathfrak{n}p^\infty,\mathcal{O}} : \mathbf{h}(\mathfrak{n}, \mathcal{O}) \times \overline{\mathcal{S}}(\mathfrak{n}p^\infty, \mathcal{O}) \longrightarrow \mathcal{O}$, which is the natural extension of the dualities $h_k(\mathfrak{n}p^r, \mathcal{O}) \cong \mathcal{S}_k(\mathfrak{n}p^r, \mathcal{O})^*$ at the finite levels.

DEFINITION 2.10.

(i) If M is an $\mathbf{h}(\mathfrak{n}, \mathcal{O})$-module, then M^{ord} will denote the largest direct summand of M upon which the Hecke operators $U_{\mathfrak{p}}$ are invertible for all $\mathfrak{p}|p$, i.e. $M^{\mathrm{ord}} := M|\mathbf{e}_{\mathrm{ord}}$ cut out by the idempotent $\mathbf{e}_{\mathrm{ord}} = \lim_{n\to\infty}(U_{p\cdot\mathcal{O}_F})^{n!}$.

(ii) For a normalised ordinary eigenform $\mathbf{f} \in \mathcal{S}_k(\mathfrak{n}p^r, \psi)$, its p-stablisation equals

$$\mathbf{f}^{(0)} \;:=\; \mathbf{f}\,\Big|\prod_{\mathfrak{p}|p}\big(1 - \beta_{\mathfrak{p}}(\mathbf{f}) \cdot V_{\mathfrak{p}}\big),$$

where by convention $\alpha_{\mathfrak{p}}(\mathbf{f})$ (resp. $\beta_{\mathfrak{p}}(\mathbf{f})$) denotes the p-unit (resp. non p-unit) root of the characteristic polynomial $X^2 - C(\mathfrak{p}, \mathbf{f})X + \psi^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-1}$.

For simplicity, let us now abbreviate the $\mathbb{Z}_p$-free subgroup $W_F(\mathfrak{n})$ using Γ instead. Assuming that $\epsilon : \Gamma \to \overline{\mathbb{Q}}_p^\times$ is any character of finite order and the weight $k \geq 2$, then $\mathrm{Ker}\big(P_{k,\epsilon}\big)$ will be a prime ideal of Λ_F. The following results are fundamental.

THEOREM 2.11 ([**10**, Corollary 4.21] and [**9**, Theorem 3.4]). *If the prime $p \nmid 3D_F$, then:*

(a) *the $\mathcal{O}$-algebra $\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})$ is a finite and free algebra over Λ_F;*

(b) *at all classical points $P_{k,\epsilon} \in \mathrm{Spec}(\Lambda_F)^{\mathrm{alg}}$, one has isomorphisms*
$$\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O})\big/P_{k,\epsilon} \cdot \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \;\cong\; h_k^{\mathrm{ord}}(\mathfrak{n}p^r, \epsilon; \mathcal{O}),$$

where r denotes any integer for which $W_{F,r}(\mathfrak{n}) \subset \mathrm{Ker}(\epsilon)$.

[1] Skipping ahead a section, we shall need this element $\mathrm{Tw}_j(\zeta_{F,p-\mathrm{adic}}^{(\mathfrak{c})}(\phi\omega_F^{j-2}))$ for the demonstration of Proposition 3.1, wherein we make a precise choice of Tate twist $j = 2 - l$ and character $\phi = \psi\eta^{-1}$.

Alternatively, if $p = 3$ or $p \big| D_F$, both (a) and (b) hold upon replacing $\mathcal{O}$ by $\mathrm{Frac}(\mathcal{O})$.

We will now explain how to find a basis of primitive forms in the Λ-adic setting. Consider a Λ_F-algebra homomorphism $\lambda : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \longrightarrow \overline{\mathcal{Q}_{\Lambda_F}}$; the fraction field of $\mathrm{Im}(\lambda)$ is a finite extension of $\mathcal{Q}_{\Lambda_F}$, and let $\mathbb{I}$ be the integral closure of Λ_F in this finite extension. By Theorem 2.11(a), the homomorphism λ will take values in $\mathbb{I}$.

Remark. The set of algebraic specialisations $\mathrm{Spec}(\mathbb{I})^{\mathrm{alg}} \subset \mathrm{Spec}(\mathbb{I})$ consists of points $P : \mathbb{I} \to \overline{\mathbb{Q}}_p$ such that $P\big|_{\Lambda_F} = P_{k,\epsilon}$ for some $\epsilon : \Gamma \to \overline{\mathbb{Q}}_p^{\times}$ and integer weight $k \geq 2$. Each composition $\lambda_P = P \circ \lambda : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \longrightarrow \overline{\mathbb{Q}}_p$ factorises through $h_k^{\mathrm{ord}}(\mathfrak{n}p^r, \epsilon; \mathcal{O})$ due to Theorem 2.11(b), hence it must correspond to a classical cusp form.

Theorem 2.12. *Let $\mathbf{f} \in \mathcal{S}_k^{\mathrm{ord}}(\mathfrak{n}p^r, \epsilon; \mathcal{O})$ denote an ordinary cuspidal eigenform. Then, there exists $\lambda : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \longrightarrow \mathbb{I}$ and a point $P \in \mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$ of type (k, ϵ) such that λ_P corresponds to $\mathbf{f}$. Furthermore, if $\mathbf{f}$ is a p-ordinary newform, then the localised algebra $\mathbb{I}_P$ is étale over $\Lambda_{F,P_{k,\epsilon}}$.*

Proof. We refer the reader to the [**9**, Theorem 3.6] and [**15**, Proof of Theorem 4.4]. □

Definition 2.13.

(i) An $\mathbb{I}$-adic Hilbert modular form $\mathcal{F}$ of tame level equal to $\mathfrak{n}$ is a collection of $\mathbb{I}$-adic coefficients $C(\mathfrak{m}, \mathcal{F})$, $C_0(\mathfrak{m}, \mathcal{F})$ indexed by ideals $\mathfrak{m} \subset \mathcal{O}_F$, such that there is a Zariski dense subset of points $P \in \mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$ with $P\big|_{\Lambda_F} = P_{k,\epsilon}$ and Hilbert modular forms $\mathcal{F}_P \in \mathcal{M}_k\big(\mathfrak{n}p^r, \epsilon; \mathcal{O}[\epsilon]\big)$, satisfying

$$P\big(C(\mathfrak{m}, \mathcal{F})\big) = C(\mathfrak{m}, \mathcal{F}_P) \quad \text{and} \quad P\big(C_0(\mathfrak{m}, \mathcal{F})\big) = C_0(\mathfrak{m}, \mathcal{F}_P) \quad \text{at every } \mathcal{O}_F\text{-ideal } \mathfrak{m}.$$

(ii) The module of $\mathbb{I}$-adic modular forms of tame level $\mathfrak{n}$ is denoted by $\mathcal{M}(\mathfrak{n}, \mathbb{I})$; similarly, $\mathcal{S}(\mathfrak{n}, \mathbb{I})$ denotes the $\mathbb{I}$-submodule, consisting of those elements $\mathcal{F}$ whose specialisations yield cusp forms at a Zariski dense subset of points $P \in \mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$.

As usual, $\mathcal{M}^{\mathrm{ord}}(\mathfrak{n}, \mathbb{I}) = \mathcal{M}(\mathfrak{n}, \mathbb{I})\big|\mathbf{e}_{\mathrm{ord}}$ and $\mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \mathbb{I}) = \mathcal{S}(\mathfrak{n}, \mathbb{I})\big|\mathbf{e}_{\mathrm{ord}}$ will indicate their ordinary components. In particular, there is a natural duality

$$\mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \mathbb{I}) \;\cong\; \mathrm{Hom}_{\Lambda_F}\big(\mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}), \mathbb{I}\big)$$

which allows one to interpret $\mathbb{I}$-adic cusp forms $\mathcal{F}$ as Λ_F-algebra homomorphisms $\lambda_{\mathcal{F}} : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}, \mathcal{O}) \longrightarrow \mathbb{I}$, and vice versa of course. Assuming the coefficient ring $\mathcal{O}$ is sufficiently large with $\mathcal{O} = \mathbb{I} \cap \overline{\mathbb{Q}}_p$, the space $\mathcal{S}^{\mathrm{ord}}(\mathfrak{n}, \mathbb{I})$ is completely diagonalisable under the Hecke algebra (which means that a basis of primitive forms can be found).

Example 2.14. Let E be a modular elliptic curve over $\mathbb{Q}$ with good ordinary or bad multiplicative reduction at p, and let f_E denote its associated newform. Assuming F is a solvable extension, then there must exist a cuspidal eigenform $\mathbf{BC}(f_E) \in \mathcal{S}_2(\mathfrak{n}_E, \mathbf{1})$ which is the base-change of f_E over F.

From the Control Theorem 2.12, such a form lifts to a family $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}\big(\mathfrak{n}_E, \mathbb{I}\big)$ with the interpolation property that $P(\mathcal{F}) \in \mathcal{S}_k^{\mathrm{ord}}(\mathfrak{n}_E p^\infty, \epsilon\omega_F^{2-k})$ when $P|_{\Lambda_F} = P_{k,\epsilon}$. In particular, at weight 2 and the trivial character, $P(\mathcal{F}) = \mathbf{BC}(f_E)^{(0)}$ will be the p-stablisation of f_E over the extension F; as a corollary, $L(P(\mathcal{F}), s) = L(E/F, s)$ if one ignores the Euler factors at the primes dividing p.

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EXAMPLE 2.15. Another nice specimen arises from the Λ-adic Eisenstein measure. Let θ_1, θ_2 be two characters of $Z_F(\mathfrak{n})_{\text{tors}}$ whose conductors divide into $\mathfrak{n}p^\infty$, and such that $\theta_1\theta_2\big|_{F\otimes\mathbb{R}} = \mathbf{1}$. Then, there exists a unique $\mathfrak{E}(\theta_1, \theta_2) \in \mathcal{M}^{\text{ord}}\big(\mathfrak{n}, \Lambda_F\big) \otimes_{\Lambda_F} \mathcal{Q}_{\Lambda_F}$ with the property that $P_{k,\epsilon}\big(\mathfrak{E}(\theta_1, \theta_2)\big)$ is equal to the p-adic HMF

$$\tau_p\left(\frac{\theta\big((-1)_\infty\big)D_F^{k-1/2}\Gamma(k)^{[F:\mathbb{Q}]}\tau_F(\theta)}{\mathcal{N}(\mathfrak{n}_\theta)(-2\pi i)^{k[F:\mathbb{Q}]}}\Bigg|_{\theta=\epsilon\theta_2\omega_F^{k-2}} \times\ E_k\big(\theta_1, \epsilon\theta_2\omega_F^{k-2}\big)^{(0)}\right),$$

where the (non-p-stabilised) Eisenstein series $E_k(\theta_1, \theta_2)$ is given in [**22**, Proposition 1.3.1]. We shall write $\mathcal{M}^{\text{ord},\dagger}(\mathfrak{n}, \mathbb{I}) = \mathcal{M}^{\text{ord}}(\mathfrak{n}, \mathbb{I}) \otimes_{\Lambda_F} \mathcal{Q}_{\Lambda_F}$ to indicate the space of $\mathbb{I}$-adic modular forms admitting poles, and likewise $\mathcal{S}^{\text{ord},\dagger}(\mathfrak{n}, \mathbb{I}) = \mathcal{S}^{\text{ord}}(\mathfrak{n}, \mathbb{I}) \otimes_{\Lambda_F} \mathcal{Q}_{\Lambda_F}$.

REMARK. Our initial hope had been to exploit properties of these elements $\mathfrak{E}(\theta_1, \theta_2)$ to construct a two-variable p-adic L-function, interpolating twists of elliptic curves. Unfortunately, the integral expressions for the associated Rankin–Selberg L-function involve Hilbert modular forms which are non-holomorphic outside the line $s = k - 1$, and the $\mathfrak{E}(\theta_1, \theta_2)$'s do not interpolate outside this line (this is a potential disaster!). The salvage is to instead try to construct by hand an $\mathbb{I}$-adic modular form, $\mathcal{H}^\pm$ say, which glues together the holomorphic projections of these $\mathcal{C}^\infty$-functions.

3. Projecting Hol($\mathcal{G}\widetilde{\mathbf{E}}$) as the weight of $\widetilde{\mathbf{E}}$ varies.

Let F be a totally real field in which the prime $p \geq 3$ is inert; we shall write $\mathfrak{p}$ for the unique ideal of $\mathcal{O}_F$ lying above p. Recall that $\Lambda = \Lambda_F$ is isomorphic to a power series ring in $1 + \delta_{F,p}$ variables, where $\delta_{F,p}$ was the defect term of Leopoldt (in particular, it is conjectured that $\delta_{F,p} = 0$ for all totally real F and primes p, and the result is certainly known for real abelian extensions of $\mathbb{Q}$).

Lastly, pick a Hilbert modular cusp form $\mathcal{G}_l \in \mathcal{S}_l\big(\mathfrak{c}\mathfrak{p}^n, \eta\big)$ of parallel weight $l \geq 1$, nebentypus η, and level $\mathfrak{c}\mathfrak{p}^n$ with $n > 0$. We can deform $\mathcal{G}_l$ along the weight-axis by multiplying it with an appropriate Λ_F-adic Eisenstein series.

PROPOSITION 3.1. *For a fixed finite order character ψ mod $\mathfrak{c}\mathfrak{p}^n$ and integer $r \geq 0$, there exist pairs of Hilbert modular forms $\mathcal{H}^\pm_{\mathfrak{p}^n,r} = \mathcal{H}^\pm_{\mathfrak{p}^n,r}\big(\mathcal{G}_l, \mathfrak{c}\mathfrak{p}^n, \psi\big) \in \mathcal{S}^{\text{ord},\dagger}\big(\mathfrak{c}, \Lambda\big)$ interpolating the data*

$$P_{k,\epsilon}\left(\mathcal{H}^\pm_{\mathfrak{p}^n,r}\right) \;=\; \text{Hol}\Big(\mathcal{G}_l \times \widetilde{\mathbf{E}}_{k-l}\big(r - (k - l - 1), \psi\eta^{-1}\omega_F^{2-k}\epsilon; \mathfrak{c}\mathfrak{p}^n\big)\Big)^{\text{ord}}$$

for every finite order character $\epsilon : \Gamma \to \overline{\mathbb{Q}}_p^\times$ such that $\epsilon \neq \omega_F^{k-2}\psi^{-1}\eta$, and at all parallel weights $k \geq r + l + 1$ satisfying the parity condition $(-1)^{k-l-1} = \pm 1$.

In fact, if $\delta_{F,p} = 0$, then the forms $\mathcal{H}^\pm_{\mathfrak{p}^n,r}$ are uniquely determined by the above interpolation formula, as a consequence of the Zariski density of each $\pm$-subset

$$\text{Spec}(\Lambda)^\pm_{\text{alg}} \;:=\; \Big\{P_{k,\epsilon}\ \text{ such that } (-1)^{k-l-1} = \pm 1 \text{ and } \epsilon \in \text{Hom}\big(\Gamma, \overline{\mathbb{Q}}_p^\times\big)_{\text{tors}}\Big\}$$

within the full set of Λ-adic specialisations (note the mapping $[w] \mapsto [w]^2$ induces an automorphism of $\Lambda \cong \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]$ as $p \neq 2$, so the even/odd weights are dense).

Proof when $r \neq 0$ To establish existence of these HMFs, for every $\lambda \in \{1, \ldots, \hat{h}_F\}$ we shall define universal ξ-expansions

$$\left(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\right)_\lambda \ := \ (-1)^{r[F:\mathbb{Q}]} \times \sum_{0 \ll \xi \in \widetilde{t}_\lambda} \lim_{N\to\infty} \left(C^{N!,\pm}\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)_\lambda\right) \times \mathbf{e}_F(\xi z),$$

where each ξ-coefficient is approximated by the finite sum

$$C^{N!,\pm}\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)_\lambda = \sum_{\mathfrak{p}^{N!}\xi=\xi_1+\xi_2} C(\xi_1\widetilde{t}_\lambda^{-1}, \mathcal{G}_l) \sum_{\substack{\widetilde{\xi}_2=\widetilde{b}\times\widetilde{c},\ b\in\widetilde{t}_\lambda \\ c\in\mathcal{O}_F-\mathfrak{p}}} \left((-1)^{[F:\mathbb{Q}]}\mathrm{sign}\mathcal{N}(\widetilde{b})\right)^{\frac{1\mp1}{2}}$$
$$\times\, \mathcal{N}(\widetilde{t}_\lambda)^{-1}\mathcal{N}\big(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1}\big)^r \ \times\ \left(\psi\eta^{-1}\right)^*(\widetilde{c})\,\mathcal{N}(\widetilde{c})^{1-l} \times \langle[\widetilde{c}]\rangle.$$

The truth of the proposition will follow, provided we can establish:

(3.1.1) $C^{\pm}\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)_\lambda = \lim_{N\to\infty} C^{N!,\pm}\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)_\lambda$ converges inside of Λ;

(3.1.2) If $P_{k,\epsilon} \in \mathrm{Spec}(\Lambda)^{\pm}_{\mathrm{alg}}$, then $(-1)^{r[F:\mathbb{Q}]}\mathcal{N}(\widetilde{t}_\lambda)^{k/2} \times P_{k,\epsilon}\left(C^{\pm}\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)_\lambda\right)$ is the ξ-coefficient of $\mathrm{Hol}(\mathcal{G}_l \times \widetilde{\mathbf{E}}_{k-l}(r-(k-l-1), \psi\eta^{-1}\omega_F^{2-k}\epsilon; \mathfrak{c}\mathfrak{p}^n))^{\mathrm{ord}}_\lambda$;

(3.1.3) $\mathcal{H}^{\pm}_{\mathfrak{p}^n,r} = \sum_{\lambda=1}^{\hat{h}_F}(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r})_\lambda$ belongs to the space of cusp forms $\mathcal{S}^{\mathrm{ord}}(\mathfrak{c}, \Lambda)$.

Let us prove these statements in order. Following Panchiskin [**16, 17**], we first give the Fourier expansion for the holomorphic projection of $\mathcal{G}_l(z) \cdot \widetilde{\mathbf{E}}_{k-l}(-\,,\,-\,;\mathfrak{c}\mathfrak{p}^n)$.

Let θ denote a (not necessarily primitive) character factoring through $\mathrm{Gal}\big(F^{\mathfrak{c}\mathfrak{p}^n}/F\big)$; attached to $\theta \neq \mathbf{1}$, we consider the normalised Eisenstein series

$$\mathbf{E}_{k-l}\big(s,\theta;\mathfrak{c}\mathfrak{p}^n\big) \ = \ \frac{2^{-[F:\mathbb{Q}]} \times D_F^{1/2} \times \Gamma(k-l+s)^{[F:\mathbb{Q}]}}{(-4\pi)^{-s[F:\mathbb{Q}]} \times (-2\pi i)^{[F:\mathbb{Q}](k-l+2s)}} \times \mathbb{L}^0_{k-l}\big(s;\mathcal{O}_F,\mathcal{O}_F;\theta\big).$$

Using the development of $\mathbb{L}^0_m(s;-)_\lambda$ given in Section 2 and putting $m = k-l$, one obtains

$$\mathbf{E}_{k-l}\big(s,\theta;\mathfrak{c}\mathfrak{p}^n\big)_\lambda \ = \ \mathcal{N}(\widetilde{t}_\lambda)^{-s-(k-l)/2-1}(4\pi)^{[F:\mathbb{Q}]s}(\mathcal{N}y)^s \sum_{\substack{\xi\,=\,0,\ \text{or} \\ 0\ll\xi\in\widetilde{t}_\lambda}} \mathcal{A}_\lambda(\xi,s,y,\theta) \times \mathbf{e}_F(\xi z)$$

at every component $\lambda \in \{1, \ldots, \hat{h}_F\}$, with real-analytic coefficients (for $\xi \gg 0$):

$$\mathcal{A}_\lambda(\xi,s,y,\theta) \ = \sum_{\substack{\widetilde{\xi}=\widetilde{b}\times\widetilde{c}, \\ b\in\widetilde{t}_\lambda,\ c\in\mathcal{O}_F}} \mathrm{sign}\mathcal{N}(\widetilde{b})^{k-l-1}\mathcal{N}(\widetilde{b})^{k-l+2s-1}\theta^*(\widetilde{c})\prod_\nu W\big(4\pi\xi_\nu y_\nu, k-l+s, s\big).$$

□

Remarks.

(i) If $s = 0$, then $\mathbf{E}_{k-l}\big(0,\theta;\mathfrak{c}\mathfrak{p}^n\big)$ is a holomorphic modular form, and its Fourier expansion was computed by Klingen (e.g. see [**18**, Section 3] or [**15**, equation (3.14)]). However, if $s \neq 0$, then $\mathbf{E}_{k-l}\big(s,\theta;\mathfrak{c}\mathfrak{p}^n\big)$ will unfortunately be non-holomorphic in z; consequently, for $s < 0$, the function $\mathcal{G}_l(z) \times \mathbf{E}_{k-l}\big(s,\theta;\mathfrak{c}\mathfrak{p}^n\big)$ is only real-analytic.

(ii) If $\theta_{\mathfrak{p}}$ denotes the Hecke character satisfying

$$\theta^*_{\mathfrak{p}}(\mathfrak{y}) = \begin{cases} \theta^*(\mathfrak{y}) & \text{if } \mathfrak{p}+\mathfrak{y}=\mathcal{O}_F \\ 0 & \text{if } \mathfrak{p}+\mathfrak{y}\neq\mathcal{O}_F \end{cases},$$

then $\widetilde{\mathbf{E}}_{k-l}(s,\theta;\mathfrak{cp}^n) = \mathbf{E}_{k-l}(s,\theta_{\mathfrak{p}};\mathfrak{cp}^n)$; it follows that the corresponding Fourier coefficients $\widetilde{\mathcal{A}}_\lambda(\xi,s,y,\theta)$ for $\widetilde{\mathbf{E}}_{k-l}$ differ from the $\mathcal{A}_\lambda(\xi,s,y,\theta)$'s in that their sum instead involves ideal-decompositions $\widetilde{\xi}=\widetilde{b}\times\widetilde{c}$, $b\in\widetilde{t}_\lambda$ which omit those $c\in\mathfrak{p}$.

Multiplying our ξ-development of the $\mathcal{C}^\infty$-modular form $\widetilde{\mathbf{E}}_{k-l}(s,\theta;\mathfrak{cp}^n)|_{s=r-(k-l-1)}$ together with the expansion $\mathcal{G}_l(z)_\lambda = \mathcal{N}(\widetilde{t}_\lambda)^{l/2}\sum_{0\ll\xi\in\widetilde{t}_\lambda} C(\xi\widetilde{t}_\lambda^{-1},\mathcal{G}_l)\times\mathbf{e}_F(\xi z)$:

$$\begin{aligned}&\left(\mathcal{G}_l\times\widetilde{\mathbf{E}}_{k-l}\big(r-(k-l-1),\theta;\mathfrak{cp}^n\big)\right)_\lambda\\ &\quad=\mathcal{N}(\widetilde{t}_\lambda)^{-s-k/2+l}\sum_{0\ll\xi\in\widetilde{t}_\lambda}\ \sum_{\xi=\xi_1+\xi_2} C\big(\xi_1\widetilde{t}_\lambda^{-1},\mathcal{G}_l\big)\times\sum_{\substack{\widetilde{\xi}_2=\widetilde{b}\times\widetilde{c},\ b\in\widetilde{t}_\lambda\\ c\in\mathcal{O}_F-\mathfrak{p}}}\mathrm{sign}\mathcal{N}(\widetilde{b})^{k-l-1}\\ &\qquad\times\mathcal{N}(\widetilde{b})^{k-l-1+2s}\ \times\ \theta^*(\widetilde{c})\times(4\pi)^{[F:\mathbb{Q}]s}(\mathcal{N}y)^s\prod_\nu W\big(4\pi\xi_{2,\nu}y_\nu,k-l+s,s\big)\times\mathbf{e}_F(\xi z)\end{aligned}$$

with the R.H.S. evaluated at the non-positive integer point $s=r-(k-l-1)$.

FACT: [**17**, p134]. *If $\beta\in-\mathbb{N}\cup\{0\}$, then the Whittaker function*

$$W(y,\alpha,\beta)\ =\ \sum_{j=0}^{-\beta}(-1)^j\binom{-\beta}{j}\frac{\Gamma(\alpha)}{\Gamma(\alpha-j)}y^{-\beta-j};$$

in particular, this holds true when $y=4\pi\xi_\nu y_\nu$, $\alpha=k-l+s$ and $\beta=+s$ as above.
Now calculating the holomorphic projection,

$$\begin{aligned}\mathrm{Hol}\Big(\mathcal{G}_l\times\widetilde{\mathbf{E}}_{k-l}\big(r-(k-l-1),\theta;\mathfrak{cp}^n\big)\Big)_\lambda &= \mathcal{N}(\widetilde{t}_\lambda)^{k/2-r-1}\textstyle\sum_{0\ll\xi\in\widetilde{t}_\lambda}\sum_{\xi=\xi_1+\xi_2}C\big(\xi_1\widetilde{t}_\lambda^{-1},\mathcal{G}_l\big)\\ \times\textstyle\sum_{\substack{\widetilde{\xi}_2=\widetilde{b}\times\widetilde{c},\ b\in\widetilde{t}_\lambda\\ c\in\mathcal{O}_F-\mathfrak{p}}}&\mathrm{sign}\mathcal{N}(\widetilde{b})^{k-l-1}\mathcal{N}(\widetilde{b})^{2r-(k-l-1)}\ \theta^*(\widetilde{c})\ \textstyle\prod_\nu P_s(\xi_{2,\nu},\xi_\nu)\Big|_{s=r-(k-l-1)}\times\mathbf{e}_F(\xi z),\end{aligned}$$

where the polynomial $P_s(\xi_{2,\nu},\xi_\nu)=\sum_{j=0}^{-s}(-1)^j\binom{-s}{j}\frac{\Gamma(r+1)}{\Gamma(r+1-j)}\frac{\Gamma(k-1-j)}{\Gamma(k-1)}\xi_{2,\nu}^{-s-j}\xi_\nu^j$ belongs to the two-variable ring $\mathbb{Z}\big[\xi_{2,\nu},\xi_\nu\big]$.

REMARKS.

(a) Since $P_s(\xi_{2,\nu},\xi_\nu)|_{s=r-(k-l-1)}\equiv\ \xi_{2,\nu}^{k-l-1-r}\mod\xi_\nu\cdot\mathbb{Z}[\xi_{2,\nu},\xi_\nu]$, there is a congruence [**17**, equation (5.9)] on the level of ξ-expansions

$$\mathrm{Hol}\Big(\mathcal{G}_l\times\widetilde{\mathbf{E}}_{k-l}\big(r-(k-l-1),\theta;\mathfrak{cp}^n\big)\Big)_\lambda\equiv\sum_{0\ll\xi\in\widetilde{t}_\lambda}\delta_\lambda(\xi)\mathbf{e}_F(\xi z)\quad\mathrm{mod}\ \mathcal{N}(\xi)\cdot\mathcal{O}_{\mathbb{C}_p}[\![\xi]\!],$$

where $\delta_\lambda(\xi) = \delta_\lambda(\xi, \mathcal{G}_l, k, r, \theta)$ is defined by

$$\delta_\lambda(\xi) := \mathcal{N}(\widetilde{t}_\lambda)^{k/2-r-1} \sum_{\xi=\xi_1+\xi_2} C\big(\xi_1\widetilde{t}_\lambda^{-1}, \mathcal{G}_l\big) \times (-1)^{(k-l-1-r)[F:\mathbb{Q}]} \times \mathcal{N}(\xi_2)^{k-l-1-r}$$
$$\times \sum_{\substack{\widetilde{\xi}_2=\widetilde{b}\times\widetilde{c},\ b\in\widetilde{t}_\lambda \\ c\in\mathcal{O}_F-\mathfrak{p}}} \mathrm{sign}\mathcal{N}(\widetilde{b})^{k-l-1} \times \mathcal{N}(\widetilde{b})^{2r-(k-l-1)} \times \theta^*(\widetilde{c}).$$

(b) Provided $N > \hat{h}_F$, then hitting $\mathrm{Hol}(\dots)_\lambda$ with the $U_{\mathfrak{p}}$-operator $N!$-times:

$$\mathrm{Hol}\Big(\mathcal{G}_l \times \widetilde{\mathbf{E}}_{k-l}\big(r-(k-l-1), \theta; \mathfrak{c}\mathfrak{p}^n\big)\Big)_\lambda \Big| U_{\mathfrak{p}}^{N!} \equiv \sum_{0\ll\xi\in\widetilde{t}_\lambda} \delta_\lambda\left(\mathfrak{p}^{N!}\xi\right) \mathbf{e}_F(\xi z) \mod \mathcal{N}(\mathfrak{p})^{N!}.$$

(c) Lastly, a bare-hands calculation[2] shows that

$$\delta_\lambda\left(\mathfrak{p}^{N!}\xi, \mathcal{G}_l, k, r, \theta\right) \equiv (-1)^{r[F:\mathbb{Q}]} \times \mathcal{N}(\widetilde{t}_\lambda)^{k/2-1} \times \sum_{\mathfrak{p}^{N!}\xi=\xi_1+\xi_2} C\big(\xi_1\widetilde{t}_\lambda^{-1}, \mathcal{G}_l\big)$$
$$\times \sum_{\substack{\widetilde{\xi}_2=\widetilde{b}\times\widetilde{c},\ b\in\widetilde{t}_\lambda \\ c\in\mathcal{O}_F-\mathfrak{p}}} \big((-1)^{[F:\mathbb{Q}]}\mathrm{sign}\mathcal{N}(\widetilde{b})\big)^{(k-l-1)} \mathcal{N}(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1})^r$$
$$\times\, \theta^*(\widetilde{c})\mathcal{N}(\widetilde{c})^{k-l-1} \mod \mathcal{N}(\mathfrak{p})^{N!}.$$

An important corollary of (a)–(c) is that whenever $(-1)^{k-l-1} = \pm 1$,

$$\delta_\lambda\left(\mathfrak{p}^{N!}\xi, \mathcal{G}_l, k, r, \theta\right)\big|_{\theta=\psi\eta^{-1}\omega_F^{2-k}\epsilon} \equiv (-1)^{r[F:\mathbb{Q}]}\mathcal{N}(\widetilde{t}_\lambda)^{k/2} P_{k,\epsilon}\Big(C^{N!,\pm}\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)_\lambda\Big)$$

modulo $\mathcal{N}(\mathfrak{p})^{N!}$, as each specialisation $P_{k,\epsilon}\big(\langle\!\langle[\widetilde{c}]\rangle\!\rangle\big) = \omega_F^{2-k}\epsilon\big(\widetilde{c}\big)\mathcal{N}(\widetilde{c})^{k-2}$ for $k \geq 2$.

For convenience, we will abbreviate $\mathcal{G}_l \times \widetilde{\mathbf{E}}_{k-l}\big(r-(k-l-1), \theta; \mathfrak{c}\mathfrak{p}^n\big)$ by writing $\mathcal{G}\widetilde{\mathbf{E}}$. From its definition $\mathrm{Hol}\big(\mathcal{G}\widetilde{\mathbf{E}}\big)^{\mathrm{ord}} = \lim_{N\to\infty} \mathrm{Hol}\big(\mathcal{G}\widetilde{\mathbf{E}}\big)\Big|U_{\mathfrak{p}}^{N!}$, whence

$$\mathrm{Hol}\big(\mathcal{G}\widetilde{\mathbf{E}}\big)^{\mathrm{ord}}_\lambda = \sum_{0\ll\xi\in\widetilde{t}_\lambda} \mathcal{N}(\widetilde{t}_\lambda)^{k/2} \lim_{N\to\infty} C\big(\mathfrak{p}^{N!}\xi\widetilde{t}_\lambda^{-1}, \mathrm{Hol}(\mathcal{G}\widetilde{\mathbf{E}})\big) \times \mathbf{e}_F(\xi z)$$
$$\overset{\text{by (b)}}{=} \sum_{0\ll\xi\in\widetilde{t}_\lambda} \lim_{N\to\infty} \delta_\lambda\left(\mathfrak{p}^{N!}\xi\right) \mathbf{e}_F(\xi z) \quad \text{at all components } \lambda \in \big\{1, \dots, \hat{h}_F\big\}.$$

The $C(\mathfrak{p}^{N!}\xi\widetilde{t}_\lambda^{-1}, \mathrm{Hol}(\mathcal{G}\widetilde{\mathbf{E}}))$'s are Cauchy under the usual p-adic topology, therefore both the sequences $\{\delta_\lambda(\mathfrak{p}^{N!}\xi)\}_{N\geq 1}$ and $\{P_{k,\epsilon}(C^{N!,\pm}(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r})_\lambda)\}_{N\geq 1}$ must also be Cauchy (via the congruences on the previous page).

Under the diagonal embedding $\Lambda \hookrightarrow \bigoplus_{P_{k,\epsilon}\in\mathrm{Spec}(\Lambda)^{\pm}_{\mathrm{alg}}} \Lambda\big/P_{k,\epsilon}$ if a Λ-adic sequence were not Cauchy, it would also fail to be Cauchy modulo some $P_{k,\epsilon} \in \mathrm{Spec}(\Lambda)^{\pm}_{\mathrm{alg}}$. The contrapositive of this statement implies $\{C^{N!,\pm}(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r})_\lambda\}_{N\geq 1}$ is Cauchy; as

[2]The $\big((-1)^{[F:\mathbb{Q}]}\mathrm{sign}\mathcal{N}(\widetilde{b})\big)^{(k-l-1)}$ term is missing from [**17**, equation (5.9)]; however, the argument of Panchiskin is not compromised in any way by its omission. In the Λ-adic setting, the presence of this sign term forces a (hypothetical) single "$\mathcal{H}_{\mathfrak{p}^n,r}$" to split into a '$+$'-part and a '$-$'-part.

Λ is complete, this sequence tends to a unique limit '$C^{\pm}(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r})_\lambda$' say, thereby establishing assertion (3.1.1) in the process.

To prove (3.1.2), we simply note that if $\theta = \psi\eta^{-1}\omega_F^{2-k}\epsilon \neq \mathbf{1}$, then

$$\mathrm{Hol}\Big(\mathcal{G}_l \times \widetilde{\mathbf{E}}_{k-l}\big(r-(k-l-1), \theta; \mathfrak{c}\mathfrak{p}^n\big)\Big)_\lambda \Big| U_{\mathfrak{p}}^{N!} \equiv \sum_{0\ll\xi\in\widetilde{t}_\lambda} \delta_\lambda\left(\mathfrak{p}^{N!}\xi, \mathcal{G}_l, k, r, \theta\right) \mathbf{e}_F(\xi z)$$

$$\equiv (-1)^{r[F:\mathbb{Q}]}\mathcal{N}(\widetilde{t}_\lambda)^{k/2} \times \sum_{0\ll\xi\in\widetilde{t}_\lambda} P_{k,\epsilon}\left(C^{N!,\pm}\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)_\lambda\right)\mathbf{e}_F(\xi z)$$

modulo $\mathcal{N}(\mathfrak{p})^{N!}$; taking the limit as $N \to \infty$, the L.H.S. tends to $\mathrm{Hol}(\mathcal{G}\widetilde{\mathbf{E}})_\lambda^{\mathrm{ord}}$ whilst the R.H.S. tends to $(-1)^{r[F:\mathbb{Q}]}\mathcal{N}(\widetilde{t}_\lambda)^{k/2} \times \sum_{0\ll\xi\in\widetilde{t}_\lambda} P_{k,\epsilon}(C^{\pm}\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)_\lambda)\mathbf{e}_F(\xi z)$.

Finally, the specialisation of $\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}$ at any point $P_{k,\epsilon} \in \mathrm{Spec}(\Lambda)^{\pm}_{\mathrm{alg}}$ with $k \geq r + l + 1$ coincides with $\sum_{\lambda=1}^{\hat{h}_F} \mathrm{Hol}(\mathcal{G}\widetilde{\mathbf{E}})_\lambda^{\mathrm{ord}} \in \mathcal{S}_k^{\mathrm{ord}}(\mathfrak{c}\mathfrak{p}^\infty, \psi\eta^{-1}\omega_F^{2-k}\epsilon)$; it follows directly that $P(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r})$ is a classical cusp form at a Zariski dense subset of points $P \in \mathrm{Spec}(\Lambda)$, therefore assertion (3.1.3) is established too.

Proof when $r = 0$ The argument is exactly the same as for $r \neq 0$, except that the Fourier coefficient $\delta_\lambda(\xi) = \delta_\lambda(\xi, \mathcal{G}_l, k, 0, \theta)$ requires the addition of the L-value

$$\mathcal{N}(\widetilde{t}_\lambda)^{k/2-1} \times 2^{-[F:\mathbb{Q}]} \cdot \zeta_F^{(\mathfrak{c})}\big(1-(k-l), \theta_{\mathfrak{p}}\big) \times C\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{G}_l\big)$$

caused by an extra term in $\mathrm{Hol}\big(\mathcal{G}\widetilde{\mathbf{E}}\big)$ which occurs only at $r = 0$ [**16**, equation (5.7)]. Thus, we need to modify the definition of $C^{\pm}(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r})_\lambda$ by adding in the factor

$$\mathcal{N}(\widetilde{t}_\lambda)^{-1} \times 2^{-[F:\mathbb{Q}]} \cdot \mathrm{Tw}_{2-l}\Big(\zeta^{(\mathfrak{c})}_{F,p\text{-adic}}\big(\psi\eta^{-1}\omega_F^{-l}\big)\Big) \times C\big(\xi\widetilde{t}_\lambda^{-1}, \mathcal{G}_l^{\mathrm{ord}}\big)$$

(the p-adic zeta-function $\zeta^{(\mathfrak{c})}_{F,p\text{-adic}}(\psi\eta^{-1}\omega_F^{-l})$ merely belongs to $\mathbb{Q} \otimes \Lambda_F\{\frac{1}{[w]-1}\}$, which explains why the family $\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}$ lives inside $\mathcal{S}^{\mathrm{ord},\dagger}(\mathfrak{c}, \Lambda)$ precisely when $r = 0$). $\square$

Recall that for $x \in \mathrm{GL}_2(\mathbb{A}_F)$, the trace mapping

$$\mathrm{Tr}^{\mathfrak{c}\mathfrak{p}^N}_{\mathfrak{c}\mathfrak{p}^m} : \mathcal{M}_k\big(\mathfrak{c}\mathfrak{p}^N, \psi\big) \longrightarrow \mathcal{M}_k\big(\mathfrak{c}\mathfrak{p}^m, \psi\big) \quad \text{sends} \quad \mathbf{h}(x) \mapsto \sum_{w\in W_{\mathfrak{c}\mathfrak{p}^N}\backslash W_{\mathfrak{c}\mathfrak{p}^m}} \psi(w^{-\iota})\mathbf{h}(xw).$$

The following (twisted) inner product will appear frequently when we calculate special values for the improved p-adic L-function.

DEFINITION 3.2. For integers $N \geq m \geq 1$, we now introduce a $\mathbb{C}$-bilinear pairing

$$\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}} : \mathcal{S}_k\big(\mathfrak{c}\mathfrak{p}^m, \psi\big) \times \mathcal{S}_k\big(\mathfrak{c}\mathfrak{p}^N, \psi\big) \longrightarrow \mathbb{C}$$

by the rule $\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}, \mathbf{h}\big) = \mathcal{N}\big(\mathfrak{p}^{N-m}\big)^{k/2-1} \times \Big\langle \mathbf{f}^\#, \mathrm{Tr}^{\mathfrak{c}\mathfrak{p}^N}_{\mathfrak{c}\mathfrak{p}^m}\big(\mathbf{h}\big|J_{\mathfrak{c}\mathfrak{p}^N}\big)\Big\rangle_{\mathfrak{c}\mathfrak{p}^m}$.

As the trace map satisfies $\mathrm{Tr}^{\mathfrak{c}\mathfrak{p}^N}_{\mathfrak{c}\mathfrak{p}^m}(\mathbf{h}) = (-1)^{[F:\mathbb{Q}]k}\mathcal{N}(\mathfrak{p}^{N-m})^{1-k/2}\mathbf{h}|J_{\mathfrak{c}\mathfrak{p}^N}U_{\mathfrak{p}}^{N-m}J_{\mathfrak{c}\mathfrak{p}^m}$, one has an alternative definition

$$\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}, \mathbf{h}\big) = \big\langle \mathbf{f}^\#, \ \mathbf{h}\big|U_{\mathfrak{p}}^{N-m}\big|J_{\mathfrak{c}\mathfrak{p}^m}\big\rangle_{\mathfrak{c}\mathfrak{p}^m}.$$

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Listed below are some basic properties, which shall certainly be required later on.

LEMMA 3.3.
(a) *Under* ∇ *the* $U_{\mathfrak{p}}$*-operator is self-adjoint, i.e.* $\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}, \mathbf{h}\big|U_{\mathfrak{p}}\big) = \nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}\big|U_{\mathfrak{p}}, \mathbf{h}\big)$;
(b) *if* $\mathbf{f} \in \mathcal{S}_k\big(\mathfrak{c}\mathfrak{p}^m, \psi\big)$ *and* $j > 0$, *then* $\nabla_{\mathfrak{c}\mathfrak{p}^{N+j,m}}\big(\mathbf{f}\,, -\big)\Big|_{\mathcal{S}_k(\mathfrak{c}\mathfrak{p}^N,\psi)} = \nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}\,\big|U_{\mathfrak{p}}^j, -\big)$;
(c) *if* $\mathbf{f}$ *is a* $\mathfrak{p}$*-stabilised newform and* $\mathbf{h} \in \mathcal{S}_k\big(\mathfrak{c}\mathfrak{p}^{N-1}, \psi\big)$, *there is an equality*

$$\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}, \mathbf{h}\big|V_{\mathfrak{p}}\big) \;=\; \mathcal{N}(\mathfrak{p})^{-k} \times \nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}\,\big|U_{\mathfrak{p}}^*, \mathbf{h}\big).$$

Proof. Because the operation '$-\big|U_{\mathfrak{p}}\big|J_{\mathfrak{c}\mathfrak{p}^m}$' has the same effect as '$-\big|J_{\mathfrak{c}\mathfrak{p}^m}\big|U_{\mathfrak{p}}^*$',

$$\big\langle \mathbf{f}^\#, (\mathbf{h}|U_{\mathfrak{p}})\big|U_{\mathfrak{p}}^{N-m}\big|J_{\mathfrak{c}\mathfrak{p}^m}\big\rangle = \big\langle \mathbf{f}^\#, \mathbf{h}\big|U_{\mathfrak{p}}^{N-m}\big|J_{\mathfrak{c}\mathfrak{p}^m}\big|U_{\mathfrak{p}}^*\big\rangle = \big\langle \mathbf{f}^\#\big|U_{\mathfrak{p}}^{**}, \mathbf{h}\big|U_{\mathfrak{p}}^{N-m}\big|J_{\mathfrak{c}\mathfrak{p}^m}\big\rangle.$$

However, $\mathbf{f}^\#\big|U_{\mathfrak{p}}^{**} = \big(\mathbf{f}|U_{\mathfrak{p}}\big)^\#$, so (a) follows from the alternative definition for $\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}$. Assertion (b) is an easy consequence of (a). It therefore remains to prove (c).

Let us decompose our cusp form as $\mathbf{h} = \sum_i c_i\mathbf{h}_i\big|V_{\mathfrak{a}_i}$ where the c_i's are scalars, each $\mathbf{h}_i \in \mathcal{S}_k\big(\mathfrak{b}_i, \psi\big)$ is a primitive eigenform, and the ideals $\mathfrak{a}_i$, $\mathfrak{b}_i$ satisfy $\mathfrak{a}_i\mathfrak{b}_i\big|\mathfrak{c}\mathfrak{p}^{N-1}$. After some algebraic manipulation

$$\begin{aligned}
\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}, \mathbf{h}\big|V_{\mathfrak{p}}\big) &= \sum_i c_i\, \nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}, \mathbf{h}_i\big|V_{\mathfrak{a}_i}\big|V_{\mathfrak{p}}\big)\\
&= \sum_i \mathcal{N}\big(\mathfrak{p}^{N-m}\big)^{k/2-1} c_i \left\langle \mathbf{f}^\#,\ \mathrm{Tr}^{\mathfrak{c}\mathfrak{p}^N}_{\mathfrak{c}\mathfrak{p}^m}\big(\mathbf{h}_i\big|V_{\mathfrak{p}\mathfrak{a}_i}\big|J_{\mathfrak{c}\mathfrak{p}^N}\big)\right\rangle_{\mathfrak{c}\mathfrak{p}^m}\\
&= \sum_i \mathcal{N}\big(\mathfrak{p}^{N-m}\big)^{k/2-1} c_i \left\langle \mathbf{f}^\#,\ \mathbf{h}_i\big|V_{\mathfrak{p}\mathfrak{a}_i}\big|J_{\mathfrak{c}\mathfrak{p}^N}\right\rangle_{\mathfrak{c}\mathfrak{p}^N}\\
&= \sum_i \mathcal{N}\big(\mathfrak{p}^{N-m}\big)^{k/2-1} c_i\, (-1)^{[F:\mathbb{Q}]k}\left\langle \mathbf{f}^\#\big|J_{\mathfrak{c}\mathfrak{p}^N},\ \mathbf{h}_i\big|V_{\mathfrak{p}\mathfrak{a}_i}\right\rangle_{\mathfrak{c}\mathfrak{p}^N}\\
&= \sum_i \mathcal{N}\big(\mathfrak{p}^{N-m}\big)^{k/2-1} c_i\, (-1)^{[F:\mathbb{Q}]k}\left\langle \mathbf{f}^\#\big|J_{\mathfrak{c}\mathfrak{p}^N},\ \mathbf{h}_i\big|V_{\mathfrak{a}_i}\right\rangle_{\mathfrak{c}\mathfrak{p}^N} \times\ \Xi^{\mathfrak{p}\mathfrak{a}_i}_{\mathfrak{a}_i}(s, \mathbf{h}_i)\Big|_{s=k},
\end{aligned}$$

where the ratio of convolution L-functions

$$\Xi^{\mathfrak{p}\mathfrak{a}_i}_{\mathfrak{a}_i}(s, \mathbf{h}_i) \;:=\; \frac{L\big(s, \big(\mathbf{f}^\#\big|J_{\mathfrak{c}\mathfrak{p}^N}\big)^\#, \mathbf{h}_i\big|V_{\mathfrak{p}\mathfrak{a}_i}\big)}{L\big(s, \big(\mathbf{f}^\#\big|J_{\mathfrak{c}\mathfrak{p}^N}\big)^\#, \mathbf{h}_i\big|V_{\mathfrak{a}_i}\big)} \;=\; \frac{\overline{\alpha_{\mathfrak{p}}\big(\mathbf{f}^\#\big|J_{\mathfrak{c}\mathfrak{p}^N}\big)}}{\mathcal{N}(\mathfrak{p})^s} \;=\; \frac{\overline{\alpha_{\mathfrak{p}}\big(\mathbf{f}\big)}}{\mathcal{N}(\mathfrak{p})^s}\,.$$

Reversing each of the steps in our algebraic manipulations:

$$\begin{aligned}
\sum_i \mathcal{N}\big(\mathfrak{p}^{N-m}\big)^{k/2-1} c_i\, (-1)^{[F:\mathbb{Q}]k}\left\langle \mathbf{f}^\#\big|J_{\mathfrak{c}\mathfrak{p}^N}, \mathbf{h}_i\big|V_{\mathfrak{a}_i}\right\rangle_{\mathfrak{c}\mathfrak{p}^N} &= \ldots\\
= \sum_i c_i\, \nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}, \mathbf{h}_i\big|V_{\mathfrak{a}_i}\big) \;&=\; \nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathbf{f}, \mathbf{h}\big)
\end{aligned}$$

hence, part(c) follows immediately, and so does the lemma. □

Recall that we fixed the cusp form $\mathcal{G}_l \in \mathcal{S}_l\big(\mathfrak{c}\mathfrak{p}^n, \eta\big)$ at the very start of this section; in addition, one now insists that $\mathcal{G}_l = \mathbf{g}_l\big|V_{\mathfrak{p}}$ for some cusp form $\mathbf{g}_l \in \mathcal{S}_l\big(\mathfrak{c}\mathfrak{p}^{m-1}, \eta\big)$. We introduce a secondary modular form $\mathcal{F}_{k,\epsilon} \in \mathcal{S}_k\big(\mathfrak{c}\mathfrak{p}^m, \psi\omega_F^{2-k}\epsilon\big)$, such that
(i) $\mathcal{F}_{k,\epsilon}$ is a $\mathfrak{p}$-stablised newform;

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(ii) $\mathcal{F}_{k,\epsilon}$ is ordinary at $\mathfrak{p}$; and
(iii) the parallel weight k is strictly greater than l.

Our aim is to calculate the $\mathcal{F}_{k,\epsilon}$-isotypic projection of the family lifting $\mathrm{Hol}(\mathcal{G}\widetilde{\mathbf{E}})$ (note the specialisation of $\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}$ at $P_{k,\epsilon}$ is a classical HMF with $\tau_p(\overline{\mathbb{Q}})$-coefficients, and so it may be paired against $\mathcal{F}_{k,\epsilon}$ via the composition $\tau_\infty \circ \tau_p^{-1}$).

THEOREM 3.4. *If $\mathcal{H}^{\pm}_{\mathfrak{p}^n,r} = \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}(\mathcal{G}_l, \mathfrak{c}\mathfrak{p}^n, \psi) \in \mathcal{S}^{\mathrm{ord},\dagger}(\mathfrak{c}, \Lambda)$ are the Λ_F-adic HMFs described in Proposition 3.2, then for all $N > n$ and $r \in \{0, \dots, k-l-1\}$*

$$\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\,,\,P_{k,\epsilon}\big(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)\Big) = \Big(\mathcal{N}(\mathfrak{c}\mathfrak{p}^m)D_F^2\Big)^{s+1-\frac{k+l}{2}} \times \left(\frac{2^{-k}\times i^{k-1}}{(-2\pi i)^{1-l}}\right)^{[F:\mathbb{Q}]}$$
$$\times\frac{\Big(1-\psi\eta^{-1}\omega_F^{2-k}\epsilon(\mathfrak{p})\cdot\big|\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})\big|_\infty^2\cdot\mathcal{N}(\mathfrak{p})^{-(s+1+r)}\Big)}{\Big(1-\psi\eta^{-1}\omega_F^{2-k}\epsilon(\mathfrak{p})\cdot\mathcal{N}(\mathfrak{p})^{-(2s+2-k-l)}\Big)}$$
$$\times\,\mathcal{N}(\mathfrak{p})^{-l/2}\times\,\mathfrak{D}\Big(s,\ \mathcal{F}_{k,\epsilon}\Big|U_{\mathfrak{p}}^{N-m}\,,\ \mathbf{g}_l\Big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\Big)\Big|_{s=l+r},$$

where the sign $\pm$ is chosen to satisfy the parity condition $(-1)^{k-l-1} = \pm 1$.

Proof. Again abbreviating $\widetilde{\mathbf{E}}_{k-l}(r-(k-l-1), \psi\eta^{-1}\omega_F^{2-k}\epsilon; \mathfrak{c}\mathfrak{p}^n)$ simply by $\widetilde{\mathbf{E}}_{k-l}$, we know from Proposition 3.1 that $\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}(\mathcal{F}_{k,\epsilon}\,,P_{k,\epsilon}(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}))$ must coincide with $\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\,,\ \mathrm{Hol}\big(\mathbf{g}_l\big|V_{\mathfrak{p}}\times\widetilde{\mathbf{E}}_{k-l}\big)^{\mathrm{ord}}\Big)$. One now appeals to the following identity:

LEMMA 3.5. *If $\mathbf{g}_l \in \mathcal{S}_l(\mathfrak{c}\mathfrak{p}^{m-1}, \eta)$ and $\theta = \psi\eta^{-1}\omega_F^{2-k}\epsilon \neq \mathbf{1}$, then*

$$\mathrm{Hol}\big(\mathbf{g}_l\big|V_{\mathfrak{p}}\times\widetilde{\mathbf{E}}_{k-l}\big)^{\mathrm{ord}}$$
$$=\mathrm{Hol}\big(\mathbf{g}_l\big|V_{\mathfrak{p}}\times\mathbf{E}_{k-l}\big)^{\mathrm{ord}} - \theta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-l-1-r}\times\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)^{\mathrm{ord}}\Big|V_{\mathfrak{p}}\,.$$

(In order not to interrupt the exposition, we defer its proof to Appendix A.)
Observe that $\mathcal{G}_l = \mathbf{g}_l\big|V_{\mathfrak{p}}$ and the form $\mathcal{F}_{k,\epsilon}$ is $\mathfrak{p}$-ordinary; as a corollary

$$\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\,,P_{k,\epsilon}\big(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big)\Big) = \nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\,,\mathrm{Hol}\big(\mathcal{G}_l\times\mathbf{E}_{k-l}\big)\Big)$$
$$-\,\theta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-l-1-r}\times\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\,,\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)^{\mathrm{ord}}\Big|V_{\mathfrak{p}}\Big).$$

The computation neatly subdivides into two separate problems.

Case I – Calculating $\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\,,\mathrm{Hol}\big(\mathcal{G}_l\times\mathbf{E}_{k-l}\big)\Big)$:

First setting $\mathbf{h} := \mathcal{G}_l\cdot\mathbf{E}_{k-l} \in \mathcal{S}_k^{\infty}\big(\mathfrak{c}\mathfrak{p}^n, \psi\omega_F^{2-k}\epsilon\big)$, we have

$$\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\big(\mathcal{F}_{k,\epsilon}\,,\mathrm{Hol}(\mathbf{h})\big) = \mathcal{N}\big(\mathfrak{p}^{N-m}\big)^{k/2-1}\Big\langle\mathcal{F}^{\#}_{k,\epsilon}\,,\,\mathrm{Tr}^{\mathfrak{c}\mathfrak{p}^N}_{\mathfrak{c}\mathfrak{p}^m}\big(\mathrm{Hol}(\mathbf{h})\big|J_{\mathfrak{c}\mathfrak{p}^N}\big)\Big\rangle_{\mathfrak{c}\mathfrak{p}^m}$$
$$=\mathcal{N}\big(\mathfrak{p}^{N-m}\big)^{k/2-1}\big\langle\mathcal{F}^{\#}_{k,\epsilon}\,,\ \mathrm{Hol}(\mathbf{h})\big|J_{\mathfrak{c}\mathfrak{p}^N}\big\rangle_{\mathfrak{c}\mathfrak{p}^N}$$
$$=\mathcal{N}\big(\mathfrak{p}^{N-m}\big)^{k/2-1}\mathcal{N}\big(\mathfrak{p}^{N-n}\big)^{k/2}\big\langle\mathcal{F}^{\#}_{k,\epsilon}\,,\mathrm{Hol}(\mathbf{h})\big|J_{\mathfrak{c}\mathfrak{p}^n}\big|V_{\mathfrak{p}^{N-n}}\big\rangle_{\mathfrak{c}\mathfrak{p}^N}$$

upon using the standard identity $-\big|_k J_{\mathfrak{c}\mathfrak{p}^N} = \mathcal{N}\big(\mathfrak{p}^{N-n}\big)^{k/2}\times\big(-\big|_k J_{\mathfrak{c}\mathfrak{p}^n}\big|_k V_{\mathfrak{p}^{N-n}}\big)$.

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Decomposing $\mathrm{Hol}(\mathbf{h})\big|J_{\mathfrak{c}\mathfrak{p}^n} = \sum_i c_i \mathbf{h}_i\big|V_{\mathfrak{a}_i}$ into a finite sum of primitive eigenforms (c.f. the proof of Lemma 3.3), the exact same method shows that

$$\frac{\left\langle \mathcal{F}^{\#}_{k,\epsilon}\ ,\ \mathbf{h}_i\big|V_{\mathfrak{a}_i}\big|V_{\mathfrak{p}^{N-n}}\right\rangle_{\mathfrak{c}\mathfrak{p}^N}}{\left\langle \mathcal{F}^{\#}_{k,\epsilon}\ ,\ \mathbf{h}_i\big|V_{\mathfrak{a}_i}\right\rangle_{\mathfrak{c}\mathfrak{p}^N}} \;=\; \frac{L\big(s,\ \mathcal{F}_{k,\epsilon}\ ,\ \mathbf{h}_i\big|V_{\mathfrak{a}_i\mathfrak{p}^{N-n}}\big)}{L\big(s,\ \mathcal{F}_{k,\epsilon}\ ,\ \mathbf{h}_i\big|V_{\mathfrak{a}_i}\big)}\Bigg|_{s=k} \;=\; \frac{\alpha_{\mathfrak{p}}\big(\mathcal{F}_{k,\epsilon}\big)^{N-n}}{\mathcal{N}\big(\mathfrak{p}^{N-n}\big)^k}$$

and consequently,

$$\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}(\mathbf{h})\Big) \;=\; \mathcal{N}\big(\mathfrak{p}\big)^{m-N+(n-m)\frac{k}{2}} \times \Big\langle \mathcal{F}^{\#}_{k,\epsilon}\ \Big|U_{\mathfrak{p}}^{N-n},\ \mathrm{Hol}(\mathbf{h})\Big|J_{\mathfrak{c}\mathfrak{p}^n}\Big\rangle_{\mathfrak{c}\mathfrak{p}^N}.$$

REMARKS

(a) Recall from Section 1, the Eisenstein series $\mathbf{E}_{k-l},\ \mathbb{L}^0_{k-l},\ \mathbb{K}^0_{k-l}$ are related by

$$\begin{aligned}\mathbf{E}_{k-l}\big(s-k+1,\theta;\mathfrak{c}\mathfrak{p}^n\big)\Big|J_{\mathfrak{c}\mathfrak{p}^n} \;&=\; \Delta_F^{(1)}(s)\times\mathbb{L}^0_{k-l}\big(s-k+1;\ \mathcal{O}_F,\mathcal{O}_F;\ \theta\big)\Big|J_{\mathfrak{c}\mathfrak{p}^n}\\ &=\; \Delta_F^{(1)}(s)\times\mathcal{N}\big(\mathfrak{c}\mathfrak{p}^n\mathfrak{d}^2\big)^{s+1-\frac{k+l}{2}}\\ &\quad\times\mathbb{K}^0_{k-l}\big(s-k+1;\ \mathfrak{c}\mathfrak{p}^n,\mathcal{O}_F;\ \theta\big),\end{aligned}$$

where the complex factor $\Delta_F^{(1)}(s) = \frac{2^{-[F:\mathbb{Q}]}\times\sqrt{D_F}\times\Gamma(s-l+1)^{[F:\mathbb{Q}]}}{(-4\pi)^{-[F:\mathbb{Q}](s-k+1)}\times(-2\pi i)^{[F:\mathbb{Q}](2s+2-k-l)}}$.

(b) It follows that if $\theta=\psi\eta^{-1}\omega_F^{2-k}\epsilon\neq\mathbf{1}$, then

$$\begin{aligned}\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}(\mathbf{h})\Big) &= \mathcal{N}\big(\mathfrak{p}\big)^{m-N+(n-m)\frac{k}{2}}\times\Big\langle\mathcal{F}^{\#}_{k,\epsilon}\Big|U_{\mathfrak{p}}^{N-n},\ \mathbf{h}\Big|J_{\mathfrak{c}\mathfrak{p}^n}\Big\rangle_{\mathfrak{c}\mathfrak{p}^N}\\ &= \mathcal{N}\big(\mathfrak{p}\big)^{m-N+(n-m)\frac{k}{2}}\ \times\ \Delta_F^{(1)}(s)\ \times\ \mathcal{N}\big(\mathfrak{c}\mathfrak{p}^n\mathfrak{d}^2\big)^{s+1-\frac{k+l}{2}}\\ &\quad\times\Big\langle\mathcal{F}^{\#}_{k,\epsilon}\ \Big|U_{\mathfrak{p}}^{N-n},\ \mathcal{G}_l\Big|J_{\mathfrak{c}\mathfrak{p}^n}\cdot\mathbb{K}^0_{k-l}\big(s-k+1;\mathfrak{c}\mathfrak{p}^n,\mathcal{O}_F;\ \theta\big)\Big\rangle_{\mathfrak{c}\mathfrak{p}^N}\end{aligned}$$

with the R.H.S. evaluated at the critical point $s=l+r$.

(c) Lastly, the inner products at level $\mathfrak{c}\mathfrak{p}^N$ can be shrunk down to level $\mathfrak{c}\mathfrak{p}^n$ via

$$\begin{aligned}\Big\langle\Big(\mathcal{F}^{\#}_{k,\epsilon}\Big)_\lambda\Big|U_{\mathfrak{p}}^{N-n},\dots\Big\rangle_{\mathfrak{c}\mathfrak{p}^N} &= \frac{\nu_\infty\big(\Gamma_\lambda(\mathfrak{c}\mathfrak{p}^N)\backslash\mathfrak{h}^{[F:\mathbb{Q}]}\big)}{\nu_\infty\big(\Gamma_\lambda(\mathfrak{c}\mathfrak{p}^n)\backslash\mathfrak{h}^{[F:\mathbb{Q}]}\big)}\times\Big\langle\Big(\mathcal{F}^{\#}_{k,\epsilon}\Big)_\lambda\Big|U_{\mathfrak{p}}^{N-n},\dots\Big\rangle_{\mathfrak{c}\mathfrak{p}^n}\\ &= \mathcal{N}(\mathfrak{p})^{N-n}\times\Big\langle\Big(\mathcal{F}^{\#}_{k,\epsilon}\Big)_\lambda\Big|U_{\mathfrak{p}}^{N-n},\ \dots\Big\rangle_{\mathfrak{c}\mathfrak{p}^n}.\end{aligned}$$

Note the Haar measure [**18**, 2.31] of a fundamental domain for $\Gamma_\lambda(\mathfrak{n})$ is precisely

$$\nu_\infty\Big(\Gamma_\lambda(\mathfrak{n})\backslash\mathfrak{h}^{[F:\mathbb{Q}]}\Big) \;=\; 2\pi^{-[F:\mathbb{Q}]}D_F^{3/2}\zeta_F(2)\Big[\mathcal{O}^\times_{F,+}:\big(\mathcal{O}_F^\times\big)^2\Big]^{-1}\mathcal{N}(\mathfrak{n})\prod_{\mathfrak{q}|\mathfrak{n}}\big(1+\mathcal{N}(\mathfrak{q})^{-1}\big).$$

Conclusion: In summary, we have so far established

$$\begin{aligned}\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}(\mathbf{h})\Big) &= \mathcal{N}\big(\mathfrak{p}\big)^{(n-m)(\frac{k}{2}-1)}\times\Delta_F^{(1)}(l+r)\times\mathcal{N}\big(\mathfrak{c}\mathfrak{p}^n\mathfrak{d}^2\big)^{r+1-\frac{k-l}{2}}\\ &\quad\times\Big\langle\mathcal{F}^{\#}_{k,\epsilon}\ \Big|U_{\mathfrak{p}}^{N-n},\ \mathcal{G}_l\Big|J_{\mathfrak{c}\mathfrak{p}^n}\cdot\mathbb{K}^0_{k-l}\big(r-(k-l-1);\ \mathfrak{c}\mathfrak{p}^n,\mathcal{O}_F;\ \theta\big)\Big\rangle_{\mathfrak{c}\mathfrak{p}^n}\end{aligned}$$

which holds true for all integers r contained within the strip $0\le r\le k-l-1$.

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We now exploit the following three key identities:

(3.4.1) $\mathcal{G}_l\big|J_{\mathfrak{cp}^n} \;=\; \mathcal{N}\big(\mathfrak{p}^{n-m}\big)^{l/2} \times \mathcal{G}_l\big|J_{\mathfrak{cp}^m}\big|V_{\mathfrak{p}^{n-m}}$;

(3.4.2) $\mathfrak{D}^{(\mathfrak{c})}(s,\mathbf{f},\mathbf{g}) \;=\; \Delta_F^{(2)}(s) \times \Big\langle \mathbf{f}^{\#},\ \mathbf{g} \times \mathbb{K}^0_{k-l}\big(s-k+1;\ \mathfrak{cp}^n, \mathcal{O}_F;\ \theta\big)\Big\rangle_{\mathfrak{cp}^n}$

with the choice of Γ-factor $\Delta_F^{(2)}(s) = \sqrt{D_F} \times \Gamma(s-l+1)^{[F:\mathbb{Q}]} \times \pi^{-[F:\mathbb{Q}]s}$;

(3.4.3) For coprime $\mathcal{O}_F$-ideals $\mathfrak{a}$, $\mathfrak{b}$: $\mathfrak{D}\big(s,\mathbf{f}|V_{\mathfrak{a}},\mathbf{g}|V_{\mathfrak{b}}\big) \;=\; \mathcal{N}(\mathfrak{ab})^{-s}\cdot\mathfrak{D}\big(s,\mathbf{f}|U_{\mathfrak{b}},\mathbf{g}|U_{\mathfrak{a}}\big)$.

Applying the above sequentially,

$$\begin{aligned}
\nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}(\mathbf{h})\Big) &\overset{\text{by (3.4.1)}}{=} \mathcal{N}(\mathfrak{p})^{(n-m)(\frac{k+l}{2}-1)}\cdot\Delta_F^{(1)}\big(l+r\big)\cdot\mathcal{N}\big(\mathfrak{cp}^n\mathfrak{d}^2\big)^{r+1-\frac{k-l}{2}} \\
&\quad\times\Big\langle \mathcal{F}^{\#}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-n},\ \mathcal{G}_l\big|J_{\mathfrak{cp}^m}\big|V_{\mathfrak{p}^{n-m}}\cdot\mathbb{K}^0_{k-l}\big(r-(k-l-1);\ \mathfrak{cp}^n,\mathcal{O}_F;\ \theta\big)\Big\rangle_{\mathfrak{cp}^n} \\
&\overset{\text{by (3.4.2)}}{=} \mathcal{N}(\mathfrak{p})^{(n-m)(\frac{k+l}{2}-1)}\cdot\Delta_F^{(1)}\big(l+r\big)\cdot\mathcal{N}\big(\mathfrak{cp}^n\mathfrak{d}^2\big)^{r+1-\frac{k-l}{2}} \\
&\quad\times\Delta_F^{(2)}\big(l+r\big)^{-1}\cdot\mathfrak{D}^{(\mathfrak{c})}\Big(l+r,\ \mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-n},\ \mathcal{G}_l\big|J_{\mathfrak{cp}^m}\big|V_{\mathfrak{p}^{n-m}}\Big) \\
&\overset{\text{by (3.4.3)}}{=} \mathcal{N}(\mathfrak{p})^{(n-m)(\frac{k+l}{2}-1)}\cdot\Delta_F^{(1)}\big(l+r\big)\cdot\mathcal{N}\big(\mathfrak{cp}^n\mathfrak{d}^2\big)^{r+1-\frac{k-l}{2}} \\
&\quad\times\Delta_F^{(2)}\big(l+r\big)^{-1}\cdot\mathcal{N}\big(\mathfrak{p}^{n-m}\big)^{-(l+r)}\cdot\mathfrak{D}^{(\mathfrak{c})}\Big(l+r,\ \mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-m},\ \mathcal{G}_l\big|J_{\mathfrak{cp}^m}\Big).
\end{aligned}$$

Cleaning up these extraneous factors, one arrives at the formula

$$\begin{aligned}
&(3.4.4)\ \nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}\big(\mathcal{G}_l\times\mathbf{E}_{k-l}\big)\Big) \\
&\quad= \mathcal{N}\big(\mathfrak{cp}^m\mathfrak{d}^2\big)^{r+1-\frac{k-l}{2}}\times\frac{\Delta_F^{(1)}\big(l+r\big)}{\Delta_F^{(2)}\big(l+r\big)}\times\mathfrak{D}^{(\mathfrak{c})}\Big(l+r,\ \mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-m},\ \mathcal{G}_l\big|J_{\mathfrak{cp}^m}\Big).
\end{aligned}$$

Lastly, the fact that $\mathcal{G}_l\big|J_{\mathfrak{cp}^m} \;=\; \mathbf{g}_l\big|V_{\mathfrak{p}}\big|J_{\mathfrak{cp}^m} = \mathcal{N}(\mathfrak{p})^{-l/2}\times\mathbf{g}_l\big|J_{\mathfrak{cp}^{m-1}}$ implies

$$\begin{aligned}
&\nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}\big(\mathcal{G}_l\times\mathbf{E}_{k-l}\big)\Big) \\
&= \mathcal{N}\big(\mathfrak{cp}^m\mathfrak{d}^2\big)^{r+1-\frac{k-l}{2}}\cdot\frac{\Delta_F^{(1)}\big(l+r\big)}{\Delta_F^{(2)}\big(l+r\big)}\cdot\mathcal{N}(\mathfrak{p})^{-l/2}\cdot\mathfrak{D}^{(\mathfrak{c})}\Big(l+r,\mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-m},\mathbf{g}_l\big|J_{\mathfrak{cp}^{m-1}}\Big)
\end{aligned}$$

and the calculation in Case I is complete.

(To deal with the second part of the computation, we will apply some formulae derived in Lemma 3.3; this case is non-empty precisely when $\psi\eta^{-1}\omega_F^{2-k}\epsilon(\mathfrak{p}) \neq 0$.)

Case II – Calculating $\theta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-l-1-r}\times\nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)^{\mathrm{ord}}\Big|V_{\mathfrak{p}}\Big)$**:**
Exploiting the properties of our ∇-pairing,

$$\begin{aligned}
&\nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)^{\mathrm{ord}}\Big|V_{\mathfrak{p}}\Big) \\
&\qquad\overset{\text{by 3.3(c)}}{=} \mathcal{N}(\mathfrak{p})^{-k}\times\nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\Big|U_{\mathfrak{p}}^{*}\ ,\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)^{\mathrm{ord}}\Big) \\
&\qquad= \mathcal{N}(\mathfrak{p})^{-k}\ \overline{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}\ \times\nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)^{\mathrm{ord}}\Big) \\
&\qquad\overset{\text{as } \mathcal{F}_{k,\epsilon}\in\mathcal{S}_k^{\mathrm{ord}}}{=} \mathcal{N}(\mathfrak{p})^{-k}\ \overline{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}\ \times\nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)\Big).
\end{aligned}$$

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We can now reduce most of Case II to the same computation outlined in Case I, by exploiting equation (3.4.4) and replacing $\mathcal{G}_l$ with $\mathbf{g}_l$ throughout – this yields

$$\text{(3.4.4')} \qquad \nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)\Big)$$
$$= \mathcal{N}\big(\mathfrak{cp}^m\mathfrak{d}^2\big)^{r+1-\frac{k-l}{2}}\times\frac{\Delta_F^{(1)}\big(l+r\big)}{\Delta_F^{(2)}\big(l+r\big)}\times\mathfrak{D}^{(\mathrm{c})}\Big(l+r,\ \mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-m},\ \mathbf{g}_l\big|J_{\mathfrak{cp}^m}\Big).$$

On the other hand, $\mathbf{g}_l\big|J_{\mathfrak{cp}^m}=\mathcal{N}(\mathfrak{p})^{l/2}\times\mathbf{g}_l\big|J_{\mathfrak{cp}^{m-1}}\big|V_{\mathfrak{p}}$, whence

$$\mathfrak{D}^{(\mathrm{c})}\Big(s,\ \mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-m},\ \mathbf{g}_l\big|J_{\mathfrak{cp}^m}\Big)=\mathcal{N}(\mathfrak{p})^{l/2}\times\mathfrak{D}^{(\mathrm{c})}\Big(s,\ \mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-m},\ \mathbf{g}_l\big|J_{\mathfrak{cp}^{m-1}}\big|V_{\mathfrak{p}}\Big)$$
$$\overset{\text{by (2.4.3)}}{=}\ \mathcal{N}(\mathfrak{p})^{l/2}\cdot\frac{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}{\mathcal{N}(\mathfrak{p})^s}\times\mathfrak{D}^{(\mathrm{c})}\Big(s,\ \mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-m},\ \mathbf{g}_l\big|J_{\mathfrak{cp}^{m-1}}\Big)$$

and we therefore obtain

$$\nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)^{\mathrm{ord}}\Big|V_{\mathfrak{p}}\Big)=\mathcal{N}(\mathfrak{p})^{-k}\ \overline{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}\times\mathcal{N}(\mathfrak{p})^{l/2}\cdot\frac{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}{\mathcal{N}(\mathfrak{p})^{l+r}}$$
$$\times\,\mathcal{N}\big(\mathfrak{cp}^m\mathfrak{d}^2\big)^{r+1-\frac{k-l}{2}}\times\frac{\Delta_F^{(1)}(l+r)}{\Delta_F^{(2)}(l+r)}\times\mathfrak{D}^{(\mathrm{c})}\Big(l+r,\ \mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-m},\ \mathbf{g}_l\big|J_{\mathfrak{cp}^{m-1}}\Big)\ .$$

As a direct consequence, one easily deduces that

$$\theta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-l-1-r}\times\nabla_{\mathfrak{cp}^{N,m}}\Big(\mathcal{F}_{k,\epsilon}\ ,\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)^{\mathrm{ord}}\Big|V_{\mathfrak{p}}\Big)$$
$$=\ \theta^*(\mathfrak{p})\cdot\Big|\alpha_{\mathfrak{p}}\big(\mathcal{F}_{k,\epsilon}\big)\Big|_{\infty}^{2}\cdot\mathcal{N}(\mathfrak{p})^{-(l+1+2r)}\times\ \mathcal{N}\big(\mathfrak{cp}^m\mathfrak{d}^2\big)^{r+1-\frac{k-l}{2}}$$
$$\times\ \frac{\Delta_F^{(1)}(l+r)}{\Delta_F^{(2)}(l+r)}\times\ \mathcal{N}(\mathfrak{p})^{-l/2}\times\mathfrak{D}^{(\mathrm{c})}\Big(l+r,\ \mathcal{F}_{k,\epsilon}\ \big|U_{\mathfrak{p}}^{N-m},\ \mathbf{g}_l\big|J_{\mathfrak{cp}^{m-1}}\Big)$$

which means the calculation in Case II is also complete.

Finally, it is a tedious but straightforward exercise to verify that the ratio

$$\frac{\Delta_F^{(1)}(l+r)}{\Delta_F^{(2)}(l+r)}\ =\ \Big(2^{-(k-l+1)}\pi^{l-1}i^{k-l}\Big)^{[F:\mathbb{Q}]}\ =\ \left(\frac{2^{-k}\times i^{k-1}}{(-2\pi i)^{1-l}}\right)^{[F:\mathbb{Q}]}.$$

The theorem follows immediately upon combining Cases I and II together. $\square$

4. Constructing the 'Improved' p-adic L-function. The deformation theory ideas in **[7, 15]** (for $F=\mathbb{Q}$ and $F\subset\mathbb{R}$, respectively) carry over well to Rankin L-functions. We shall examine in detail some useful consequences of the $\mathfrak{p}$-stabilised newform $\mathcal{F}_{k,\epsilon}$ varying inside an analytic family.

Let us now assume the cusp form $\mathbf{g}_l$ is a primitive HMF of character η, of parallel weight $l>0$, and whose conductor $\mathfrak{n}(\mathbf{g}_l)$ has tame part $\mathfrak{n}(\mathbf{g}_l)'$, i.e. $\mathfrak{n}(\mathbf{g}_l)'+\mathfrak{p}=\mathcal{O}_F$. In addition, suppose that the Hecke polynomial of $\mathbf{g}_l$ at $\mathfrak{p}$ factorises (over $\mathbb{C}$) into

$$X^2-C(\mathfrak{p},\mathbf{g}_l)X+\eta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{l-1}\ =\ \big(X-\alpha_{\mathfrak{p}}(\mathbf{g}_l)\big)\times\big(X-\beta_{\mathfrak{p}}(\mathbf{g}_l)\big).$$

Following the notation of [**15**, Section 5.1], one defines

$$\mathbf{g}_l^0 \;:=\; \mathbf{g}_l - \beta_{\mathfrak{p}}(\mathbf{g}_l) \times \mathbf{g}_l \big| V_{\mathfrak{p}} \;\in\; \mathcal{S}_l\big(\mathfrak{n}(\mathbf{g}_l)\mathfrak{p}, \eta\big)$$

and similarly,

$$\mathbf{g}_l^{00} := \mathbf{g}_l^0 - \alpha_{\mathfrak{p}}(\mathbf{g}_l) \times \mathbf{g}_l^0 \big| V_{\mathfrak{p}} \;\in\; \mathcal{S}_l\big(\mathfrak{n}(\mathbf{g}_l)\mathfrak{p}^2, \eta\big)$$

(of course, these definitions depend on how we labelled the roots $\alpha_{\mathfrak{p}}\big(\mathbf{g}_l\big), \beta_{\mathfrak{p}}\big(\mathbf{g}_l\big)$).

LEMMA 4.1. *Considering the four Hilbert cusp forms* $\mathbf{g}_l^{00}, \; \mathbf{g}_l^0, \; \mathbf{g}_l, \; \mathbf{g}_l^{\#}$ *in tandem, their Rankin convolutions are related by the formulae*

$$\text{(a)}\quad \mathfrak{D}\Big(s,\; \mathcal{F}_{k,\epsilon}\big|U_{\mathfrak{p}}^{N-m},\; \mathbf{g}_l^{00}\big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\Big) = \left(1 - \frac{\alpha_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}\mathcal{N}(\mathfrak{p})^{s-l}\right) \times \mathfrak{D}\Big(s,\; \mathcal{F}_{k,\epsilon}\big|U_{\mathfrak{p}}^{N-m},\; \mathbf{g}_l^{0}\big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\Big);$$

$$\text{(b)}\quad \mathfrak{D}\Big(s,\; \mathcal{F}_{k,\epsilon}\big|U_{\mathfrak{p}}^{N-m},\; \mathbf{g}_l^{0}\big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\Big) = \left(1 - \frac{\beta_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}\mathcal{N}(\mathfrak{p})^{s-l}\right) \times \mathfrak{D}\Big(s,\; \mathcal{F}_{k,\epsilon}\big|U_{\mathfrak{p}}^{N-m},\; \mathbf{g}_l\big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\Big);$$

$$\text{(c)}\quad \mathfrak{D}\Big(s,\; \mathcal{F}_{k,\epsilon}\big|U_{\mathfrak{p}}^{N-m},\; \mathbf{g}_l\big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\Big) = \mathcal{N}\left(\frac{\mathfrak{c}\mathfrak{p}^{m-1}}{\mathfrak{n}(\mathbf{g}_l)}\right)^{\frac{l}{2}-s} \cdot \varpi\big(\mathbf{g}_l\big) \cdot C\big(\mathfrak{n}(\mathcal{F}_{k,\epsilon})', \mathcal{F}_{k,\epsilon}\big) \times \alpha_{\mathfrak{p}}\big(\mathcal{F}_{k,\epsilon}\big)^{N-1-ord_{\mathfrak{p}}\mathfrak{n}(\mathbf{g}_l)} \cdot \mathfrak{D}\Big(s,\; \mathcal{F}_{k,\epsilon},\; \mathbf{g}_l^{\#}\Big).$$

Here, we have written $\mathfrak{n}(\mathcal{F}_{k,\epsilon})'$ to denote the prime-to-$\mathfrak{p}$ part of the level of $\mathcal{F}_{k,\epsilon}$, and have fixed the $\mathcal{O}_F$-ideal $\mathfrak{c} := \mathfrak{n}(\mathcal{F}_{k,\epsilon})' \times \mathfrak{n}(\mathbf{g}_l)'$ which is clearly coprime to $\mathfrak{p}$.

REMARK. The two Euler factors occurring in (a) and (b) vanish at $s = l$, under the special circumstances that $\alpha_{\mathfrak{p}}(\mathbf{g}_l) = \alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})$ and $\beta_{\mathfrak{p}}(\mathbf{g}_l) = \alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})$ respectively. In particular, if $s = l = 1$ and $\mathcal{F}_{k,\epsilon}$ is the weight $k = 2$ newform associated to a split multiplicative elliptic curve defined over F, then we are in the precise situation covered by the generalised Mazur–Tate–Teitelbaum conjecture.

Indeed, the vanishing of these rather innocuous Euler factors (when viewed as rigid analytic functions over the weight-space), turns out to be the main building block in deriving the exceptional zero formula.

The cautious reader will have spotted if the quantity $C\big(\mathfrak{n}(\mathcal{F}_{k,\epsilon})', \mathcal{F}_{k,\epsilon}\big)$ is zero, then the above lemma represents none other than the formula '0 = 0' three times. To guard against this eventuality, let us henceforth assume

HYPOTHESIS (SS). The Fourier coefficients $C\big(\mathfrak{n}(\mathcal{F}_{k,\epsilon})', \mathcal{F}_{k,\epsilon}\big) \neq 0$.

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Proof of Lemma 4.1. To show (a), we exploit the following basic identities

$$\begin{aligned}\mathfrak{D}\big(s,\ -\ ,\mathbf{g}_l^{00}\big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\big) &= \mathfrak{D}\big(s,\ -\ ,\mathbf{g}_l^{0}\big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\big) - \alpha_{\mathfrak{p}}(\mathbf{g}_l)\cdot \mathfrak{D}\big(s,\ -\ ,\mathbf{g}_l^{0}\big|V_{\mathfrak{p}}\big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\big)\\ &= \mathcal{N}(\mathfrak{p})^{l/2}\cdot\mathfrak{D}\big(s,\ -\ ,\mathbf{g}_l^{0}\big|J_{\mathfrak{c}\mathfrak{p}^{m-2}}\big|V_{\mathfrak{p}}\big) - \alpha_{\mathfrak{p}}(\mathbf{g}_l)\mathcal{N}(\mathfrak{p})^{-l/2}\cdot\mathfrak{D}\big(s,\ -\ ,\mathbf{g}_l^{0}\big|J_{\mathfrak{c}\mathfrak{p}^{m-2}}\big)\\ &= \mathcal{N}(\mathfrak{p})^{l/2}\cdot\left(1-\frac{\alpha_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}\mathcal{N}(\mathfrak{p})^{s-l}\right)\cdot\mathfrak{D}\big(s,\ -\ ,\ \mathbf{g}_l^{0}\big|J_{\mathfrak{c}\mathfrak{p}^{m-2}}\big|V_{\mathfrak{p}}\big)\\ &= \left(1-\frac{\alpha_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}\mathcal{N}(\mathfrak{p})^{s-l}\right)\cdot\mathfrak{D}\big(s,\ -\ ,\ \mathbf{g}_l^{0}\big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\big)\end{aligned}$$

which can be deduced from equations (3.4.1) and (3.4.3). Part (b) follows similarly.

To establish statement (c), one already knows that

$$\begin{aligned}\mathbf{g}_l\Big|J_{\mathfrak{c}\mathfrak{p}^{m-1}} &= \mathcal{N}\left(\frac{\mathfrak{c}\mathfrak{p}^{m-1}}{\mathfrak{n}(\mathbf{g}_l)}\right)^{l/2}\times\mathbf{g}_l\Big|J_{\mathfrak{n}(\mathbf{g}_l)}\Big|V_{\frac{\mathfrak{c}\mathfrak{p}^{m-1}}{\mathfrak{n}(\mathbf{g}_l)}}\\ &= \mathcal{N}\left(\frac{\mathfrak{c}\mathfrak{p}^{m-1}}{\mathfrak{n}(\mathbf{g}_l)}\right)^{l/2}\cdot\varpi(\mathbf{g}_l)\times\mathbf{g}_l^{\#}\Big|V_{\frac{\mathfrak{c}\mathfrak{p}^{m-1}}{\mathfrak{n}(\mathbf{g}_l)}},\end{aligned}$$

where $\varpi(\mathbf{g}_l)$ denoted the pseudo-eigenvalue under Shimura's J-operator; however,

$$\begin{aligned}\mathfrak{D}\left(s,\ \mathcal{F}_{k,\epsilon}\big|U_{\mathfrak{p}}^{N-m},\mathbf{g}_l^{\#}\big|V_{\frac{\mathfrak{c}\mathfrak{p}^{m-1}}{\mathfrak{n}(\mathbf{g}_l)}}\right) &= \alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})^{m-1-\mathrm{ord}_{\mathfrak{p}}\mathfrak{n}(\mathbf{g}_l)}\times C\big(\mathfrak{n}(\mathcal{F}_{k,\epsilon})',\mathcal{F}_{k,\epsilon}\big)\\ &\quad\times\mathcal{N}\left(\frac{\mathfrak{c}\mathfrak{p}^{m-1}}{\mathfrak{n}(\mathbf{g}_l)}\right)^{-s}\times\ \mathfrak{D}\big(s,\ \mathcal{F}_{k,\epsilon}\big|U_{\mathfrak{p}}^{N-m},\mathbf{g}_l^{\#}\big)\end{aligned}$$

upon using (3.4.3) again, which implies (c) must also be true. □

We now shift attention to gluing this information together along points of $\mathrm{Spec}(\Lambda)$. Choose any finite extension $\mathbb{I}$ of the algebra Λ_F which is integrally closed in $\overline{\mathcal{Q}_{\Lambda_F}}$, and write $\mathcal{O}=\mathbb{I}\cap\overline{\mathbb{Q}}_p$. Let $\lambda_{\mathcal{F}}:\mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}),\mathcal{O}\big)\otimes_{\Lambda_F}\mathbb{I}\to\mathbb{I}$ denote the homomorphism corresponding to an $\mathbb{I}$-adic cusp form $\mathcal{F}\in\mathcal{S}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}),\mathbb{I}\big)$; in particular, one knows each specialisation $P(\mathcal{F})$ is classical at points $P\in\mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$ by applying [**10**, 4.21].

Moreover, if $\lambda_{\mathcal{F}}$ is a *primitive* homomorphism, then the localisation $\mathbb{I}_P$ is étale over $\Lambda_{F,P_{k,\epsilon}}$ where $P\big|_{\Lambda_F}=P_{k,\epsilon}$; in fact, every specialisation $P(\mathcal{F})$ will then be the ξ-expansion of a $\mathfrak{p}$-stabilised newform $\mathcal{F}_{k,\epsilon}=P(\mathcal{F})$ of tame level $\mathfrak{n}(\mathcal{F}_{k,\epsilon})'=\mathfrak{n}(\mathcal{F})$. Using [**9**, Corollary 3.7], one obtains a decomposition

$$\mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}),\mathcal{O}\big)\otimes_{\Lambda_F}\mathcal{Q}_{\mathbb{I}}\ =\ \mathcal{Q}_{\mathbb{I}}\oplus\mathcal{B}$$

into a direct sum of algebras, and projection to the first factor is induced by $\lambda_{\mathcal{F}}$. Under the diagonal mapping

$$\mathrm{diag}:\mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}),\mathcal{O}\big)\otimes_{\Lambda_F}\mathbb{I}\ \longrightarrow\ \mathbb{I}\oplus\mathrm{pr}_{\mathcal{B}}\Big(\mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}),\mathcal{O}\big)\otimes_{\Lambda_F}\mathbb{I}\Big),$$

the cokernel is (by definition) the congruence module '$C_0(\lambda_{\mathcal{F}})$' introduced in [**9**]; it detects mod P congruences between pairs $\lambda_{\mathcal{F}_1},\lambda_{\mathcal{F}_2}:\mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}),\mathcal{O}\big)\otimes_{\Lambda_F}\mathbb{I}\longrightarrow\mathbb{I}$. In a similar vein, the differential module $C_1(\lambda_{\mathcal{F}}):=\Omega^1_{\mathbf{h}^{\mathrm{ord}}/\mathbb{I}}\otimes_{\mathbf{h}^{\mathrm{ord}}}\mathbb{I}$ measures the failure of the component in $\mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}),\mathcal{O}\big)$ containing $\lambda_{\mathcal{F}}$, to be étale over Λ_F.

Hida has shown [**9**, Corollary 3.8] both $C_0(\lambda_{\mathcal{F}})$ and $C_1(\lambda_{\mathcal{F}})$ are torsion $\mathbb{I}$-modules, with the $\mathrm{supp}_{\mathbb{I}}(C_0(\lambda_{\mathcal{F}})) = \mathrm{supp}_{\mathbb{I}}(C_1(\lambda_{\mathcal{F}}))$. In addition, $\Omega^1_{\mathbf{h}^{\mathrm{ord}}/\mathbb{I}} \otimes_{\mathbf{h}^{\mathrm{ord}}} (\mathbb{I}/P) = 0$ at all points $P \in \mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$ since the specialisations $\mathbb{I}_P$ were étale over $\Lambda_{F,P_{k,\epsilon}}$. As a corollary, the idempotent cutting out the first factor in the decomposition $\mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}), \mathcal{O}\big) \otimes_{\Lambda_F} \mathcal{Q}_{\mathbb{I}} = \mathcal{Q}_{\mathbb{I}} \oplus \mathcal{B}$ can only have poles at non-arithmetic points.

Remark. Defining $\Sigma^{\mathfrak{c}}_{\mathfrak{n}(\mathcal{F})'} := \Big\{\text{ideal pairs } (\mathfrak{a}, \mathfrak{b}) \text{ with } \mathfrak{n}(\mathbf{g}_l)' \subset \mathfrak{ab} \text{ and } \mathfrak{b} \neq \mathfrak{n}(\mathbf{g}_l)'\Big\}$, clearly there is an injection

$$\pi = \oplus_{(\mathfrak{a},\mathfrak{b})} V_{\mathfrak{a}} : \bigoplus_{(\mathfrak{a},\mathfrak{b}) \in \Sigma^{\mathfrak{c}}_{\mathfrak{n}(\mathcal{F})'}} \mathcal{S}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F})\mathfrak{b}, \mathbb{I}\big) \hookrightarrow \text{ the old part of } \mathcal{S}^{\mathrm{ord}}\big(\mathfrak{c}, \mathbb{I}\big)$$

because the action of the $V_{\mathfrak{a}}$'s extends $\mathbb{I}$-adically (provided $\mathfrak{a}$ is coprime to $\mathfrak{p}\mathfrak{n}(\mathcal{F})'$). Under the isomorphism $\mathcal{S}^{\mathrm{ord}}\big(\mathfrak{c}, \mathbb{I}\big) \cong \mathrm{Hom}_{\Lambda_F}\Big(\mathbf{h}^{\mathrm{ord}}\big(\mathfrak{c}, \mathcal{O}\big), \mathbb{I}\Big)$, its dual π^* induces

$$\begin{array}{ccccc} & & & & \mathcal{Q}_{\mathbb{I}} \oplus \mathcal{B} \\ & & & & \Big\| \\ \mathbf{h}^{\mathrm{ord}}\big(\mathfrak{c}, \mathcal{O}\big) \otimes_{\Lambda_F} \mathcal{Q}_{\mathbb{I}} & \xrightarrow{\pi^*} & \displaystyle\bigoplus_{(\mathfrak{a},\mathfrak{b}) \in \Sigma^{\mathfrak{c}}_{\mathfrak{n}(\mathcal{F})'}} \mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F})\mathfrak{b}, \mathcal{O}\big) \otimes_{\Lambda_F} \mathcal{Q}_{\mathbb{I}} & \xrightarrow{\mathrm{proj}} & \mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}), \mathcal{O}\big) \otimes_{\Lambda_F} \mathcal{Q}_{\mathbb{I}}, \end{array}$$

where 'proj' maps down to the summand involving the ideal pair $(\mathfrak{a}, \mathfrak{b}) = (\mathcal{O}_F, \mathcal{O}_F)$. The composition $\lambda_{\mathcal{F}} \circ \mathrm{proj} \circ \pi^* : \mathbf{h}^{\mathrm{ord}}\big(\mathfrak{c}, \mathcal{O}\big) \otimes_{\Lambda_F} \mathcal{Q}_{\mathbb{I}} \longrightarrow \mathcal{Q}_{\mathbb{I}}$ factorises through a Hecke algebra at level $\mathfrak{n}(\mathcal{F}) \cdot \mathfrak{p}^\infty$, so is imprimitive as an $\mathbb{I}$-algebra homomorphism (in fact the $P(\mathcal{F})$'s will be oldforms at level $\mathfrak{c} = \mathfrak{n}(\mathcal{F})'\mathfrak{n}(\mathbf{g}_l)'$ because $\mathfrak{n}(\mathbf{g}_l)' \neq \mathcal{O}_F$).

The corresponding idempotent $t_{\mathcal{F}} \in \mathbf{h}^{\mathrm{ord}}\big(\mathfrak{c}, \mathcal{O}\big) \otimes_{\Lambda_F} \mathcal{Q}_{\mathbb{I}}$ cuts out the $\mathcal{F}$-isotypic component from the old part of $\mathcal{S}^{\mathrm{ord}}\big(\mathfrak{c}, \mathbb{I}\big)$, and has no poles lying along $\mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$. Extending the perfect pairing $[-,-]_{\mathfrak{c},\mathbb{I}} : \mathbf{h}^{\mathrm{ord}}(\mathfrak{c}, \mathcal{O}) \times \mathcal{S}^{\mathrm{ord}}(\mathfrak{c}, \mathbb{I}) \longrightarrow \mathbb{I}$ over $\mathrm{Frac}(\mathbb{I})$, we now determine the image of $\big[t_{\mathcal{F}}\, , \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\big]_{\mathfrak{c},\mathbb{I}} \in \mathcal{Q}_{\mathbb{I}}$ at its arithmetic specialisations.

Definition 4.2. For all algebraic points $P \in \mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$ satisfying $P\big|_{\Lambda_F} = P_{k,\epsilon}$, let us denote the $\mathbb{I}$-adic period associated to $\lambda_{\mathcal{F}}$ at P by

$$\Omega_{\mathbb{I}}\big(\lambda_{\mathcal{F}}, P\big) = \frac{C\big(\mathfrak{n}(\mathcal{F}_{k,\epsilon})', \mathcal{F}_{k,\epsilon}\big)}{D_F^k \cdot \big(2^k \cdot i^{1-k}\big)^{[F:\mathbb{Q}]}} \cdot \alpha_{\mathfrak{p}}\big(\mathcal{F}_{k,\epsilon}\big)^{m-1} \cdot \mathcal{N}\big(\mathfrak{p}^m\big)^{1-k/2} \cdot \frac{\big\langle \mathcal{F}_{k,\epsilon}\, , \mathcal{F}_{k,\epsilon}\big\rangle_{\mathfrak{n}(\mathcal{F}_{k,\epsilon})}}{\nabla_{\mathfrak{c}\mathfrak{p}^{m,m}}\big(\mathcal{F}_{k,\epsilon}\, , \mathcal{F}_{k,\epsilon}\big)},$$

where the eigenform $\mathcal{F}_{k,\epsilon} = P(\mathcal{F})$, and the integer $m \geq \mathrm{ord}_{\mathfrak{p}}\mathfrak{n}(\mathcal{F}_{k,\epsilon})$.

To see why this quantity is independent of m, it is enough to show for $j > 0$ that

$$\nabla_{\mathfrak{c}\mathfrak{p}^{m+j,m+j}}\big(\mathcal{F}_{k,\epsilon}\, , \mathcal{F}_{k,\epsilon}\big) = \mathcal{N}\big(\mathfrak{p}^j\big)^{1-k/2} \times \nabla_{\mathfrak{c}\mathfrak{p}^{m,m}}\big(\mathcal{F}_{k,\epsilon}\big| U_{\mathfrak{p}}^j\, , \mathcal{F}_{k,\epsilon}\big)$$

which we leave as an exercise for the reader.

The periods $\Omega_{\mathbb{I}}\big(\lambda_{\mathcal{F}}, P\big)$ themselves are algebraic numbers; they are error terms measuring discrepancies between the $\mathbb{I}$-adic deformation of the automorphic period, and the original value $\big\langle \mathcal{F}_{k,\epsilon}\, , \mathcal{F}_{k,\epsilon}\big\rangle$. Not only do they appear in the two-variable p-adic L-function (interpolating critical values of the whole family $\mathcal{F}$), but also in the one-variable L-functions interpolating $\mathcal{F}$ at a single fixed value $s = l + r$.

PROPOSITION 4.3. *If $P \in \mathrm{Spec}(\mathbb{I})^{alg}$ with $P\big|_{\Lambda_F} = P_{k,\epsilon}$ and $r \in \{0, \ldots, k-l-1\}$, then each specialisation $P([t_{\mathcal{F}}\ ,\ \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}(\mathbf{g}_l | V_{\mathfrak{p}}, \mathfrak{c}\mathfrak{p}^n, \psi)]_{\mathfrak{c},\mathbb{I}})$ is equal to*

$$\frac{\left(\mathcal{N}(\mathfrak{c}\mathfrak{p}^m) D_F^2\right)^{r+1-\frac{k-l}{2}}}{\mathcal{N}(\mathfrak{p})^{l/2}\mathcal{N}(\mathfrak{p}^m)^{1-k/2}} \times \frac{\left(1 - \psi\eta^{-1}\omega_F^{2-k}\epsilon(\mathfrak{p})\cdot\left|\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})\right|_\infty^2 \cdot \mathcal{N}(\mathfrak{p})^{-(l+2r+1)}\right)}{\left(1 - \psi\eta^{-1}\omega_F^{2-k}\epsilon(\mathfrak{p})\cdot\mathcal{N}(\mathfrak{p})^{-(l+2r+2-k)}\right)}$$

$$\times\ \alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})^{1-N} \times \frac{D_F^k\cdot\Omega_{\mathbb{I}}(\lambda_{\mathcal{F}}, P)}{C\left(\mathfrak{n}(\mathcal{F}_{k,\epsilon})', \mathcal{F}_{k,\epsilon}\right)} \times \frac{\mathfrak{D}\left(l+r,\ \mathcal{F}_{k,\epsilon}\Big|U_{\mathfrak{p}}^{N-m}\ ,\ \mathbf{g}_l\Big|J_{\mathfrak{c}\mathfrak{p}^{m-1}}\right)}{(-2\pi i)^{(1-l)[F:\mathbb{Q}]}\cdot\left\langle\mathcal{F}_{k,\epsilon}\ ,\mathcal{F}_{k,\epsilon}\right\rangle_{\mathfrak{n}(\mathcal{F}_{k,\epsilon})}},$$

where again the sign $\pm$ is chosen so that $(-1)^{k-l-1} = \pm 1$.

Proof. The majority of the hard work was done in Section 3 (in particular, Theorem 3.4). We first remark that the pairing $[t_{\mathcal{F}}\ , -]_{\mathfrak{c},\mathbb{I}}$ respects specialisation along $\mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$; namely, at such arithmetic primes P, one has $P([t_{\mathcal{F}}\ , \mathcal{H}]_{\mathfrak{c},\mathbb{I}}) = [t_{P(\mathcal{F})}\ , P(\mathcal{H})]_{\mathfrak{c}\mathfrak{p}^\infty,\mathcal{O}}$ under the $Q_{\mathcal{O}}$-extension of the duality

$$[-,-]_{\mathfrak{c}\mathfrak{p}^\infty,\mathcal{O}}:\ h_k^{\mathrm{ord}}(\mathfrak{c}\mathfrak{p}^\infty, \epsilon; \mathcal{O}) \times \mathcal{S}_k^{\mathrm{ord}}(\mathfrak{c}\mathfrak{p}^\infty, \epsilon; \mathcal{O}) \longrightarrow \mathcal{O}.$$

Here, $t_{P(\mathcal{F})} \in h_k^{\mathrm{ord}}(\mathfrak{c}\mathfrak{p}^\infty, \epsilon; Q_{\mathcal{O}}[\epsilon])$ denotes the Hecke idempotent which decomposes $h_k^{\mathrm{ord}}(\mathfrak{c}\mathfrak{p}^\infty, \epsilon; Q_{\mathcal{O}}[\epsilon]) = Q_{\mathcal{O}}[\epsilon] \oplus B$, where projection to the first factor is $\lambda_{\mathcal{F}_{k,\epsilon}} \circ \mathrm{proj}$. It follows directly that

$$P\left(\left[t_{\mathcal{F}}\ , \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right) = \left[t_{P(\mathcal{F})}\ , P_{k,\epsilon}(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r})\right]_{\mathfrak{c}\mathfrak{p}^\infty,\mathcal{O}} = \frac{\left\langle\mathcal{F}^{\#}_{k,\epsilon}\ , P_{k,\epsilon}(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r})\big|J_{\mathfrak{c}\mathfrak{p}^N}\right\rangle_{\mathfrak{c}\mathfrak{p}^N}}{\left\langle\mathcal{F}^{\#}_{k,\epsilon}\ , \mathcal{F}_{k,\epsilon}\big|J_{\mathfrak{c}\mathfrak{p}^N}\right\rangle_{\mathfrak{c}\mathfrak{p}^N}}$$

provided the integer N is chosen sufficiently large.

Ensuring $N > n \geq m \gg 1$, this last ratio can be rewritten in terms of ∇ via

$$P\left(\left[t_{\mathcal{F}}\ , \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right) = \frac{\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\left(\mathcal{F}_{k,\epsilon}\ , P_{k,\epsilon}(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r})\right)}{\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\left(\mathcal{F}_{k,\epsilon}\ , \mathcal{F}_{k,\epsilon}\right)} \overset{\text{by 3.3(b)}}{=} \frac{\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\left(\mathcal{F}_{k,\epsilon}\ , P_{k,\epsilon}(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r})\right)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})^{N-m}\cdot\nabla_{\mathfrak{c}\mathfrak{p}^{m,m}}\left(\mathcal{F}_{k,\epsilon}\ , \mathcal{F}_{k,\epsilon}\right)}.$$

Plugging in the Definition 4.2 (which relates $\nabla_{\mathfrak{c}\mathfrak{p}^{m,m}}(\mathcal{F}_{k,\epsilon}\ , \mathcal{F}_{k,\epsilon})$ with $\Omega_{\mathbb{I}}(\lambda_{\mathcal{F}}, P)$) then after some easy algebra, one deduces

$$P\left(\left[t_{\mathcal{F}}\ , \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right) = \frac{D_F^k\cdot\left(2^k\cdot i^{1-k}\right)^{[F:\mathbb{Q}]}}{C\left(\mathfrak{n}(\mathcal{F}_{k,\epsilon})', \mathcal{F}_{k,\epsilon}\right)} \times \alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})^{1-N}\cdot\mathcal{N}(\mathfrak{p}^m)^{k/2-1}$$
$$\times\frac{\Omega_{\mathbb{I}}(\lambda_{\mathcal{F}}, P)}{\left\langle\mathcal{F}_{k,\epsilon}\ , \mathcal{F}_{k,\epsilon}\right\rangle_{\mathfrak{n}(\mathcal{F}_{k,\epsilon})}} \times \nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}\left(\mathcal{F}_{k,\epsilon}\ , P_{k,\epsilon}(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r})\right).$$

Now using Theorem 3.4 to evaluate $\nabla_{\mathfrak{c}\mathfrak{p}^{N,m}}(\mathcal{F}_{k,\epsilon}\ , P_{k,\epsilon}(\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}))$, the result follows. $\square$

It makes sense to shorten some of our notations; we shall introduce

$$\mathcal{H}^{\pm}_{\mathfrak{p}^n,r} = \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\left(\mathbf{g}_l \middle| V_{\mathfrak{p}}, \mathfrak{c}\mathfrak{p}^n, \psi\right),$$
$$\mathcal{H}^{\pm,0}_{\mathfrak{p}^n,r} = \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\left(\mathbf{g}^0_l \middle| V_{\mathfrak{p}}, \mathfrak{c}\mathfrak{p}^n, \psi\right)$$
$$\text{and} \quad \mathcal{H}^{\pm,00}_{\mathfrak{p}^n,r} = \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\left(\mathbf{g}^{00}_l \middle| V_{\mathfrak{p}}, \mathfrak{c}\mathfrak{p}^n, \psi\right).$$

The following summarises the delicate effect switching amongst $\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}$, $\mathcal{H}^{\pm,0}_{\mathfrak{p}^n,r}$, $\mathcal{H}^{\pm,00}_{\mathfrak{p}^n,r}$ has on the specialisations of $[t_{\mathcal{F}}\ ,\ -]_{\mathfrak{c},\mathbb{I}}$ at arithmetic primes.

COROLLARY 4.4. *If $P \in \mathrm{Spec}(\mathbb{I})^{alg}$ with $P\big|_{\Lambda_F} = P_{k,\epsilon}$ and $(-1)^{k-l-1} = \pm 1$, then*

$$(a) \quad P\left(\left[t_{\mathcal{F}}\ ,\mathcal{H}^{\pm,00}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right) = \left(1 - \frac{\alpha_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}\,\mathcal{N}(\mathfrak{p})^r\right) \times P\left(\left[t_{\mathcal{F}}\ ,\mathcal{H}^{\pm,0}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right);$$

$$(b) \quad P\left(\left[t_{\mathcal{F}}\ ,\mathcal{H}^{\pm,0}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right) = \left(1 - \frac{\beta_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon})}\,\mathcal{N}(\mathfrak{p})^r\right) \times P\left(\left[t_{\mathcal{F}}\ ,\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right);$$

$$(c) \quad P\left(\left[t_{\mathcal{F}}\ ,\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right) = D_F^{2r+2+l} \cdot \frac{\mathcal{N}\big(\mathfrak{n}(\mathbf{g}_l)\big)^{l/2+r}\mathcal{N}(\mathfrak{p})^r}{\mathcal{N}(\mathfrak{c})^{k/2-1}} \cdot \frac{\varpi(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}\left(\mathcal{F}_{k,\epsilon}\right)^{\mathrm{ord}_{\mathfrak{p}}\mathfrak{n}(\mathbf{g}_l)}}$$
$$\times \frac{\left(1 - \psi\eta^{-1}\omega_F^{2-k}\epsilon(\mathfrak{p})\cdot\left|\alpha_{\mathfrak{p}}\left(\mathcal{F}_{k,\epsilon}\right)\right|^2_{\infty}\cdot\mathcal{N}(\mathfrak{p})^{-(l+2r+1)}\right)}{\left(1 - \psi\eta^{-1}\omega_F^{2-k}\epsilon(\mathfrak{p})\cdot\mathcal{N}(\mathfrak{p})^{-(l+2r+2-k)}\right)}$$
$$\times\ \Omega_{\mathbb{I}}\left(\lambda_{\mathcal{F}}, P\right) \times \frac{\mathfrak{D}\left(l+r,\ \mathcal{F}_{k,\epsilon}\ ,\ \mathbf{g}^{\#}_l\right)}{\left(-2\pi i\right)^{(1-l)[F:\mathbb{Q}]}\cdot\left\langle\mathcal{F}_{k,\epsilon}\ ,\mathcal{F}_{k,\epsilon}\right\rangle_{\mathfrak{n}(\mathcal{F}_{k,\epsilon})}}.$$

Proof. To show (c), one combines Proposition 4.3 with the formula in Lemma 4.1(c). To establish (b), one simply combines Proposition 4.3 for $\mathcal{H}^{\pm,0}_{\mathfrak{p}^n,r}$ with Lemma 4.1(b). Lastly, to prove (a), one combines Proposition 4.3 for $\mathcal{H}^{\pm,00}_{\mathfrak{p}^n,r}$ with Lemma 4.1(a). □

The final task in Section 4 is to interpret these results in a rigid-analytic context. A crucial feature of the deformation rings we have been considering is that their localisations at arithmetic primes tend to be unramified over the weight algebra. Consequently, in some local neighbourhood, they behave like affinoid $Q_{\mathcal{O}}$-algebras; the underlying Galois representations also interpolate seamlessly over such rings. Fix a base weight $k_0 \geq 2$ and character ϵ_0. Write $\mathcal{R}$ for the subring of $\overline{\mathbb{Q}}_p[\![w-k_0]\!]$ consisting of formal power series with positive radius of convergence about k_0.

The natural homomorphism $\mathcal{P}_{k_0,\epsilon_0} : \Lambda_F \longrightarrow \mathcal{R}$, sending an element $\langle[\mathfrak{l}]\rangle$ to the power series representing $w \mapsto \epsilon_0([\mathfrak{l}])\langle\mathfrak{l}\rangle_F^{w-k_0}$, extends uniquely to give a mapping $\widetilde{\mathcal{P}}_{k_0,\epsilon_0} : \mathbb{I}_P \longrightarrow \mathcal{R}$; this follows because $\mathcal{R}$ is Henselian, and the localisation $\mathbb{I}_P$ is étale over $\Lambda_{F,P_{k_0,\epsilon_0}}$ where P lies over P_{k_0,ϵ_0}. Hitting the ξ-expansion of $\mathcal{F}$ with the Mellin transform $\widetilde{\mathcal{P}}_{k_0,\epsilon_0}$, one may view its Fourier coefficients as rigid-analytic functions convergent on some closed p-adic disk '$\mathbb{U}_{k_0,\epsilon_0}$' (centred at the point $w = k_0$).

NOTATION. *We also write $\widetilde{\mathcal{F}}(w)$ for the image of the family $\mathcal{F}$ under $\widetilde{\mathcal{P}}_{k_0,\epsilon_0}$, so that*

$$\widetilde{\mathcal{F}}(k) = \mathcal{F}_{k,\epsilon_0} \ \in\ \mathcal{S}_k^{\mathrm{ord}}\left(\mathfrak{n}(\mathcal{F})\mathfrak{p}^{\infty}, \psi\omega_F^{2-k}\epsilon_0\right) \quad \textit{at all weights } k \in \mathbb{U}_{k_0,\epsilon_0} \cap \mathbb{Z}_{\geq 2},$$

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where ψ indicates the tame character associated to $\lambda_{\mathcal{F}} : \mathbf{h}^{\mathrm{ord}}(\mathfrak{n}(\mathcal{F}), \mathcal{O}) \otimes_{\Lambda_F} \mathbb{I} \longrightarrow \mathbb{I}$. Similarly, one denotes by $\alpha_{\mathfrak{p}}(\mathcal{F}, w)$ the power series interpolating $k \mapsto \alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon_0})$.

DEFINITION 4.5. For a fixed pair (k_0, ϵ_0) with $k_0 > l$, and choosing $\pm = (-1)^{k_0-l-1}$, let us define functions on $\mathbb{U}_{k_0,\epsilon_0}$ by

$$\text{(i)} \quad \mathbf{L}^{(00)}_{p,k_0,\epsilon_0}(\mathcal{F}, r) := D_F^{l-2} \cdot \mathcal{N}(\mathfrak{p})^{-r} \times \widetilde{\mathcal{P}}_{k_0,\epsilon_0}\left(\left[t_{\mathcal{F}}\, , \mathcal{H}^{\pm,00}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right);$$

$$\text{(ii)} \quad \mathbf{L}^{(0)}_{p,k_0,\epsilon_0}(\mathcal{F}, r) := D_F^{l-2} \cdot \mathcal{N}(\mathfrak{p})^{-r} \times \widetilde{\mathcal{P}}_{k_0,\epsilon_0}\left(\left[t_{\mathcal{F}}\, , \mathcal{H}^{\pm,0}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right);$$

$$\text{(iii)} \quad \mathbf{L}^{\mathrm{imp}}_{p,k_0,\epsilon_0}(\mathcal{F}, r) := D_F^{l-2} \cdot \mathcal{N}(\mathfrak{p})^{-r} \times \widetilde{\mathcal{P}}_{k_0,\epsilon_0}\left(\left[t_{\mathcal{F}}\, , \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\right]_{\mathfrak{c},\mathbb{I}}\right).$$

Note if $r \neq 0$, then $\mathbf{L}^{(00)}_{p,k_0,\epsilon_0}$, $\mathbf{L}^{(0)}_{p,k_0,\epsilon_0}$ and $\mathbf{L}^{\mathrm{imp}}_{p,k_0,\epsilon_0}$ are analytic functions on $\mathbb{U}_{k_0,\epsilon_0}$ as each of $\mathcal{H}^{\pm,00}_{\mathfrak{p}^n,r}$, $\mathcal{H}^{\pm,0}_{\mathfrak{p}^n,r}$, $\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}$ live in the space of Λ-adic cusp forms $\mathcal{S}^{\mathrm{ord}}(\mathfrak{c}, \Lambda_F)$. However, if $r = 0$, then $\mathbf{L}^{(0)}_{p,k_0,\epsilon_0}(\mathcal{F}, 0)$ and $\mathbf{L}^{\mathrm{imp}}_{p,k_0,\epsilon_0}(\mathcal{F}, 0)$ might inherit simple poles, due to the constant term

$$2^{-[F:\mathbb{Q}]} \times \mathrm{Tw}_{2-l}\left(\zeta^{(\mathfrak{c})}_{F,p\text{-adic}}\left(\psi\eta^{-1}\omega_F^{-l}\right)\right)$$

associated to the Λ-adic family $\{\widetilde{\mathbf{E}}_{k-l}(0, \theta; \mathfrak{c}\mathfrak{p}^n)\}_{k \in \mathbb{U}_{k_0,\epsilon_0}}$. Since the zeta-functions $\zeta_F^{(\mathfrak{c})}(1 - (k-l), \theta_{\mathfrak{p}})$ can only exhibit a pole where $k - l = 0$ and we know $k_0 > l$, there exists a (possibly smaller) neighbourhood $\mathbb{U}^{\star}_{k_0,\epsilon_0} \subset \mathbb{U}_{k_0,\epsilon_0}$ upon which each of the functions $\mathbf{L}^{(0)}_{p,k_0,\epsilon_0}(\mathcal{F}, 0)$ and $\mathbf{L}^{\mathrm{imp}}_{p,k_0,\epsilon_0}(\mathcal{F}, 0)$ is rigid-analytic.

(Observe the function $\mathbf{L}^{(00)}_{p,k_0,\epsilon_0}(\mathcal{F}, r)$ is rigid-analytic on $\mathbb{U}_{k_0,\epsilon_0}$ even if r equals 0; in fact, the additional term $2^{-[F:\mathbb{Q}]} \cdot \zeta_F^{(\mathfrak{c})}(1 - (k-l), \theta_{\mathfrak{p}}) \times C(\xi\widetilde{t}_\lambda^{-1}, (\mathbf{g}_l^{00}|V_{\mathfrak{p}})^{\mathrm{ord}})$ in the ξ-expansion of $\mathcal{H}^{\pm,00}_{\mathfrak{p}^n,0}$ is identically zero, because $\mathbf{g}_l^{00}|V_{\mathfrak{p}}$ is killed by the $U_{\mathfrak{p}}$-operator whence $(\mathbf{g}_l^{00}|V_{\mathfrak{p}})^{\mathrm{ord}} = 0$.)

THEOREM 4.6.

(a) *If $\alpha_{\mathfrak{p}}(\mathbf{g}_l) \neq \alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})$ and $\beta_{\mathfrak{p}}(\mathbf{g}_l) \neq \alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})$, then*

$$\mathbf{L}^{(00)}_{p,k_0,\epsilon_0}(\mathcal{F}, 0)\Big|_{w=k_0} = \left(1 - \frac{\alpha_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}\right) \cdot \left(1 - \frac{\beta_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}\right) \times \mathbf{L}^{\mathrm{imp}}_{p,k_0,\epsilon_0}(\mathcal{F}, 0)\Big|_{w=k_0};$$

(b) *if $\alpha_{\mathfrak{p}}(\mathbf{g}_l) = \alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})$ but $\beta_{\mathfrak{p}}(\mathbf{g}_l) \neq \alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})$, then as $w \to k_0$ inside of $\mathbb{U}_{k_0,\epsilon_0}$, the* $\mathrm{order}_{w=k_0}\left(\mathbf{L}^{(00)}_{p,k_0,\epsilon_0}(\mathcal{F}, 0)\right) \geq 1$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}w}\mathbf{L}^{(00)}_{p,k_0,\epsilon_0}(\mathcal{F}, 0)\Big|_{w=k_0} = \frac{\alpha'_{\mathfrak{p}}(\mathcal{F}, k_0)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})} \times \left(1 - \frac{\beta_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}\right) \times \mathbf{L}^{\mathrm{imp}}_{p,k_0,\epsilon_0}(\mathcal{F}, 0)\Big|_{w=k_0};$$

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(c) *if both* $\alpha_{\mathfrak{p}}(\mathbf{g}_l)=\beta_{\mathfrak{p}}(\mathbf{g}_l)=\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})$*, then again allowing* $w\to k_0$ *inside* $\mathbb{U}_{k_0,\epsilon_0}$*, the* $\mathrm{order}_{w=k_0}\left(\mathbf{L}^{(00)}_{p,k_0,\epsilon_0}(\mathcal{F},0)\right)\geq 2$ *and*

$$\frac{\mathrm{d}^2}{\mathrm{d}w^2}\mathbf{L}^{(00)}_{p,k_0,\epsilon_0}(\mathcal{F},0)\bigg|_{w=k_0} \;=\; 2\cdot\left(\frac{\alpha'_{\mathfrak{p}}(\mathcal{F},k_0)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}\right)^2\times\mathbf{L}^{\mathrm{imp}}_{p,k_0,\epsilon_0}(\mathcal{F},0)\bigg|_{w=k_0}.$$

Proof. By the first two parts in Corollary 4.4, for $P\in\mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$ with $P\big|_{\Lambda_F}=P_{k_0,\epsilon_0}$:

$$P\Big(\big[t_{\mathcal{F}}\,,\mathcal{H}^{\pm,00}_{\mathfrak{p}^n,0}\big]_{\mathfrak{c},\mathbb{I}}\Big) \;=\; \left(1-\frac{\alpha_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}\right)\cdot\left(1-\frac{\beta_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}\right)\times P\Big(\big[t_{\mathcal{F}}\,,\mathcal{H}^{\pm}_{\mathfrak{p}^n,0}\big]_{\mathfrak{c},\mathbb{I}}\Big).$$

Thus, assertion (a) follows immediately from Definition 4.5.

To establish statement (b), if we restrict to p-adic weights $w\in\mathbb{U}^{\star}_{k_0,\epsilon_0}$ then

$$\widetilde{\mathcal{P}}_{k_0,\epsilon_0}\Big(\big[t_{\mathcal{F}}\,,\mathcal{H}^{\pm,00}_{\mathfrak{p}^n,0}\big]_{\mathfrak{c},\mathbb{I}}\Big)\;\overset{\text{by }4.4(\text{a})}{=}\;\left(1-\frac{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}{\alpha_{\mathfrak{p}}(\mathcal{F},w)}\right)\times\widetilde{\mathcal{P}}_{k_0,\epsilon_0}\Big(\big[t_{\mathcal{F}}\,,\mathcal{H}^{\pm,0}_{\mathfrak{p}^n,0}\big]_{\mathfrak{c},\mathbb{I}}\Big).$$

Furthermore, the analytic function $\Upsilon(w)=1-\frac{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}{\alpha_{\mathfrak{p}}(\mathcal{F},w)}$ admits the Taylor series

$$\Upsilon(w)\;=\;0\times\big(w-k_0\big)^0\;+\;\frac{\alpha'_{\mathfrak{p}}(\mathcal{F},k_0)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}\times\big(w-k_0\big)^1\;+\;O\Big(\big(w-k_0\big)^2\Big);$$

since $\Upsilon(w)$ has a zero of order ≥ 1 at $w=k_0$, the second part of our theorem is a direct consequence of the value for the linear term above.

Finally, to show (c) once more, we restrict to those $w\in\mathbb{U}^{\star}_{k_0,\epsilon_0}$, in which case

$$\widetilde{\mathcal{P}}_{k_0,\epsilon_0}\Big(\big[t_{\mathcal{F}}\,,\mathcal{H}^{\pm,00}_{\mathfrak{p}^n,0}\big]_{\mathfrak{c},\mathbb{I}}\Big)\;\overset{\text{by }4.4(\text{a,b})}{=}\;\left(1-\frac{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}{\alpha_{\mathfrak{p}}(\mathcal{F},w)}\right)^2\times\widetilde{\mathcal{P}}_{k_0,\epsilon_0}\Big(\big[t_{\mathcal{F}}\,,\mathcal{H}^{\pm}_{\mathfrak{p}^n,0}\big]_{\mathfrak{c},\mathbb{I}}\Big).$$

Clearly, $\Upsilon(w)^2$ has a zero of order ≥ 2 at $w=k_0$ with quadratic term $\left(\frac{\alpha'_{\mathfrak{p}}(\mathcal{F},k_0)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k_0,\epsilon_0})}\right)^2$, and the truth of assertion (c) is now evident. □

5. Measures associated to two-variable deformations. We consider the same scenario as the previous two sections. Again, the primitive homomorphism $\lambda_{\mathcal{F}}:\mathbf{h}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}),\mathcal{O}\big)\otimes_{\Lambda_F}\mathbb{I}\to\mathbb{I}$ corresponds to an $\mathbb{I}$-adic cusp form $\mathcal{F}\in\mathcal{S}^{\mathrm{ord}}\big(\mathfrak{n}(\mathcal{F}),\mathbb{I}\big)$, and $P\in\mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$ ranges over the arithmetic specialisations. In due course, we will insist that the primitive cusp form $\mathbf{g}_l$ has parallel weight one; however, for the initial discussion we can allow $\mathbf{g}_l$ to be of positive weight $(l,\ldots,l)$.

It is worthwhile to remind the reader of the running assumption we made:

HYPOTHESIS (SS). The Fourier coefficients $C\big(\mathfrak{n}(\mathcal{F}),P(\mathcal{F})\big)$ are all non-zero.

In particular, the periods $\Omega_{\mathbb{I}}\big(\lambda_{\mathcal{F}},P\big)$ are non-vanishing at every $P\in\mathrm{Spec}(\mathbb{I})^{\mathrm{alg}}$.

In Section 4, we showed that pairing the idempotent $t_{\mathcal{F}}$ with the family $\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}$ produced a rigid analytic function, convergent on some p-adic neighbourhood $\mathbb{U}_{k_0,\epsilon_0}$ of $w=k_0$. We now wish to introduce a second cyclotomic variable, 's' say, to the deformation; the resulting two-variable power series will be analytic for all

$(s, w) \in \mathbb{Z}_p \times \mathbb{U}_{k_0,\epsilon_0}$, and will satisfy a functional equation along its line of symmetry $s = w/2$.

REMARKS.

(i) Those elements $\chi \in \mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Gal}(F^{\mathrm{ab}}/F), \mathbb{C}_p^\times\big)$ of finite order can be identified with Hecke characters over F, via the sequence of maps

$$\mathbb{A}_F^\times \xrightarrow{\mathrm{CFT}} \mathrm{Gal}(F^{\mathrm{ab}}/F) \xrightarrow{\chi} \tau_p\big(\overline{\mathbb{Q}}^\times\big) \xrightarrow{\tau_p^{-1}} \overline{\mathbb{Q}}^\times \overset{\tau_\infty}{\hookrightarrow} \mathbb{C}^\times,$$

where the symbol 'CFT' denotes the homomorphism of global class field theory. We shall frequently jump between finite order Hecke characters χ, and elements of the dual group $\mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Gal}(F^{\mathrm{ab}}/F), \mathbb{C}_p^\times\big)_{\mathrm{tors}}$, without any change in notation.

(ii) The maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$ is certainly contained inside of F^{ab}; one denotes by $\mathcal{N}x_p \in \mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Gal}(F^{\mathrm{ab}}/F), \mathbb{C}_p^\times\big)$ the mapping induced by

$$\mathrm{Gal}(F^{\mathrm{ab}}/F) \overset{\mathrm{rest}}{\twoheadrightarrow} \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/F) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \prod_{\text{primes } l} \mathbb{Z}_l^\times \twoheadrightarrow \mathbb{Z}_p^\times \hookrightarrow \mathbb{C}_p^\times.$$

(iii) The special characters of the form $\chi \cdot \mathcal{N}x_p^j$ where χ has finite order and $j \in \mathbb{Z}$, are p-adically dense inside $\mathrm{Hom}_{\mathrm{cont}}\big(\mathrm{Gal}(F^{\mathrm{ab}}/F), \mathbb{C}_p^\times\big)$.

Rather than considering all characters of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$, let us instead restrict to the cyclotomic part $F_\infty := \bigcup_{n\geq 1} F(\mu_{p^n})$. Its Galois group is an open subgroup of $\mathbb{Z}_p^\times$,

$$\text{i.e.} \quad \mathcal{X}_{p,F} := \mathrm{Gal}\big(F_\infty/F\big) \cong \mathrm{Gal}\big(F_\infty/F\big)_{\mathrm{tors}} \times \big(1 + p\mathbb{Z}_p\big)$$

thence, the torsion subgroup of $\mathcal{X}_{p,F}$ above is finite, and of cardinality dividing $p-1$ (finite order characters of $\mathcal{X}_{p,F}$ correspond to characters of the ideal group over F whose conductors are powers of $\mathfrak{p} = p \cdot \mathcal{O}_F$).

DEFINITION 5.1. For $r \geq 0$, we define $\mathcal{S}^{\mathrm{ord}}\big(\mathfrak{c}, \Lambda\big)$-valued distributions $\mathrm{d}\mu^\pm = \mathrm{d}\mu^\pm_{\mathbf{g}_l,r}$ on the p-adic Lie group $\mathcal{X}_{p,F}$ by

$$\int_{x\in\mathcal{X}_{p,F}} \chi(x) \cdot \mathrm{d}\mu^\pm_{\mathbf{g}_l,r}(x) := \mathcal{H}^\pm_{\mathfrak{p}^n,r}\Big(\big(\mathbf{g}_l^{00} \otimes \chi\big)\big| V_{\mathfrak{p}}, \mathfrak{c}\mathfrak{p}^n, \psi\Big),$$

where the integer $n \gg 0$, and the homomorphisms $\chi \in \mathrm{Hom}\big(\mathcal{X}_{p,F}, \mathbb{C}_p^\times\big)_{\mathrm{tors}}$.

PROPOSITION 5.2.

(a) *Each distribution* $\mathrm{d}\mu^\pm_{\mathbf{g}_l,r}$ *is a bounded measure on* $\mathcal{X}_{p,F}$*;*

(b) *the measures* $\mathrm{d}\mu^\pm_{\mathbf{g}_l,r}$ *are uniquely determined by the data in Definition 5.1;*

(c) *for all* $r \geq 0$*, we have relations* $\mathrm{d}\mu^\pm_{\mathbf{g}_l,r} = (-1)^{r[F:\mathbb{Q}]}\mathcal{N}(\mathfrak{p})^r \times \big(\mathcal{N}x_p\big)^r \cdot \mathrm{d}\mu^\pm_{\mathbf{g}_l,0}$.

Proof. Let us begin with some brief observations, concerning the twist of the cusp form $\mathbf{g}_l^{00}$ by an element $\chi \in \mathrm{Hom}_{\mathrm{cont}}\big(\mathcal{X}_{p,F}, \mathbb{C}_p^\times\big)_{\mathrm{tors}}$ of conductor $\mathfrak{p}^{\mathfrak{f}_\chi}$, say.

(5.2.1) The (twisted) Hecke eigenform $\mathbf{g}_l^{00} \otimes \chi$ is killed by the $U_{\mathfrak{p}}$-operator;

(5.2.2) $\big(\mathbf{g}_l^{00} \otimes \chi\big)\big|V_{\mathfrak{p}}$ belongs to $\mathcal{S}_l\big(\mathfrak{n}(\mathbf{g}_l)\mathfrak{p}^{\max\{2\mathfrak{f}_\chi,2\}}, \eta\chi^2\big)\Big|\big(1 - \lim_{N\to\infty} U_{\mathfrak{p}}^{N!}\big)$;

(5.2.3) $C\big(\xi\widetilde{t}_\lambda^{-1}, \mathbf{g}_l^{00} \otimes \chi\big) = 0$ for all elements $\xi \gg 0$ such that $\xi \in \mathfrak{p}$.

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It follows that if the integer $r=0$, there is no need to adjust the ξ-expansion of $\mathcal{H}^{\pm}_{\mathfrak{p}^n,0}((\mathbf{g}_l^{00}\otimes\chi)|V_{\mathfrak{p}},\mathfrak{c}\mathfrak{p}^n,\psi)$ by the additional factor

$$\mathcal{N}(\tilde{t}_\lambda)^{-1}\times 2^{-[F:\mathbb{Q}]}\cdot \mathrm{Tw}_{2-l}\Big(\zeta^{(\mathfrak{c})}_{F,p\text{-adic}}\big(\psi\eta^{-1}\omega_F^{-l}\chi^{-2}\big)\Big)\times C\Big(\xi\widetilde{t}_\lambda^{-1},\,(\mathbf{g}_l^{00}\otimes\chi\,|\,V_{\mathfrak{p}})^{\mathrm{ord}}\Big)$$

because by (5.2.2) above, this term is identically zero anyway.

We now abbreviate the Λ-adic form $\mathcal{H}^{\pm}_{\mathfrak{p}^n,r}((\mathbf{g}_l^{00}\otimes\chi)|V_{\mathfrak{p}},\mathfrak{c}\mathfrak{p}^n,\psi)$ with $\mathcal{H}^{\pm,(\chi)}_{\mathfrak{p}^n,r}$. Quoting verbatim from its definition (c.f. the demonstration of Proposition 3.1),

$$\Big(\mathcal{H}^{\pm,(\chi)}_{\mathfrak{p}^n,r}\Big)_\lambda\;:=\;(-1)^{r[F:\mathbb{Q}]}\times\sum_{0\ll\xi\in\widetilde{t}_\lambda}\lim_{N\to\infty}\Big(C^{N!,\pm}\big(\xi\widetilde{t}_\lambda^{-1},\mathcal{H}^{\pm,(\chi)}_{\mathfrak{p}^n,r}\big)_\lambda\Big)\times\mathbf{e}_F(\xi z)$$

at all components $\lambda\in\{1,\ldots,\hat{h}_F\}$, where each ξ-coefficient was the Λ-adic limit of

$$C^{N!,\pm}\Big(\xi\widetilde{t}_\lambda^{-1},\mathcal{H}^{\pm,(\chi)}_{\mathfrak{p}^n,r}\Big)_\lambda\;=\;\mathcal{N}(\widetilde{t}_\lambda)^{-1}\;\times\textstyle\sum_{\mathfrak{p}^{N!}\xi=\xi_1+\xi_2}C\Big(\xi_1\widetilde{t}_\lambda^{-1},\,(\mathbf{g}_l^{00}\otimes\chi)\,\big|\,V_{\mathfrak{p}}\Big)$$
$$\times\sum_{\substack{\widetilde{\xi}_2=\widetilde{b}\times\widetilde{c},\;b\in\widetilde{t}_\lambda\\ c\in\mathcal{O}_F-\mathfrak{p}}}\Big((-1)^{[F:\mathbb{Q}]}\mathrm{sign}\mathcal{N}(\widetilde{b})\Big)^{\frac{1\mp1}{2}}\mathcal{N}\big(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1}\big)^r\times\big(\psi\eta^{-1}\chi^{-2}\big)^*(\widetilde{c})\,\mathcal{N}(\widetilde{c})^{1-l}\times\langle[\widetilde{c}]\rangle.$$

To prove 5.2(a)–(c), it is enough to prove analogous statements for the distributions associated to each coefficient $C^{N!,\pm}(\xi\widetilde{t}_\lambda^{-1},\mathcal{H}^{\pm,(\chi)}_{\mathfrak{p}^n,r})$, then take the limit as $N\to\infty$. The following facts are deduced from (5.2.1–3), and also from the rules of twisting:

(5.2.4) if $\chi\neq\mathbf{1}$, then $\mathbf{g}_l^{00}\otimes\chi$ coincides with $\mathbf{g}_l\otimes\chi$;

(5.2.5) if $\chi\neq\mathbf{1}$, then $C\big(\xi\widetilde{t}_\lambda^{-1},(\mathbf{g}_l^{00}\otimes\chi)\big|V_{\mathfrak{p}}\big)=\chi^*\big(\mathfrak{p}^{-1}\xi\widetilde{t}_\lambda^{-1}\big)\cdot C\big(\mathfrak{p}^{-1}\xi\widetilde{t}_\lambda^{-1},\mathbf{g}_l\big)$;

(5.2.6) if $\chi=\mathbf{1}$, then $C\big(\xi\widetilde{t}_\lambda^{-1},(\mathbf{g}_l^{00}\otimes\mathbf{1})\big|V_{\mathfrak{p}}\big)=\begin{cases}C\big(\mathfrak{p}^{-1}\xi\widetilde{t}_\lambda^{-1},\mathbf{g}_l\big) & \text{if }\xi\in\mathfrak{p},\ \xi\notin\mathfrak{p}^2\\ 0 & \text{otherwise.}\end{cases}$

For instance, one can easily see that $C\big(\xi_1\widetilde{t}_\lambda^{-1},(\mathbf{g}_l^{00}\otimes\chi)\big|V_{\mathfrak{p}}\big)=0$ unless $\xi_1\in\mathfrak{p}-\mathfrak{p}^2$, irrespective of whether χ is the trivial character or not. As a consequence, most of the terms we are summing over in the expression defining $C^{N!,\pm}\big(\xi\widetilde{t}_\lambda^{-1},\mathcal{H}^{\pm,(\chi)}_{\mathfrak{p}^n,r}\big)_\lambda$ are equal to zero; in fact, we only need to sum over totally positive $\xi_1,\xi_2\in\mathfrak{p}-\mathfrak{p}^2$.

Exploiting the identities (5.2.5) and (5.2.6), one obtains the tidier expression

$$C^{N!,\pm}\Big(\xi\widetilde{t}_\lambda^{-1},\mathcal{H}^{\pm,(\chi)}_{\mathfrak{p}^n,r}\Big)_\lambda\;=\;\mathcal{N}(\widetilde{t}_\lambda)^{-1}\;\times\sum_{\substack{\mathfrak{p}^{N!-1}\xi=\xi_1'+\xi_2'\\ \mathfrak{p}+\widetilde{\xi}_i'=\mathcal{O}_F}}\chi^*\big(\xi_1'\widetilde{t}_\lambda^{-1}\big)\cdot C\big(\xi_1'\widetilde{t}_\lambda^{-1},\mathbf{g}_l\big)$$
$$\times\sum_{\substack{\widetilde{\xi}_2'=\widetilde{b}\times\widetilde{c},\;pb\in\widetilde{t}_\lambda\\ c\in\mathcal{O}_F-\mathfrak{p}}}\Big((-1)^{[F:\mathbb{Q}]}\mathrm{sign}\mathcal{N}(\mathfrak{p}\widetilde{b})\Big)^{\frac{1\mp1}{2}}\mathcal{N}\big(\widetilde{c}^{-1}\mathfrak{p}\widetilde{b}\widetilde{t}_\lambda^{-1}\big)^r\cdot\big(\psi\eta^{-1}\chi^{-2}\big)^*(\widetilde{c})\,\mathcal{N}(\widetilde{c})^{1-l}\cdot\langle[\widetilde{c}]\rangle$$

after performing the twin substitutions $\xi_1'=\mathfrak{p}^{-1}\xi_1$ and $\xi_2'=\mathfrak{p}^{-1}\xi_2$.

Clearly, $\chi^*\big(\xi_1'\widetilde{t}_\lambda^{-1}\big)=\chi^*\big(\xi_2'\widetilde{t}_\lambda^{-1}\big)=\chi^*\big(\widetilde{b}\widetilde{c}\widetilde{t}_\lambda^{-1}\big)$ as χ has $\mathfrak{p}$-power conductor, thus the preceding quantity equals

$$\mathcal{N}(\mathfrak{p})^r\sum_{\substack{\mathfrak{p}^{N!-1}\xi=\xi_1'+\xi_2'\\ \mathfrak{p}+\widetilde{\xi}_i'=\mathcal{O}_F}}C\big(\xi_1'\widetilde{t}_\lambda^{-1},\mathbf{g}_l\big)\sum_{\substack{\widetilde{\xi}_2'=\widetilde{b}\times\widetilde{c},\;pb\in\widetilde{t}_\lambda\\ c\in\mathcal{O}_F-\mathfrak{p}}}\Xi^{N!,\pm}(b,c)\cdot\chi^*\big(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1}\big)\mathcal{N}\big(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1}\big)^r\cdot\langle[\widetilde{c}]\rangle$$

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where $\Xi^{N!,\pm}(b,c) = \mathcal{N}(\widetilde{t}_\lambda)^{-1} \times \Big((-1)^{[F:\mathbb{Q}]} \cdot \mathrm{sign}\mathcal{N}(\mathfrak{p}\widetilde{b})\Big)^{\frac{1\mp 1}{2}} \times (\psi\eta^{-1})^*(\widetilde{c})\, \mathcal{N}(\widetilde{c})^{1-l}$ will be p-integral, and independent of both r and the ideal character χ^*.

At every component λ, one can define a distribution $\mathrm{d}\nu_{r,\lambda}^{N!,\pm} = \mathrm{d}\nu_{r,\lambda}^{N!}(\xi, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r})$ on $\mathcal{X}_{p,F}$ through the integrals $\int_{x\in\mathcal{X}_{p,F}} \chi(x)\cdot \mathrm{d}\nu_{r,\lambda}^{N!,\pm}(x) := C^{N!,\pm}(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm,(\chi)}_{\mathfrak{p}^n,r})_\lambda$. Then, to establish (a),(b) and (c) hold for the $\mathcal{S}^{\mathrm{ord}}(\mathfrak{c},\Lambda)$-valued distributions $\mathrm{d}\mu^{\pm}_{\mathbf{g}_l,r}$, we reduce the problem to showing similar properties hold for each of the $\mathrm{d}\nu_{r,\lambda}^{N!,\pm}$'s.

(a) The boundedness of $\mathrm{d}\nu_{r,\lambda}^{N!,\pm}$: Identifying $\mathcal{X}_{p,F}$ with an open subgroup of $\mathbb{Z}_p^\times$, pick any neighbourhood $e + p^t\mathbb{Z}_p \subset \mathcal{X}_{p,F}$ where $\gcd(e,p) = 1$ and the integer $t > 0$. The characteristic function of $e + p^t\mathbb{Z}_p$ can be written as a summation

$$\mathrm{char}_{e+p^t\mathbb{Z}_p}(x) \;=\; \frac{1}{p^t - p^{t-1}} \times \sum_{\chi:(\mathbb{Z}/p^t\mathbb{Z})^\times \to \mathbb{C}_p^\times} \chi^{-1}(e)\chi(x)$$

whence

$$\int_{x\in e+p^t\mathbb{Z}_p} \mathrm{d}\nu_{r,\lambda}^{N!,\pm}(x) \;=\; \mathcal{N}(\mathfrak{p})^r \sum_{\substack{\mathfrak{p}^{N!-1}\xi=\xi_1'+\xi_2' \\ \mathfrak{p}+\widetilde{\xi}_i'=\mathcal{O}_F}} C(\xi_1'\widetilde{t}_\lambda^{-1}, \mathbf{g}_l) \sum_{\substack{\widetilde{\xi}_2'=\widetilde{b}\times\widetilde{c},\ pb\in\widetilde{t}_\lambda \\ c\in\mathcal{O}_F-\mathfrak{p}}} \Xi^{N!,\pm}(b,c)$$

$$\times \left(\frac{1}{p^t-p^{t-1}} \times \sum_{\chi:(\mathbb{Z}/p^t\mathbb{Z})^\times\to\mathbb{C}_p^\times} \chi^{-1}(e)\chi^*(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1})\right) \mathcal{N}(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1})^r \cdot \langle[\widetilde{c}]\rangle.$$

The bracketed term equals 1 or 0, depending on whether the image (under CFT) of the idele associated to $\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1}$ coincides with e modulo p^t. Defining the constant

$$d_p \;=\; d_p(\mathbf{g}_l, r, \lambda) \;:=\; \Big|\mathcal{N}(\mathfrak{p})\Big|_p^r \times \max_{\xi\gg 0} \Big|C(\xi\widetilde{t}_\lambda^{-1}, \mathbf{g}_l)\Big|_p \;\in\; p^{\mathbb{Z}}$$

we conclude that $\int_{x\in e+p^t\mathbb{Z}_p} \mathrm{d}\nu_{r,\lambda}^{N!,\pm}(x) \in d_p^{-1}\cdot\Lambda_F$ for all ξ, λ and $e + p^t\mathbb{Z}_p \subset \mathcal{X}_{p,F}$.

(b) The uniqueness of $\mathrm{d}\nu_{r,\lambda}^{N!,\pm}$: It is a standard fact that a bounded measure is completely determined by the integrals $\int \chi(x)\cdot \mathrm{d}\nu(x)$ at finite order characters χ, which means there is nothing to prove here.

(c) Tate twisting $\mathrm{d}\nu_{-,\lambda}^{N!,\pm}$ by $(\mathcal{N}x_p)^r$: Replacing $\chi(x)$ instead with $\chi(x)\cdot(\mathcal{N}x_p)$ is equivalent to sending $\chi^*(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1}) \mapsto \chi^*(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1})\cdot\mathcal{N}(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1})$ in our expression for $C^{N!,\pm}(\xi\widetilde{t}_\lambda^{-1}, \mathcal{H}^{\pm,(\chi)}_{\mathfrak{p}^n,r})_\lambda$. After some simple algebra, one deduces

$$\int_{x\in\mathcal{X}_{p,F}} \chi(x)\cdot \mathrm{d}\nu_{r+1,\lambda}^{N!,\pm}(x) \;=\; \mathcal{N}(\mathfrak{p}) \times \int_{x\in\mathcal{X}_{p,F}} \chi(x)(\mathcal{N}x_p)\cdot \mathrm{d}\nu_{r,\lambda}^{N!,\pm}(x)$$

so that $\mathrm{d}\mu^{\pm}_{\mathbf{g}_l,r+1} = (-1)^{[F:\mathbb{Q}]}\mathcal{N}(\mathfrak{p}) \times (\mathcal{N}x_p)\cdot \mathrm{d}\mu^{\pm}_{\mathbf{g}_l,r}$. The result follows by induction. $\square$

The preparatory work is over. We can now define a two-variable p-adic L-function[3] interpolating the standard versions of Panchiskin et al [**2, 5, 16, 17**] over $\mathrm{Spec}(\mathbb{I})$.

[3]In fact, our two-variable L-function on $\mathbb{U}_{k_0,\epsilon_0}\times\mathbb{Z}_p$ is the rigid-analytic image, for the weight $(l,\ldots,l)$ branch, of the element $\mathcal{D}\in\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathbb{J}$ described in [**8**, Theorem 5.1] up to some normalisations, of course.

Again fix a base weight $k_0 \geq 2$ and character ϵ_0. Recall from the previous section, the Mellin transform $\widetilde{\mathcal{P}}_{k_0,\epsilon_0}$ extended the mapping sending $\langle[\mathfrak{l}]\rangle$ to the Iwasawa function representing $w \mapsto \epsilon_0([\mathfrak{l}])\langle\mathfrak{l}\rangle_F^{w-k_0}$, and converged on the disk $\mathbb{U}_{k_0,\epsilon_0} \subset \mathbb{Z}_p$.

DEFINITION 5.3. For a finite order character χ of $\mathfrak{p}$-power conductor, one defines

$$\mathbf{L}_{p,k_0,\epsilon_0}(\mathcal{F}, \mathbf{g}_l, \chi;\ w, s)\ :=\ \widetilde{\mathcal{P}}_{k_0,\epsilon_0} \otimes \mathrm{id}\left(\int_{x\in\mathcal{X}_{p,F}} \langle\mathcal{N}x_p\rangle^{s-l}\chi(x)\cdot\Big[t_{\mathcal{F}},\ \mathrm{d}\mu^{\pm}_{\mathbf{g}_l,0}(x)\Big]_{\mathfrak{c},\mathbb{I}}\right)$$

under the proviso the weight variable $w \in \mathbb{U}_{k_0,\epsilon_0}$, the cyclotomic variable $s \in \mathbb{Z}_p$, and the sign satisfies $\pm 1 = (-1)^{k_0-l-1}$.

Does this formula actually make sense?

First note that $\mathrm{d}\mu^{\pm}_{\mathbf{g}_l,0}$ is a bounded measure on $\mathcal{X}_{p,F}$ with values in $\mathcal{S}^{\mathrm{ord}}(\mathfrak{c}, \Lambda)$, therefore pairing it with $t_{\mathcal{F}}$ yields a measure (on $\mathcal{X}_{p,F}$) taking values in $\mathbb{I} \otimes \mathbb{Z}[1/p]$. We can certainly integrate special characters of the form $\langle\mathcal{N}x_p\rangle^{s-l}\chi(x)$ against it; the resulting integral of this function belongs to the affinoid algebra $\mathbb{I}\widehat{\otimes}_{\mathcal{O}}\overline{\mathbb{Q}}_p\langle\langle s\rangle\rangle$. Lastly the Mellin transform induces

$$\widetilde{\mathcal{P}}_{k_0,\epsilon_0} \otimes \mathrm{id} :\ \mathbb{I}\widehat{\otimes}_{\mathcal{O}}\overline{\mathbb{Q}}_p\langle\langle s\rangle\rangle\ \longrightarrow\ \Big\{f \in \overline{\mathbb{Q}}_p\langle\langle w, s\rangle\rangle\ \Big|\ f \text{ is rigid-analytic on } \mathbb{U}_{k_0,\epsilon_0} \times \mathbb{Z}_p\Big\}$$

so the answer is yes, this formula does make sense.

REMARK. The construction of the one-variable p-adic L-function is entirely similar. One instead performs the integration

$$\mathbf{L}_p(\mathbf{f}_k, \mathbf{g}_l, \chi; s)\ \ :=\ \int_{x\in\mathcal{X}_{p,F}} \langle\mathcal{N}x_p\rangle^{s-l}\chi(x)\cdot \mathrm{d}\mu^{\mathrm{Pan}}_{\mathbf{f}_k,\ \mathbf{g}_l^{00}}(x)\ \in\ \overline{\mathbb{Q}}_p\langle\langle s\rangle\rangle,$$

where '$\mathrm{d}\mu^{\mathrm{Pan}}_{\mathbf{f}_k,\ \mathbf{g}_l^{00}}$' denotes the bounded p-adic measure in **[16, 17]**, interpolating the algebraic part of $\mathfrak{D}(s, \mathbf{f}_k, \mathbf{g}_l^{00} \otimes \chi)$ at its critical points $s = l, \ldots, k-1$.

THEOREM 5.4. *For all integers $k \geq 2$ satisfying $k \in \mathbb{U}_{k_0,\epsilon_0}$ with $k \equiv k_0 \pmod 2$, and at all critical twists $r \in \{0, \ldots, k-l-1\}$:*

$$\begin{aligned}
\mathbf{L}_{p,k_0,\epsilon_0}(\mathcal{F}, \mathbf{g}_l, \chi;\ k, l+r)\ &=\ (-1)^{r[F:\mathbb{Q}]}\cdot D_F^{2r+2+l}\ \times\ \eta^*(\mathfrak{p}^{\mathfrak{f}_{\chi,r}})\ \times\ \varpi(\mathbf{g}_l)\\
&\times\ (\chi\omega_F^{-r})^*(\mathfrak{n}(\mathbf{g}_l))\cdot\frac{\mathcal{N}(\mathfrak{n}(\mathbf{g}_l))^{r+1-\frac{k-l}{2}}}{\mathcal{N}(\mathfrak{n}(\mathcal{F}))^{k/2-1}}\ \times\ \Omega_{\mathbb{I}}(\lambda_{\mathcal{F}}, P)\ \times\ \mathbf{Eul}_{\mathfrak{p}}(k, l, r)\\
&\times\ \frac{\tau(\chi\omega_F^{-r})^2\cdot\mathcal{N}(\mathfrak{p}^{\mathfrak{f}_{\chi,r}})^{l+2r-1}}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon_0})^{2\mathfrak{f}_{\chi,r}}}\times\frac{\mathfrak{D}\Big(l+r,\ \mathcal{F}_{k,\epsilon_0}\ ,\ \mathbf{g}_l^{\#}\otimes(\chi\omega_F^{-r})^{-1}\Big)}{(-2\pi i)^{(1-l)[F:\mathbb{Q}]}\cdot\big\langle\mathcal{F}_{k,\epsilon_0}, \mathcal{F}_{k,\epsilon_0}\big\rangle_{\mathfrak{n}(\mathcal{F}_{k,\epsilon_0})}},
\end{aligned}$$

where $\chi\omega_F^{-r}$ has conductor $\mathfrak{p}^{\mathfrak{f}_{\chi,r}}$ say, while the modified $\mathfrak{p}$-Euler factor is given by

$$\begin{aligned}
\mathbf{Eul}_{\mathfrak{p}}(k, l, r)\ &:=\ \Big(1-\chi\omega_F^{-r}(\mathfrak{p})\tfrac{\alpha_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon_0})}\,\mathcal{N}(\mathfrak{p})^r\Big)\Big(1-\chi\omega_F^{-r}(\mathfrak{p})\tfrac{\beta_{\mathfrak{p}}(\mathbf{g}_l)}{\alpha_{\mathfrak{p}}(\mathcal{F}_{k,\epsilon_0})}\,\mathcal{N}(\mathfrak{p})^r\Big)\\
&\times\ \frac{\Big(1-\psi\eta^{-1}\chi^{-2}\omega_F^{2r+2-k}\epsilon\big(\mathfrak{p}\big)\cdot\big|\alpha_{\mathfrak{p}}\big(\mathcal{F}_{k,\epsilon}\big)\big|_{\infty}^{2}\cdot\mathcal{N}(\mathfrak{p})^{-(l+2r+1)}\Big)}{\Big(1-\psi\eta^{-1}\chi^{-2}\omega_F^{2r+2-k}\epsilon\big(\mathfrak{p}\big)\cdot\mathcal{N}(\mathfrak{p})^{-(l+2r+2-k)}\Big)}\ .
\end{aligned}$$

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Proof. We begin by pointing out that the constraint $k \equiv k_0 \pmod 2$ implies the parity condition $\pm 1 = (-1)^{k_0-l-1} = (-1)^{k-l-1}$ holds true, at all such integers k. Focusing on the value $s = l + r$, the two-variable p-adic L-function satisfies

$$\mathbf{L}_{p,k_0,\epsilon_0}(\mathcal{F}, \mathbf{g}_l, \chi; w, l+r) = \widetilde{\mathcal{P}}_{k_0,\epsilon_0}\left(\int_{x\in\mathcal{X}_{p,F}} (\mathcal{N}x_p)^r \chi\omega_F^{-r}(x) \cdot \left[t_{\mathcal{F}}, \mathrm{d}\mu^{\pm}_{\mathbf{g}_l,0}(x)\right]_{\mathfrak{c},\mathbb{I}}\right)$$

$$\overset{\text{by 4.2(c)}}{=} (-1)^{r[F:\mathbb{Q}]}\mathcal{N}(\mathfrak{p})^{-r} \times \widetilde{\mathcal{P}}_{k_0,\epsilon_0}\left(\int_{x\in\mathcal{X}_{p,F}} \chi\omega_F^{-r}(x) \cdot \left[t_{\mathcal{F}}, \mathrm{d}\mu^{\pm}_{\mathbf{g}_l,r}(x)\right]_{\mathfrak{c},\mathbb{I}}\right)$$

$$\overset{\text{by 4.1}}{=} (-1)^{r[F:\mathbb{Q}]}\mathcal{N}(\mathfrak{p})^{-r} \times \widetilde{\mathcal{P}}_{k_0,\epsilon_0}\left(\left[t_{\mathcal{F}}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\left(\left(\mathbf{g}_l^{00} \otimes \chi\omega_F^{-r}\right)\middle| V_{\mathfrak{p}}, \mathfrak{c}\mathfrak{p}^n, \psi\right)\right]_{\mathfrak{c},\mathbb{I}}\right)$$

and furthermore,

$$\widetilde{\mathcal{P}}_{k_0,\epsilon_0}\left(\left[t_{\mathcal{F}}, \mathcal{H}^{\pm}_{\mathfrak{p}^n,r}\left(\left(\mathbf{g}_l^{00} \otimes \chi\omega_F^{-r}\right)\middle| V_{\mathfrak{p}}, \mathfrak{c}\mathfrak{p}^n, \psi\right)\right]_{\mathfrak{c},\mathbb{I}}\right)\Bigg|_{w=k}$$

$$\overset{\text{by 4.4(a,b,c)}}{=} D_F^{2r+2+l} \times \mathbf{Eul}_{\mathfrak{p}}(k, l, r) \times \frac{\mathcal{N}\left(\mathfrak{n}(\mathbf{g}_l)\cdot\mathfrak{p}^{2\mathfrak{f}_{\chi,r}}\right)^{l/2+r}\mathcal{N}(\mathfrak{p})^r}{\mathcal{N}(\mathfrak{c})^{k/2-1}}$$

$$\times \frac{\varpi\left(\mathbf{g}_l\otimes\chi\omega_F^{-r}\right)\cdot\Omega_{\mathbb{I}}\left(\lambda_{\mathcal{F}}, P\right)}{\alpha_{\mathfrak{p}}\left(\mathcal{F}_{k,\epsilon_0}\right)^{\mathrm{ord}_{\mathfrak{p}}\left(\mathfrak{n}(\mathbf{g}_l)\cdot\mathfrak{p}^{2\mathfrak{f}_{\chi,r}}\right)}} \times \frac{\mathfrak{D}\left(l+r, \mathcal{F}_{k,\epsilon_0}, \left(\mathbf{g}_l\otimes\chi\omega_F^{-r}\right)^{\#}\right)}{(-2\pi i)^{(1-l)[F:\mathbb{Q}]}\cdot\left\langle\mathcal{F}_{k,\epsilon_0}, \mathcal{F}_{k,\epsilon_0}\right\rangle_{\mathfrak{n}(\mathcal{F}_{k,\epsilon_0})}}.$$

However (through some low-brow computation), one finds that

(i) $\varpi\left(\mathbf{g}_l \otimes \chi\omega_F^{-r}\right) = \eta^*\left(\mathfrak{p}^{\mathfrak{f}_{\chi,r}}\right) \cdot \left(\chi\omega_F^{-r}\right)^*\left(\mathfrak{n}(\mathbf{g}_l)\right) \cdot \tau\left(\chi\omega_F^{-r}\right)^2 \cdot \mathcal{N}\left(\mathfrak{p}^{\mathfrak{f}_{\chi,r}}\right)^{-1} \times \varpi(\mathbf{g}_l)$;

(ii) $\mathcal{N}\left(\mathfrak{n}(\mathbf{g}_l)\right)^{l/2+r} \times \mathcal{N}(\mathfrak{c})^{1-k/2} = \mathcal{N}\left(\mathfrak{n}(\mathbf{g}_l)\right)^{l/2+r+1-k/2} \times \mathcal{N}\left(\mathfrak{n}(\mathcal{F})\right)^{1-k/2}$.

The theorem follows upon plugging these two identities into the expression for $\mathbf{L}_{p,k_0,\epsilon_0}(\mathcal{F}, \mathbf{g}_l, \chi; w, l+r)\Big|_{w=k}$ obtained on this page. □

5.1. The functional equation at parallel weight $l = 1$. At present, it is unclear how to obtain a nice p-adic functional equation when the weight of $\mathbf{g}_l$ is greater than 1, so we are forced to make simplifying assumptions. Essentially, we will only treat the situation where $l = 1$, and the primitive form $\mathbf{g} = \mathbf{g}_1$ arises by inducing down a grössencharakter over a CM extension K/F.

HYPOTHESIS (H1). $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}\left(\mathfrak{n}(\mathcal{F}), \mathbb{I}\right)$ has square-free level $\mathfrak{n}(\mathcal{F})$ as an $\mathcal{O}_F$-ideal.

HYPOTHESIS (H2). For all integer weights κ inside a disk $\mathbb{U} \subset \mathbb{Z}_p$ containing 2, each specialisation $\mathcal{F}_\kappa$ is the $\mathfrak{p}$-stabilisation for the base-change lift (from $\mathbb{Q}$ to F) of a classical Hecke eigenform $f_\kappa \in \mathcal{S}_\kappa^{\mathrm{new}}\left(\Gamma_0(Np^\infty), \omega_{\mathbb{Q}}^{2-\kappa}\right)$.

The advantage of assuming (H2) holds is to ensure whenever $\kappa \equiv 2 \pmod{p-1}$, the complex L-function for each form $\mathcal{F}_\kappa$ arises from a motive realisable over $\mathbb{Q}$. A consequence of (H1) is that at weight two, the specialisation $\mathcal{F}_2$ is Steinberg at every prime $\mathfrak{q}$ dividing the tame level $\mathfrak{n}(\mathcal{F})$; as a corollary $C\left(\mathfrak{n}(\mathcal{F}), \mathcal{F}_2\right) = \pm 1$. However, $C\left(\mathfrak{n}(\mathcal{F}), \mathcal{F}_\kappa\right)$ is a rigid-analytic function of κ, so it must be non-zero in a p-adic neighbourhood of $\kappa = 2$. **It follows that (H1) implies Hypothesis (SS).** One can further deduce $C\left(\mathfrak{n}(\mathcal{F}), \mathcal{F}_\kappa\right) = \pm\left\langle\mathcal{N}\left(\mathfrak{n}(\mathcal{F})\right)\right\rangle^{\kappa/2-1}$ via [**11**, Proposition 5.2].

Attached to the system $\{\mathcal{F}_\kappa\}_{\kappa\in\mathbb{U}\cap\mathbb{Z}_{\geq 2}}$, one has an analytic family (for $\kappa\in\mathbb{U}$) of representations $\pi_{\kappa,\nu}: G_F\longrightarrow \mathrm{GL}_2(\overline{F}_\nu)$, which are themselves restrictions of Deligne's $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations $V_q(f_\kappa)$ associated to the classical forms $f_\kappa/\mathbb{Q}$ (up to semisimplicity, the latter are uniquely determined by the condition that if a prime $l\nmid pN$, then both $\mathrm{trace}(\mathrm{Frob}_l^{-1})=a_l(f_\kappa)$ and $\det(\mathrm{Frob}_l^{-1})=\omega_{\mathbb{Q}}^{2-\kappa}(l)l^{\kappa-1}$). Let us denote by '$V_\nu(\mathbf{f}_\kappa)$' the underlying representation space for $\pi_{\kappa,\nu}$ over $\overline{F}_\nu$.

We now turn our attention to imposing suitable conditions on $\mathbf{g}\in\mathcal{S}_1\big(\mathfrak{n}(\mathbf{g}),\eta\big)$:

HYPOTHESIS (H3). There exists a CM extension K/F and a Hecke character $\Phi_{/K}$, such that $\mathbf{g}=\mathbf{g}_\rho$ is the weight one primitive form associated to $\rho=\mathrm{Ind}_K^F\big(\Phi_{/K}\big)$.

HYPOTHESIS (H4). The Artin representation $\rho: G_F\to\mathrm{GL}_2(\mathbb{C})$ is isomorphic to its contragredient i.e. $\rho\cong\rho^\vee$, and therefore also $\mathbf{g}_\rho=\mathbf{g}_\rho^\#$ on the level of HMF's.

In terms of the corresponding $\mathcal{O}_K$-ideal character $\Phi_{/K}^*$, the normalised newform $\mathbf{g}_\rho$ has nebentypus $\eta=\det(\rho)$ given by

$$\eta^*(\mathfrak{a})=\varphi_{K/F}(\mathfrak{a})\cdot\Phi_{/K}^*\big(\mathfrak{a}\mathcal{O}_K\big)\quad\text{where}\quad \varphi_{K/F}(\mathfrak{q})\;=\;\begin{cases}+1 & \text{if } \mathfrak{q} \text{ splits in } K/F\\ -1 & \text{if } \mathfrak{q} \text{ is inert in } K/F\\ \;\;0 & \text{if } \mathfrak{q} \text{ ramifies in } K/F.\end{cases}$$

Fixing a primitive cusp form $\mathbf{f}$ of parallel weight ≥ 2 and assuming that $\mathrm{Re}(s)\gg 0$, its ρ-twisted L-function is defined by the infinite product

$$L\big(\mathbf{f},\rho,s\big)\;:=\;\prod_{\text{finite primes }\mathfrak{q}}\det\Big(1-\mathcal{N}_{F/\mathbb{Q}}(\mathfrak{q})^{-s}\cdot\mathrm{Frob}_{\mathfrak{q}}^{-1}\Big|\big(V_\nu(\mathbf{f})\otimes_{\overline{F}_\nu}V_\nu(\rho)\big)^{I_{\mathfrak{q}}}\Big).$$

HYPOTHESIS (H5). $\big(V_\nu(\mathbf{f})\otimes_{\overline{F}_\nu}V_\nu(\rho)\big)^{I_{\mathfrak{q}}}\cong\; V_\nu(\mathbf{f})^{I_{\mathfrak{q}}}\otimes_{\overline{F}_\nu}V_\nu(\rho)^{I_{\mathfrak{q}}}$ for all primes $\mathfrak{q}$.

Upon multiplying $L\big(\mathbf{f},\rho,s\big)$ through by the appropriate gamma factor at infinity, this last assumption (H5) permits us to identify the ρ-twisted L-series attached to $\mathbf{f}$ with the convolution $\mathfrak{D}\big(s,\mathbf{f},\mathbf{g}_\rho\big)$ (whose critical values were interpolated by $d\mu^{\pm}_{\mathbf{g}_\rho,0}$). As the latter extends to an entire function [**18**, Proposition 4.13], clearly so must $L\big(\mathbf{f},\rho,s\big)$.

REMARKS.

(i) Assuming $\mathbf{f}$ is a primitive HMF of scalar weight k with $\mathfrak{n}(\mathbf{f})+\mathfrak{p}=\mathcal{O}_F$, the functional equation for the $\big(\rho\otimes\chi^{-1}\omega_F^r\big)$-twisted L-series becomes

$$\begin{aligned}(5.2.7)\quad & D_F^{2s}\cdot\Big(\mathcal{N}\big(\mathfrak{n}(\mathbf{f},\rho)\big)\cdot\mathcal{N}(\mathfrak{p})^{4\mathfrak{f}_{\chi,r}}\Big)^{s/2}\cdot\left(\frac{\Gamma(s)}{(2\pi)^s}\right)^{2[F:\mathbb{Q}]}\cdot L\big(\mathbf{f},\rho\otimes\chi^{-1}\omega_F^r,s\big)\\ &=(-1)^{[F:\mathbb{Q}]k/2}\cdot w_\infty(k)\times\chi^{-1}\omega_F^r\big(\mathfrak{n}(\mathbf{f},\rho)\big)\times\tau\big(\chi^{-1}\omega_F^r\big)^2\cdot\tau\big(\chi\omega_F^{-r}\big)^{-2}\\ &\quad\times D_F^{2(k-s)}\cdot\Big(\mathcal{N}\big(\mathfrak{n}(\mathbf{f},\rho)\big)\cdot\mathcal{N}(\mathfrak{p})^{4\mathfrak{f}_{\chi,r}}\Big)^{(k-s)/2}\\ &\quad\times\left(\frac{\Gamma(k-s)}{(2\pi)^{k-s}}\right)^{2[F:\mathbb{Q}]}\cdot L\big(\mathbf{f}^\#,\rho^\vee\otimes\chi\omega_F^{-r},k-s\big),\end{aligned}$$

where $\mathfrak{n}\big(\mathbf{f},\rho\big)$ is the F-conductor of the ν-adic system $\big\{V_\nu(\mathbf{f})\otimes_{\overline{F}_\nu}V_\nu(\rho)\big\}_{\nu\in\mathrm{Spec}\mathcal{O}_F}$.

(ii) The quantity $w_\infty(k)$ is called the *root number* and is of absolute value one. In the case where $\mathbf{f}$ is the base-change of a classical primitive form f over $\mathbb{Q}$ with rational coefficients and trivial character, plus given that the Artin

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representation $\mathrm{Ind}_F^{\mathbb{Q}}(\rho)$ was self-dual, one can further say that the root number $w_\infty(k) \in \{-1, +1\}$.

(iii) The rational integer value $\mathcal{N}(\mathfrak{n}(\mathbf{f}, \rho) \times \mathfrak{p}^{4\mathfrak{f}_{\chi,r}})$ above will coincide exactly with the $\mathbb{Q}$-conductor of the twisted system $\{V_q(f) \otimes_{\overline{\mathbb{Q}}_q} V_q(\mathrm{Ind}_F^{\mathbb{Q}}(\rho \otimes \chi^{-1}\omega_F^r))\}_{q \in \mathrm{Spec}\mathbb{Z}}$.

(iv) The support of the ideal $\mathfrak{n}(\mathbf{g}_\rho)$ is strictly contained in the support of $\mathfrak{n}(\mathbf{f}, \rho)$, as at least one of $\mathbb{V}(\mathbf{f}) = \{V_\nu(\mathbf{f})\}_{\nu \in \mathrm{Spec}\mathcal{O}_F}$ and $\rho : G_F \to \mathrm{GL}_2(\mathbb{C})$ is unramified at every place of F. One then defines $\mathfrak{N} = \mathfrak{N}(\mathcal{F}, \rho) := \mathfrak{n}(\mathbf{f}, \rho)/\mathfrak{n}(\mathbf{g}_\rho)^2$, which depends on $\{\mathcal{F}_\kappa\}_{\kappa \in \mathbb{U} \cap \mathbb{Z}_{\geq 2}}$ but is independent of the initial choice for $\mathbf{f} = \mathbf{f}_k$.

THEOREM 5.5. *Restricting to weights $k \in \mathbb{U} \cap \mathbb{Z}_{>2}$ satisfying $k \equiv 2 (\bmod\ p-1)$:*

$$\mathbf{L}_p(\mathcal{F}, \mathbf{g}_\rho, \chi;\ k, 1+r) = w_p(k) \times \chi^{-1}(\mathfrak{N}) \cdot \langle \mathfrak{N} \rangle_F^{k/2-1-r} \times \mathbf{L}_p(\mathcal{F}, \mathbf{g}_\rho, \chi^{-1};\ k, k-1-r),$$

where $w_p(k) = (-1)^{[F:\mathbb{Q}]k/2} \cdot w_\infty(k) \cdot \omega_F(\mathfrak{n}(\mathcal{F}, \rho))^{k/2-1}$ takes values inside $\{-1, +1\}$, and is therefore locally constant.

Both $\mathbf{L}_p(\mathcal{F}, \mathbf{g}_\rho, - ; \kappa, -)$ and $\langle \mathfrak{N} \rangle_F^{\kappa/2-1-r}$ are rigid-analytic in the variable $\kappa \in \mathbb{U}$, hence the function $w_p(k) \in \{-1, +1\}$ extends continuously to weight $k = 2$ as well. To extract data about the zeroes of $\mathbf{L}_p(\mathcal{F}, \mathbf{g}_\rho, -; k, s)$ at the point $(k, s) = (2, 1)$, the functional equation should involve (on both of its sides) the same p-adic L. Thus, we only need focus upon the trivial χ-branch above.

COROLLARY 5.6. *Under the additional assumption that χ is the trivial character,*

$$\mathbf{L}_p(\mathcal{F}, \mathbf{g}_\rho, \mathbf{1};\ k, s) \quad = \quad w_p(2) \times \langle \mathfrak{N} \rangle_F^{k/2-s} \times \mathbf{L}_p(\mathcal{F}, \mathbf{g}_\rho, \mathbf{1};\ k, k-s)$$

for all weights k inside a sufficiently small p-adic neighbourhood of 2.

The demonstration of Theorem 5.5 will now occupy the remainder of this section. Let us begin by obtaining a more succinct expression for the critical values of $\mathbf{L}_p(\mathcal{F}, \mathbf{g}_\rho, \chi;\ k, s)$; we explicitly relate these to the primitive Rankin L-function. Recall also the epsilon factor of the Artin representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ can be written as a product of local terms $\varepsilon_F(\rho, s) = \prod_{\text{places } \nu} \varepsilon_{F_\nu}(\rho_\nu, s)$.

LEMMA 5.7. *Noting each $\mathcal{F}_k$ is the $\mathfrak{p}$-stabilisation of a primitive cusp form $\mathbf{f}_k$ say, the special value $\mathbf{L}_p(\mathcal{F}, \mathbf{g}_\rho, \chi;\ k, 1+r)$ equals*

$$(-1)^{r[F:\mathbb{Q}]} \cdot i^{-[F:\mathbb{Q}]} \cdot \eta^*(\mathfrak{p}^{\mathfrak{f}_{\chi,r}}) \ \times\ \varepsilon_F(\rho, 0) \cdot \Omega_{\mathbb{I}}(\lambda_{\mathcal{F}}, P_k) \ \times\ (\chi\omega_F^{-r})\mathcal{N}^r(\mathfrak{n}(\mathbf{g}_\rho))$$
$$\times\ \frac{\tau(\chi\omega_F^{-r})^2 \cdot \mathcal{N}(\mathfrak{p}^{2\mathfrak{f}_{\chi,r}})^r \cdot D_F^{2r+2}}{\alpha_\mathfrak{p}(\mathcal{F}_k)^{2\mathfrak{f}_{\chi,r}} \cdot \mathcal{N}(\mathfrak{n}(\mathbf{g}_\rho)\mathfrak{n}(\mathcal{F}))^{\frac{k-2}{2}}} \ \times\ \mathcal{E}_\mathfrak{p}(k, r, \chi) \ \times\ \frac{\mathfrak{D}\left(1+r,\ \mathbf{f}_k\ ,\ \mathbf{g}_\rho \otimes \chi^{-1}\omega_F^r\right)}{\langle \mathcal{F}_k\ , \mathcal{F}_k \rangle_{\mathfrak{n}(\mathcal{F}_k)}}$$

where the degree four $\mathfrak{p}$-Euler factor is given by

$$\mathcal{E}_\mathfrak{p}(k, r, \chi) \ = \ \left(1 - \chi\omega_F^{-r}(\mathfrak{p})\frac{\alpha_\mathfrak{p}(\mathbf{g}_\rho)}{\alpha_\mathfrak{p}(\mathbf{f}_k)}\,\mathcal{N}(\mathfrak{p})^r\right)\left(1 - \chi\omega_F^{-r}(\mathfrak{p})\frac{\beta_\mathfrak{p}(\mathbf{g}_\rho)}{\alpha_\mathfrak{p}(\mathbf{f}_k)}\,\mathcal{N}(\mathfrak{p})^r\right)$$
$$\times \left(1 - \chi^{-1}\omega_F^r(\mathfrak{p})\alpha_\mathfrak{p}(\mathbf{g}_\rho)\beta_\mathfrak{p}(\mathbf{f}_k)\,\mathcal{N}(\mathfrak{p})^{-1-r}\right)\left(1 - \chi^{-1}\omega_F^r(\mathfrak{p})\beta_\mathfrak{p}(\mathbf{g}_\rho)\beta_\mathfrak{p}(\mathbf{f}_k)\,\mathcal{N}(\mathfrak{p})^{-1-r}\right).$$

Proof. There are two possibilities to consider: the cusp form $\mathcal{F}_k$ is Steinberg at $\mathfrak{p}$, or alternatively $\mathcal{F}_k$ is the $\mathfrak{p}$-stabilisation of a primitive form $\mathbf{f}_k$ of exact level $\mathfrak{n}(\mathcal{F})$. Let us deal with the latter possibility first, in other words

$$\mathcal{F}_k \;=\; \mathbf{f}_k \;-\; \beta_{\mathfrak{p}}\big(\mathbf{f}_k\big)\cdot \mathbf{f}_k\Big| V_{\mathfrak{p}} \quad \text{where } \beta_{\mathfrak{p}}\big(\mathbf{f}_k\big) \text{ is the non-unit root of Frobenius.}$$

We start by simplifying the factor $\mathbf{Eul}_{\mathfrak{p}}\big(k, l, r\big)$ occurring in Theorem 5.4 (at $l = 1$). By direct inspection, the ratio

$$\frac{\Big(1-\eta^{-1}\chi^{-2}\omega_F^{2r+2-k}(\mathfrak{p})\cdot\big|\alpha_{\mathfrak{p}}\big(\mathcal{F}_k\big)\big|_\infty^2\cdot\mathcal{N}(\mathfrak{p})^{-(l+2r+1)}\Big)}{\Big(1-\eta^{-1}\chi^{-2}\omega_F^{2r+2-k}(\mathfrak{p})\cdot\mathcal{N}(\mathfrak{p})^{-(l+2r+2-k)}\Big)}$$

equals one because $\big|\alpha_{\mathfrak{p}}(\mathcal{F}_k)\big|_\infty^2 = \alpha_{\mathfrak{p}}\big(\mathbf{f}_k\big)\times\beta_{\mathfrak{p}}\big(\mathbf{f}_k\big) = \mathcal{N}(\mathfrak{p})^{k-1}$, whence

$$\mathbf{Eul}_{\mathfrak{p}}\big(k, 1, r\big) \;=\; \left(1-\chi\omega_F^{-r}(\mathfrak{p})\frac{\alpha_{\mathfrak{p}}(\mathbf{g}_\rho)}{\alpha_{\mathfrak{p}}(\mathbf{f}_k)}\,\mathcal{N}(\mathfrak{p})^r\right)\left(1-\chi\omega_F^{-r}(\mathfrak{p})\frac{\beta_{\mathfrak{p}}(\mathbf{g}_\rho)}{\alpha_{\mathfrak{p}}(\mathbf{f}_k)}\,\mathcal{N}(\mathfrak{p})^r\right).$$

It is then a straightforward exercise to verify the identity

$$\mathbf{Eul}_{\mathfrak{p}}\big(k, 1, r\big)\cdot\mathfrak{D}\Big(1+r,\ \mathcal{F}_k\,,\mathbf{g}_\rho\otimes\big(\chi\omega_F^{-r}\big)^{-1}\Big) = \mathcal{E}_{\mathfrak{p}}\big(k, r, \chi\big)\cdot\mathfrak{D}\Big(1+r,\ \mathbf{f}_k, \mathbf{g}_\rho\otimes\chi^{-1}\omega_F^r\Big).$$

The interpolation formulae stated in our lemma are a simple consequence of the corresponding formulae in Theorem 5.4 (with $l = 1$, $\psi = \mathbf{1}$ and $\mathbf{g}_l = \mathbf{g}_\rho = \mathbf{g}_\rho^{\#}$), albeit the $\varpi(\mathbf{g}_\rho)$-term will appear in place of the $\varepsilon_F(\rho, 0)$-factor.

Lastly, the following result allows us to replace the pseudo-eigenvalue of the $J_{\mathfrak{n}(\mathbf{g}_\rho)}$-operator, with the global epsilon factor associated to the representation ρ (note that a classical Gauss sum can always be interpreted as the global ε-factor associated to a one-dimensional G_F-representation, i.e. to a Hecke character over F).

LEMMA B.1 $\quad \varepsilon_F(\rho) \;:=\; \varepsilon_F(\rho, s)\Big|_{s=0} \;=\; i^{[F:\mathbb{Q}]}\cdot D_F\cdot\sqrt{\mathcal{N}\big(\mathfrak{n}(\mathbf{g}_\rho)\big)}\;\times\;\varpi(\mathbf{g}_\rho).$

To avoid interruption, the proof of this lemma has been left until Appendix B, wherein a more detailed discussion of both local and global ε-factors is included.

The treatment of the case where $\mathbf{f}_k$ is Steinberg at $\mathfrak{p}$ follows very similar lines. Here, we note that $\alpha_{\mathfrak{p}}(\mathcal{F}_k)^2 = C(\mathfrak{p}, \mathbf{f}_k)^2 = \mathcal{N}(\mathfrak{p})^{k-2}$; because $\alpha_{\mathfrak{p}}(\mathcal{F}_k)$ is a p-adic unit, this situation can only occur at weight $k = 2$. It follows immediately that

$$\mathbf{Eul}_{\mathfrak{p}}\big(k, 1, r\big)\Big|_{(k,r)=(2,0)} = \mathbf{Eul}_{\mathfrak{p}}\big(2, 1, 0\big) = \begin{cases} 1 & \text{if } \chi\neq\mathbf{1} \\ 0 & \text{if } \chi=\mathbf{1}\end{cases} \;=\; \mathcal{E}_{\mathfrak{p}}\big(k, r, \chi\big)\Big|_{(k,r)=(2,0)}$$

and the remainder of the argument proceeds as in the non-Steinberg case. □

Returning to the proof of Theorem 5.5, we will establish the two-variable functional equation at critical pairs (k, s), where $\mathbf{f}_k$ is non-Steinberg at $\mathfrak{p}$ and $s \in \{1, \ldots, k-1\}$. Since this excludes at worst $k = 2$, and as the remaining weights (greater than two and congruent to 2 modulo $p-1$) are Zariski dense inside of the weight-space $\mathbb{U}$, this would be sufficient to prove the formula in general.

We start by supposing only that k is even. To be consistent with our earlier notation, denote by $\mathfrak{p}^{\mathfrak{f}_{\overline{\chi},r}}$ the conductor of the Hecke character $\chi\omega_F^{k-2-r}$ over F. Assuming for simplicity that both its numerator and denominator do not vanish, then applying

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Lemma 5.7 twice to the following quotient yields

$$\begin{aligned}
&\frac{\mathbf{L}_p\big(\mathcal{F}, \mathbf{g}_\rho, \chi^{-1};\ k, (k-r-2)+1\big)}{\mathbf{L}_p\big(\mathcal{F}, \mathbf{g}_\rho, \chi;\ k, r+1\big)} \\
&= \frac{\chi^{-2}\omega_F^{2r+2-k}\big(\mathfrak{n}(\mathbf{g}_\rho)\big)}{\mathcal{N}\big(\mathfrak{n}(\mathbf{g}_\rho)\big)^{2r+2-k}} \times \frac{\eta^*\big(\mathfrak{p}^{\mathfrak{f}_{\overline{\chi},r}}\big)}{\eta^*\big(\mathfrak{p}^{\mathfrak{f}_{\chi,r}}\big)} \cdot \alpha_{\mathfrak{p}}\left(\mathcal{F}_k\right)^{2\mathfrak{f}_{\chi,r}-2\mathfrak{f}_{\overline{\chi},r}} \times \frac{\mathcal{E}_{\mathfrak{p}}\big(k, k-r-2, \chi^{-1}\big)}{\mathcal{E}_{\mathfrak{p}}\big(k, r, \chi\big)} \\
&\quad \times \frac{D_F^{2(k-r-1)}\mathcal{N}\big(\mathfrak{p}^{2\mathfrak{f}_{\overline{\chi},r}}\big)^{k-r-2}\tau\big(\chi^{-1}\omega_F^{2+r-k}\big)^2}{D_F^{2(r+1)}\mathcal{N}\big(\mathfrak{p}^{2\mathfrak{f}_{\chi,r}}\big)^{r}\tau\big(\chi\omega_F^{-r}\big)^2} \times \frac{\mathfrak{D}\big(k-1-r,\ \mathbf{f}_k,\ \mathbf{g}_\rho\otimes\chi\omega_F^{k-2-r}\big)}{\mathfrak{D}\big(r+1,\ \mathbf{f}_k,\ \mathbf{g}_\rho\otimes\chi^{-1}\omega_F^{r}\big)}.
\end{aligned}$$

However, if one now restricts to $k \equiv 2(\bmod\ p-1)$, then $\chi\omega_F^{k-2-r} = \big(\chi^{-1}\omega_F^r\big)^{-1}$ whence $\mathfrak{p}^{\mathfrak{f}_{\overline{\chi},r}} = \mathfrak{p}^{\mathfrak{f}_{\chi,r}}$. Furthermore, it is a tedious exercise canceling out Euler factors to deduce $\mathcal{E}_{\mathfrak{p}}(k, k-r-2, \chi^{-1}) = \mathcal{E}_{\mathfrak{p}}(k, r, \chi)$, in which case

$$\frac{\mathbf{L}_p\big(\mathcal{F}, \mathbf{g}_\rho, \chi^{-1};\ k, k-1-r\big)}{\mathbf{L}_p\big(\mathcal{F}, \mathbf{g}_\rho, \chi;\ k, r+1\big)} \ = \ \chi^{-2}\big(\mathfrak{n}(\mathbf{g}_\rho)\big)\cdot\big\langle\mathfrak{n}(\mathbf{g}_\rho)\big\rangle_F^{k-2-2r} \times \Theta_{k,r},$$

where $\Theta_{k,r} := \dfrac{D_F^{2(k-r-1)}\mathcal{N}\big(\mathfrak{p}^{2\mathfrak{f}_{\chi,r}}\big)^{k-1-r}\tau\big(\chi^{-1}\omega_F^{r}\big)^2}{D_F^{2(r+1)}\mathcal{N}\big(\mathfrak{p}^{2\mathfrak{f}_{\chi,r}}\big)^{r+1}\tau\big(\chi\omega_F^{-r}\big)^2}\cdot\dfrac{\mathfrak{D}\big(k-1-r,\ \mathbf{f}_k,\ \mathbf{g}_\rho\otimes\chi\omega_F^{-r}\big)}{\mathfrak{D}\big(r+1,\ \mathbf{f}_k,\ \mathbf{g}_\rho\otimes\chi^{-1}\omega_F^{r}\big)}.$

Exploiting the fact $\mathfrak{D}\big(s,\ \mathbf{f}_k,\ \mathbf{g}_\rho\otimes\chi^{-1}\omega_F^r\big) = \big((2\pi)^{-s}\Gamma(s)\big)^{2[F:\mathbb{Q}]}\cdot L\big(\mathbf{f}_k, \rho\otimes\chi^{-1}\omega_F^r, s\big)$, the complex functional equation (5.2.7) implies that

$$\Theta_{k,r} \ = \ (-1)^{[F:\mathbb{Q}]k/2}\cdot w_\infty(k)\cdot\omega_F\big(\mathfrak{n}(\mathcal{F},\rho)\big)^{1-k/2} \times \chi\big(\mathfrak{n}(\mathcal{F},\rho)\big)\cdot\big\langle\mathfrak{n}(\mathcal{F},\rho)\big\rangle_F^{r+1-k/2}$$

and the p-adic functional equation along $(k, r+1) \in \mathbb{U}\times\mathbb{Z}_p$ follows readily.

REMARK. Conversely if one of $\mathbf{L}_p\big(\mathcal{F}, \mathbf{g}_\rho, \chi^{-1};\ k, k-1-r\big)$ or $\mathbf{L}_p\big(\mathcal{F}, \mathbf{g}_\rho, \chi;\ k, 1+r\big)$ does in fact vanish, so must other one (courtesy of the equation (5.2.7) again). Under this scenario the desired identity

$$\mathbf{L}_p\big(\mathcal{F}, \mathbf{g}_\rho, \chi;\ k, 1+r\big) = w_p(k)\times\chi^{-1}\big(\mathfrak{N}\big)\cdot\big\langle\mathfrak{N}\big\rangle_F^{k/2-1-r}\times\mathbf{L}_p\big(\mathcal{F}, \mathbf{g}_\rho, \chi^{-1}; k, k-1-r\big)$$

is vacuously true, as it represents none other than '$0 = 0$' in disguise.

6. An application to semistable elliptic curves. Let E be an elliptic curve over $\mathbb{Q}$, and $f_E \in \mathcal{S}_2^{\mathrm{new}}\big(\Gamma_0(N_E)\big)$ its associated newform. We shall assume that E is semistable over the totally real field F, in other words its F-conductor $\mathfrak{n}_E$ is a square-free ideal. Lastly, we suppose E has split multiplicative reduction at the prime $\mathfrak{p}$, which yields a Tate parametrisation $E(F_{\mathfrak{p}}) \cong F_{\mathfrak{p}}^\times/\mathbf{q}_{E,\mathfrak{p}}^{\mathbb{Z}}$.

Notice that f_E is the weight two representative of some Hida family $\{f_\kappa^0\}_{\kappa\in\mathbb{U}\cap\mathbb{Z}_{\geq 2}}$; in particular, f_κ^0 has trivial character and $\mathbb{Q}$-coefficients whenever $\kappa \equiv 2(\bmod\ p-1)$. Base-changing each one of these p-stabilised newforms $f_\kappa^0/\mathbb{Q}$ up to the field F, one obtains a family of HMF's $\mathcal{F}_\kappa = \mathbf{BC}(f_k^0) \in \mathcal{S}_k^{\mathrm{ord}}(\mathfrak{n}_E\mathfrak{p}^\infty, \omega_F^{2-\kappa})$ with $\kappa \in \mathbb{U}\cap\mathbb{Z}_{\geq 2}$. The elliptic curve E was assumed to be semistable over F so the tame level of the $\mathbb{I}$-adic cusp form $\mathcal{F}$ must be square-free, and Hypothesis (H1) is therefore satisfied. Similarly, (H2) also trivially holds as this $\mathbb{I}$-adic form is constructed via base-change.

At this stage, we shall still need to assume both the Hypotheses (H3) and (H4). However, (H5) is true under the proviso that $\mathfrak{n}_E$ and $\mathfrak{n}(\mathbf{g}_\rho)$ are coprime $\mathcal{O}_F$-ideals, and in Section 5 we already discussed at length why (H1) $\Longrightarrow$ (SS).

DEFINITION 6.1.

$$\mathbf{L}_p(E, \rho\otimes\chi, s) \;:=\; \frac{D_F^{-2}\cdot\langle\mathcal{F}_2\,,\mathcal{F}_2\rangle_{\mathfrak{n}_E}}{\left(\Omega_E^+\Omega_E^-\right)^{[F:\mathbb{Q}]}} \times \frac{\mathbf{L}_p\left(\mathcal{F},\mathbf{g}_\rho,\chi;2,s\right)}{i^{-[F:\mathbb{Q}]}\cdot\Omega_{\mathbb{I}}\left(\lambda_{\mathcal{F}},P_2\right)}\,.$$

From our interpolation rules (see Lemma 5.7), if χ has conductor $\mathfrak{p}^{\mathfrak{f}_\chi}\neq 1$, then

$$\mathbf{L}_p(E,\rho\otimes\chi,1) \;=\; \varepsilon_F(\rho)\times\tau(\chi)^2\cdot\det\rho^*\left(\mathfrak{p}^{\mathfrak{f}_\chi}\right)\cdot\chi\left(\mathfrak{n}(\mathbf{g}_\rho)\right)\times\frac{L(E,\rho\otimes\chi^{-1},1)}{\left(\Omega_E^+\Omega_E^-\right)^{[F:\mathbb{Q}]}}$$

whilst at the trivial character,

$$\mathbf{L}_p(E,\rho,1) \;=\; \varepsilon_F(\rho)\times\left(1-\alpha_{\mathfrak{p}}(\mathbf{g}_\rho)\right)\left(1-\beta_{\mathfrak{p}}(\mathbf{g}_\rho)\right)\times\frac{L(E,\rho,1)}{\left(\Omega_E^+\Omega_E^-\right)^{[F:\mathbb{Q}]}}\,.$$

Precisely when either of $\alpha_{\mathfrak{p}}(\mathbf{g}_\rho)$ or $\beta_{\mathfrak{p}}(\mathbf{g}_\rho)$ equals one, we have an exceptional zero.

Case I – The eigenvalue $\alpha_{\mathfrak{p}}(\mathbf{g}_\rho)=1$ but $\beta_{\mathfrak{p}}(\mathbf{g}_\rho)\neq 1$:
This scenario is accessible using the method of Greenberg, Stevens et al [**7, 15**]. As a general comment, the sign term $\epsilon_{\infty,\mathbb{Q}}(E,\rho)$ in the complex functional equation for the Rankin L-function (attached to the pure motive $h^1(E)\otimes\mathrm{Ind}_F^{\mathbb{Q}}(\rho)$ over $\mathbb{Q}$) is related to its p-adic counterpart $w_p(2)$, via the simple formula

$$w_p(2) = (-1)^{e_{\mathfrak{p}}}\times\epsilon_{\infty,\mathbb{Q}}(E,\rho),$$

where $e_{\mathfrak{p}}$ counts the multiplicity of 1 inside the $\mathrm{Frob}_{\mathfrak{p}}^{-1}$-eigenvalues $\left\{\alpha_{\mathfrak{p}}(\mathbf{g}_\rho),\beta_{\mathfrak{p}}(\mathbf{g}_\rho)\right\}$. In Case I, clearly $e_{\mathfrak{p}}=1$, therefore the complex and p-adic signs must be different.

THEOREM 6.2. *The p-adic L-function vanishes at $s=1$, with derivative formula*

$$\left.\frac{\mathrm{d}\mathbf{L}_p(E,\rho,s)}{\mathrm{d}s}\right|_{s=1} \;=\; -2\times\left.\frac{\mathrm{d}\alpha_{\mathfrak{p}}(\mathcal{F}_k)}{\mathrm{d}k}\right|_{k=2}\cdot\left(1-\beta_{\mathfrak{p}}(\mathbf{g}_\rho)\right)\times\frac{\varepsilon_F(\rho)\cdot L(E,\rho,1)}{\left(\Omega_E^+\Omega_E^-\right)^{[F:\mathbb{Q}]}}\,.$$

This is the ρ-twisted analogue of the results in [**7**, Theorem 1.3] and [**15**, Theorem 1.1]. Provided the derivative of $\alpha_{\mathfrak{p}}(\mathcal{F}_k)$ does not vanish at $k=2$, one conjectures that $\mathrm{order}_{s_p=1}\mathbf{L}_p(E,\rho,s_p)\overset{?}{=}1+\mathrm{order}_{s_\infty=1}L(E,\rho,s_\infty)$, but its proof is a long way off.

Proof. First, if $\epsilon_{\infty,\mathbb{Q}}(E,\rho)=-1$, then $L(E,\rho,1)=0$, so the right-hand side is zero. On the other hand $\mathbf{L}_p(E,\rho,1)=0$ and $w_p(2)=+1$, hence the p-adic L-function $\mathbf{L}_p(E,\rho,s)$ has a zero of order ≥ 2 at $s=1$; the formula just collapses to '$0=0$'.

The interesting situation occurs when both $\epsilon_{\infty,\mathbb{Q}}(E,\rho)=+1$ and $w_p(2)=-1$. Developing our two-variable L-function about $(k,s)=(2,1)$:

$$\mathbf{L}_p\left(\mathcal{F},\mathbf{g}_\rho,\mathbf{1};\,k,s\right) \;=\; 0\;+\;c_1(s-1)\;+\;c_2(k-2)\;+\;\text{higher order terms.}$$

Moreover, Corollary 5.6 implies the above vanishes along the central line $s=k/2$, in which case its Taylor series coefficients satisfy $c_1=-2c_2$. Defining for the moment a

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temporary scalar $c_3 = \frac{i^{[F:\mathbb{Q}]}\cdot D_F^{-2}\cdot\langle\mathcal{F}_2\,,\mathcal{F}_2\rangle_{\mathfrak{n}_E}}{\Omega_{\mathbb{I}}(\lambda_{\mathcal{F}},P_2)\cdot(\Omega_E^+\Omega_E^-)^{[F:\mathbb{Q}]}} \in \mathbb{C}_p^\times$, we apply the reasoning

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{L}_p(E,\rho,s)}{\mathrm{d}s}\bigg|_{s=1} &= c_3 \times \frac{\mathrm{d}\mathbf{L}_p\big(\mathcal{F},\mathbf{g}_\rho,\mathbf{1};\ 2,s\big)}{\mathrm{d}s}\bigg|_{s=1} = c_3\times c_1\\
&= -2c_3\times c_2 = -2c_3 \times \frac{\partial\mathbf{L}_p\big(\mathcal{F},\mathbf{g}_\rho,\mathbf{1};\ k,s\big)}{\partial k}\bigg|_{(k,s)=(1,2)}\\
&\overset{\text{by 5.3\&4.5(i)}}{=} -2c_3 \times D_F\cdot\frac{\mathrm{d}\mathbf{L}_{p,2}^{(00)}(\mathcal{F},0)}{\mathrm{d}k}\bigg|_{k=2}\\
&\overset{\text{by 4.6(b)}}{=} -2c_3\times\frac{\alpha'_{\mathfrak{p}}(\mathcal{F},2)}{\alpha_{\mathfrak{p}}(\mathcal{F},2)}\cdot\left(1-\frac{\beta_{\mathfrak{p}}(\mathbf{g}_\rho)}{\alpha_{\mathfrak{p}}(\mathcal{F},2)}\right)\times D_F\cdot\mathbf{L}_{p,2}^{\mathrm{imp}}(\mathcal{F},0)\Big|_{k=2}.
\end{aligned}$$

But $D_F\cdot\mathbf{L}_{p,2}^{\mathrm{imp}}(\mathcal{F},0)\Big|_{k=2} = \varepsilon_F(\rho)\cdot L(E,\rho,1)\times\big(i^{-[F:\mathbb{Q}]}\cdot D_F^2\cdot\Omega_{\mathbb{I}}(\lambda_{\mathcal{F}},P_2)\big/\langle\mathcal{F}_2\,,\mathcal{F}_2\rangle_{\mathfrak{n}_E}\big)$ whilst the eigenvalue $\alpha_{\mathfrak{p}}(\mathcal{F},2) = a_p(E) = +1$. The result follows immediately. $\square$

Case II – The eigenvalues $\alpha_{\mathfrak{p}}(\mathbf{g}_\rho) = \beta_{\mathfrak{p}}(\mathbf{g}_\rho) = 1$:
Unfortunately, this situation appears more complicated than the preceding scenario. One can make progress towards proving a formula for the double derivative here, by first establishing the correct lower bound in the order of vanishing.

THEOREM 6.3. *The function $\mathbf{L}_p(E,\rho,s)$ has at least a double zero at $s=1$.*

Proof. As both of $\alpha_{\mathfrak{p}}(\mathbf{g}_\rho)$ and $\beta_{\mathfrak{p}}(\mathbf{g}_\rho)$ equal one, clearly $w_p(2) = (-1)^2\times\epsilon_{\infty,\mathbb{Q}}(E,\rho)$. If $w_p(2) = \epsilon_{\infty,\mathbb{Q}}(E,\rho) = +1$, then $\mathbf{L}_p(E,\rho,s)$ has even order of vanishing at $s=1$; however, $\mathbf{L}_p(E,\rho,1) = 0$ thence the order of vanishing must be ≥ 2.

Alternatively, if $w_p(2) = \epsilon_{\infty,\mathbb{Q}}(E,\rho) = -1$, it is enough to show that there is no simple zero in the p-adic L-function at $s=1$. By the reasoning employed earlier, the Taylor series for $\mathbf{L}_p\big(\mathcal{F},\mathbf{g}_\rho,\mathbf{1};\ k,s\big)$ about $(2,1)$ had the form

$$\mathbf{L}_p\big(\mathcal{F},\mathbf{g}_\rho,\mathbf{1};\ k,s\big) = 0 + c_1(s-1) + c_2(k-2) + \text{higher order terms},$$

and again $c_1 = -2c_2$ because the two-variable L-function vanishes along $s = k/2$. Fixing the value $s=1$ and using Theorem 4.6(c), its derivative with respect to k must then vanish at weight two since $\mathbf{L}_{p,2}^{(00)}(\mathcal{F},0)$ has at least a double zero there. Consequently, $c_2 = 0$ in which case $c_1 = 0$, so $\mathbf{L}_p(E,\rho,s)$ cannot have a simple zero. $\square$

Bearing in mind the expression we obtained for the double derivative with respect to the weight variable k (c.f. Theorem 4.6), it is tempting to hope that the vanishing along $s = k/2$ will yield an analogous formula for the cyclotomic variable $s\in\mathbb{Z}_p$. This is regrettably not the case. Indeed, the quadratic terms in the Taylor series for $\mathbf{L}_p\big(\mathcal{F},\mathbf{g}_\rho,\mathbf{1};\ k,s\big)$ are of the form $c_4\times(k-2)^2 + c_5\times(s-1)^2 + c_6\times(k-2)(s-1)$, and we lack derivative formulae along $s = k/2$ that would allow us to determine c_6.

CONJECTURE 6.4.

$$\frac{\mathrm{d}^2\mathbf{L}_p(E,\rho,s)}{\mathrm{d}s^2}\bigg|_{s=1} = 8\cdot\left(\frac{\mathrm{d}\alpha_{\mathfrak{p}}(\mathcal{F}_k)}{\mathrm{d}k}\bigg|_{k=2}\right)^2\cdot\frac{\varepsilon_F(\rho)\cdot L(E,\rho,1)}{\big(\Omega_E^+\Omega_E^-\big)^{[F:\mathbb{Q}]}}.$$

For instance, if the derivative of $\alpha_{\mathfrak{p}}(\mathcal{F}_k)$ is non-zero at $k=2$ one speculates that there should be a strict equality $\mathrm{order}_{s_p=1}\mathbf{L}_p(E,\rho,s_p) \overset{?}{=} 2+\mathrm{order}_{s_\infty=1}L(E,\rho,s_\infty)$, although again its proof lies well beyond the reach of current technology.

6.1. Behaviour over the number fields $\mathbb{Q}\big(\mu_{q^n}, \sqrt[q^n]{m}\big)$. We shall conclude by revisiting the situation first mentioned in the Introduction. Let us recall that $q \neq p$ denoted an auxiliary prime, and $E_{/\mathbb{Q}}$ an elliptic curve. We consider fields $L_n = \mathbb{Q}\big(\mu_{q^n}, \sqrt[q^n]{m}\big)$, $K_n = \mathbb{Q}\big(\mu_{q^n}\big)$ and $F = F_n = \mathbb{Q}\big(\mu_{q^n}\big) \cap \mathbb{R}$. Throughout, we assume the prime $p \geq 5$, and that p remains inert in F_n.

Let $\pi_n : \mathrm{Gal}(L_n/K_n) \longrightarrow \mu_{q^n}$ denote the character sending $\sigma \mapsto \sigma\big(\sqrt[q^n]{m}\big)\big/\sqrt[q^n]{m}$. The irreducible Artin representations factoring through $L_n/\mathbb{Q}$ all have the form

$$\rho_{t,\mathbb{Q}} \otimes \psi \quad \text{such that } 0 \leq t \leq n, \quad \rho_{t,\mathbb{Q}} = \mathrm{Ind}_{K_t}^{\mathbb{Q}}(\pi_t) \quad \text{and} \quad \psi : \mathrm{Gal}(K_n/\mathbb{Q}) \to \mathbb{C}^\times.$$

Working over the totally real field $F = F_t$, the corresponding representations are $\rho_t^{(\Psi)} := \rho_t \otimes \Psi = \mathrm{Ind}_{K_t}^{F_t}\big(\pi_t \otimes \mathrm{Res}_{K_t}(\psi)\big)$ with $\rho_t = \mathrm{Ind}_{K_t}^{F_t}(\pi_t)$ and $\Psi = \mathrm{Res}_{F_t}(\psi)$.

Remark. A basic calculation shows $\rho_{t,\mathbb{Q}} \otimes \psi$ occurs inside the regular representation of $\mathrm{Gal}(L_n/\mathbb{Q})$ exactly $(q-1)\cdot q^{t-1}$-times, which also coincides with its actual degree. It follows that the Hasse–Weil L-function for E over $\mathbb{Q}\big(\mu_{q^n}, \sqrt[q^n]{m}\big)$ splits into

$$L\big(E/L_n, s\big) \;=\; \prod_{\theta} L\big(E,\theta,s\big) \;\times\; \prod_{t=1}^{n} \prod_{\Psi:\mathrm{Gal}(F_n/F_t)\to\overline{\mathbb{Q}}^\times} \Big(L\big(E/F_t,\ \rho_t^{(\Psi)}, s\big)\Big)^{(q-1)q^{t-1}},$$

where θ ranges over Dirichlet characters whose conductors divide q^n.

Definition 6.5. The p-adic L-function for E over $\mathbb{Q}\big(\mu_{q^n}, \sqrt[q^n]{m}\big)$ is given by

$$\mathbf{L}_p\big(E/L_n, s\big) \;:=\; \prod_{\theta} \mathbf{L}_p^{(\theta)}\big(E,s\big) \times \prod_{t=1}^{n} \prod_{\Psi:\mathrm{Gal}(F_n/F_t)\to\overline{\mathbb{Q}}^\times} \Big(\mathbf{L}_p\big(E,\ \rho_t^{(\Psi)}, s\big)\Big)^{(q-1)q^{t-1}}$$

which is an Iwasawa function defined at all points $s \in \mathbb{Z}_p$.

The θ-twists are the p-adic L-functions (over $\mathbb{Q}$) of Mazur–Tate–Teitelbaum [**14**]; their derivatives at $s=1$ are described by the formula of Greenberg and Stevens. We are left with the task of inputting the information from Theorems 6.2 and 6.3, in order to bound below the order of vanishing for $\mathbf{L}_p\big(E/L_n, s\big)$.

Let us first verify that our running assumptions are valid over these number fields. We only need check Hypothesis (H4) at the orthogonal representations $\rho_t^{(\mathbf{1})} = \rho_t$ because it is precisely these which exhibit the exceptional zeroes.

Lemma 6.6. *For all $t \leq n$, Hypotheses (SS) and (H1)–(H5) hold over F_t.*

Proof. Let us check through the conditions (H1)–(H5) carefully.

(i) The curve E will be semistable over $F = F_t$, hence the base-change of f_E has square-free conductor as an $\mathcal{O}_{F_t}$-ideal, and (H1) must be true;

(ii) both (H2) and (H3) are automatic from the construction of $\{\mathcal{F}_\kappa\}_{\kappa\geq 2}$ and $\rho_t^{(\Psi)}$;

(iii) the self-duality of $\rho_t^{(\mathbf{1})} = \mathrm{Ind}_{K_t}^{F_t}(\pi_t)$ follows because π_t is orthogonal;

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(iv) since E has good reduction at the unique prime ideal of F_t lying above q, the level $\mathfrak{n}(\mathbf{g}_{\rho_t})$ is coprime to $\mathfrak{n}_E = \mathfrak{n}_{E/F_t}$ so (H5) holds;

(v) finally, we have already seen that (H1) $\Longrightarrow$ (SS) for a sufficiently small p-adic neighbourhood of $k_0 = 2$ in weight-space.

$\square$

Let ϕ be a Dirichlet character of conductor $p^{\mathfrak{f}_\phi}$, which we interpret as a finite order character of $\mathbb{Z}_p^\times$. By Mazur–Tate–Teitelbaum [**14**, Section 14], at all such characters ϕ each individual θ-twist interpolates

$$\mathbf{L}_p^{(\theta)}\big(E,\phi,1\big) \;=\; \theta(p)^{-\mathfrak{f}_\phi}\cdot\tau_{\mathbb{Q}}\big(\theta^{-1}\phi\big)\cdot\big(1-\theta^{-1}\phi(p)\big)\cdot\frac{L\big(E,\ \theta\times\phi^{-1},\ 1\big)}{\Omega_E^{\mathrm{sign}(\theta\times\phi)}}$$

which gives rise to a trivial p-adic zero when $\theta(p)=1$ and $\phi=\mathbf{1}_{\mathbb{Q}}$.

Let $\mathfrak{p}_t$ denote the $\mathcal{O}_{F_t}$-ideal generated by p. Provided $\chi=\mathrm{Res}_{F_t}(\phi)\neq\mathbf{1}_{F_t}$, then

$$\mathbf{L}_p(E,\rho_t^{(\Psi)}\otimes\chi,1)=\varepsilon_{F_t}\big(\rho_t^{(\Psi)}\big)\cdot\tau_{F_t}(\chi)^2\cdot\det\rho_t^{(\Psi)*}\big(\mathfrak{p}_t^{\mathfrak{f}_\chi}\big)\cdot\chi\big(\mathfrak{n}(\mathbf{g}_{\rho_t^{(\Psi)}})\big)$$
$$\cdot\frac{L\big(E,\rho_t^{(\Psi)}\otimes\chi^{-1},1\big)}{\big(\Omega_E^+\Omega_E^-\big)^{[F_t:\mathbb{Q}]}};$$

alternatively, if $\mathrm{Res}_{F_t}(\phi)=\mathbf{1}_{F_t}$, one has the interpolation

$$\mathbf{L}_p(E,\rho_t^{(\Psi)},1)\;=\;\varepsilon_{F_t}\big(\rho_t^{(\Psi)}\big)\cdot\Big(1-\Psi\big(\mathfrak{p}_t\big)\alpha_{\mathfrak{p}}(\mathbf{g}_{\rho_t})\Big)\Big(1-\Psi\big(\mathfrak{p}_t\big)\beta_{\mathfrak{p}}(\mathbf{g}_{\rho_t})\Big)\cdot\frac{L\big(E,\rho_t^{(\Psi)},1\big)}{\big(\Omega_E^+\Omega_E^-\big)^{[F_t:\mathbb{Q}]}}$$

which itself yields a trivial p-adic zero when $\Psi^{-1}\big(\mathfrak{p}_t\big)\in\big\{\alpha_{\mathfrak{p}}(\mathbf{g}_{\rho_t}),\beta_{\mathfrak{p}}(\mathbf{g}_{\rho_t})\big\}$.

DEFINITION 6.7. We define a p-adic multiplier $\mathfrak{M}_p(L_n,\phi)$ by the local factor

$$\prod_\theta\theta(p)^{-\mathfrak{f}_\phi}\cdot\tau_{\mathbb{Q}}\big(\theta^{-1}\phi\big)\times\prod_{t=1}^{n}\prod_\Psi\Big(\varepsilon_{F_t}\big(\rho_t^{(\Psi)}\big)\tau_{F_t}(\chi)^2\,\det\rho_t^{(\Psi)*}\big(\mathfrak{p}_t^{\mathfrak{f}_\chi}\big)\;\chi\big(\mathfrak{n}(\mathbf{g}_{\rho_t^{(\Psi)}})\big)\Big)^{(q-1)q^{t-1}},$$

where $\chi=\mathrm{Res}_{F_t}(\phi)$, and the right-hand product ranges over $\Psi:\mathrm{Gal}(F_n/F_t)\to\overline{\mathbb{Q}}^\times$.

Proof of Theorem 1.1. To prove this result, we multiply together our interpolation formulae at ϕ above, first over the one-dimensional θ-twists, and secondly over all the Artin twists $\rho_t^{(\Psi)}$. A little bookkeeping shows that

(i) $$L\big(E/L_n,\phi,1\big)\;=\;\prod_\theta L\big(E,\theta\phi,1\big)\;\times\;\prod_{t=1}^{n}\prod_\Psi L\big(E,\rho_t^{(\Psi)}\otimes\mathrm{Res}_{F_t}(\phi),1\big)^{(q-1)q^{t-1}}$$

(ii) $$\big(\Omega_E^+\Omega_E^-\big)^{[L_n:\mathbb{Q}]/2}\;=\;\prod_\theta\;\Omega_E^{\mathrm{sign}(\theta\times\phi)}\;\times\;\prod_{t=1}^{n}\prod_\Psi\;\Big(\big(\Omega_E^+\Omega_E^-\big)^{[F_t:\mathbb{Q}]}\Big)^{(q-1)q^{t-1}}$$

and the rest of the theorem follows readily. $\square$

Proof of Theorem 1.2. Assume first that $\mathfrak{p}_n$ remains inert in the CM extension K_n/F_n; it is an easy exercise to show for all $t\le n$, that each prime ideal $\mathfrak{p}_t\subset\mathcal{O}_{F_t}$ cannot split in K_t/F_t either. Let $\{\mathfrak{P}_1,\ldots,\mathfrak{P}_{\mathfrak{e}_p}\}$ denote the set of primes of $L_n=\mathbb{Q}\big(\mu_{q^n},\sqrt[q^n]{m}\big)$ lying

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above p, so that $\mathfrak{e}_p = \left[\mathbb{Q}\left(\mu_{q^n}, \sqrt[q^n]{m}\right) : \mathbb{Q}\left(\mu_{q^n}\right)\right] / \left[\mathbb{Q}_p\left(\mu_{q^n}, \sqrt[q^n]{m}\right) : \mathbb{Q}_p\left(\mu_{q^n}\right)\right] = q^{n_0}$ say. Only one of the θ-twists satisfies the exceptional zero condition (the branch $\theta = \mathbf{1}_{\mathbb{Q}}$), whilst the Artin twists $\rho_t^{(\Psi)} = \rho_t \otimes \Psi$ do so precisely when $1 \le t \le n_0$ and $\Psi = \mathbf{1}_{F_t}$. We immediately deduce

$$\operatorname{order}_{s=1}\mathbf{L}_p\left(E/L_n, s\right) \;\ge\; \operatorname{order}_{s=1}\mathbf{L}_p\left(E, s\right) + \sum_{t=1}^{n_0}(q-1)\, q^{t-1} \cdot \operatorname{order}_{s=1}\mathbf{L}_p\left(E, \rho_t, s\right)$$

which is bounded below by $1 + \sum_{t=1}^{n_0}(q-1)\, q^{t-1} \times 1 \;=\; q^{n_0} = \mathfrak{e}_p(L_n)$.

Conversely if $\mathfrak{p}_n$ splits in the CM extension K_n/F_n, each $\mathfrak{p}_t$ must split in K_t/F_t. In this situation, there are precisely $\mathfrak{e}_p = 2 \times q^{n_0}$ primes of L_n lying above p. Moreover, $\theta(p) = 1$ for exactly two choices of θ: the trivial character obviously, but also for the quadratic character θ' of conductor q (as p splits in the associated quadratic extension). If we write $\mathfrak{p}_t \cdot \mathcal{O}_{K_t} = \wp_t \cdot \overline{\wp}_t$, then provided $1 \le t \le n_0$, both eigenvalues $\alpha_{\mathfrak{p}}(\mathbf{g}_{\rho_t}) = \pi_t\left(\mathrm{Frob}_{\wp_t}^{-1}\right)$ and $\beta_{\mathfrak{p}}(\mathbf{g}_{\rho_t}) = \pi_t\left(\mathrm{Frob}_{\overline{\wp}_t}^{-1}\right)$ will equal one. It follows that $\operatorname{order}_{s=1}\mathbf{L}_p\left(E/L_n, s\right)$ is bounded below by

$$\operatorname{order}_{s=1}\mathbf{L}_p\left(E, s\right) + \;\operatorname{order}_{s=1}\mathbf{L}_p^{(\theta')}\left(E, s\right) + \;\sum_{t=1}^{n_0}\,(q-1)\, q^{t-1} \cdot \operatorname{order}_{s=1}\mathbf{L}_p\left(E, \rho_t, s\right)$$

which upon applying Theorem 6.3, must itself be bounded below by the quantity $1 + 1 + \sum_{t=1}^{n_0}\,(q-1)\, q^{t-1} \times 2 \;=\; 2 \times q^{n_0} = \mathfrak{e}_p(L_n)$. □

Proof of Theorem 1.3. The task is to calculate the coefficient of $(s-1)^{\mathfrak{e}_p(L_n)}$ occurring in the Taylor series expansion of $\mathbf{L}_p\left(E/L_n, s\right)$ about $s = 1$. Applying an identical argument to the demonstration of 1.2 above, this coefficient is given by the product

$$\frac{1}{\mathfrak{e}_p!}\,\frac{\mathrm{d}^{\mathfrak{e}_p}\mathbf{L}_p\left(E/L_n, s\right)}{\mathrm{d}s^{\mathfrak{e}_p}}\bigg|_{s=1} = \;\mathbf{L}_p'\left(E, 1\right) \times \prod_{\theta \ne \mathbf{1}_{\mathbb{Q}}} \mathbf{L}_p^{(\theta)}\left(E, 1\right) \times \prod_{t=1}^{n_0}\left(\mathbf{L}_p'\left(E, \rho_t, 1\right)\right)^{(q-1)q^{t-1}}$$
$$\times \prod_{t=n_0+1}^{n}\left(\mathbf{L}_p\left(E, \rho_t, 1\right)\right)^{(q-1)q^{t-1}} \times \prod_{t=1}^{n}\;\prod_{\Psi \ne \mathbf{1}_{F_t}}\left(\mathbf{L}_p\left(E, \rho_t^{(\Psi)}, 1\right)\right)^{(q-1)q^{t-1}}.$$

REMARKS.

(a) The term $\mathbf{L}_p'\left(E, 1\right)$ is computed via the Greenberg–Stevens formula.

(b) The terms $\mathbf{L}_p^{(\theta)}\left(E, 1\right)$ when $\theta \ne \mathbf{1}_{\mathbb{Q}}$ are already described on the previous page, and so are the special values $\mathbf{L}_p\left(E, \rho_t^{(\Psi)}, 1\right)$ when either $t > n_0$ or $\Psi \ne \mathbf{1}_{F_t}$.

(c) Finally, if $1 \le t \le n_0$ and $\Psi = \mathbf{1}_{F_t}$ then $\alpha_{\mathfrak{p}}(\mathbf{g}_{\rho_t}) = 1 = -\beta_{\mathfrak{p}}(\mathbf{g}_{\rho_t})$ because the local L-factor of ρ_t above p is $\left(1 - \pi_t(\mathrm{Frob}_{\wp}^{-1})\mathcal{N}_{K_t/\mathbb{Q}}(\wp)^{-s}\right) = \left(1 - \mathcal{N}_{F_t/\mathbb{Q}}(\mathfrak{p})^{-2s}\right)$; Theorem 6.2 then allows us to obtain the derivative of $\mathbf{L}_p\left(E, \rho_t, s\right)$ at $s = 1$.

Combining these remarks (a)–(c) together, one then arrives at the formula

$$\frac{1}{\mathfrak{e}_p!}\,\frac{\mathrm{d}^{\mathfrak{e}_p}\mathbf{L}_p\left(E/L_n, s\right)}{\mathrm{d}s^{\mathfrak{e}_p}}\bigg|_{s=1} \;=\; \mathcal{L}_p(E/L_n) \;\times\; \mathcal{E}_p\left(p^{-s}\right)\Big|_{s=0} \times\; \frac{\mathfrak{M}_p\left(L_n, \mathbf{1}\right) \cdot L\left(E/L_n, 1\right)}{\left(\Omega_E^+\Omega_E^-\right)^{[L_n:\mathbb{Q}]/2}},$$

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where the $\mathcal{L}$-invariant term is defined by

$$\mathcal{L}_p(E/L_n) \;=\; \frac{\log_p(\mathbf{q}_{E,p})}{\mathrm{ord}_p(\mathbf{q}_{E,p})} \;\times\; \prod_{t=1}^{n_0}\left(-2\times\frac{\mathrm{d}\alpha_{\mathfrak{p}_t}(\mathcal{F}_k)}{\mathrm{d}k}\bigg|_{k=2}\right)^{(q-1)q^{t-1}}.$$

If one sets $d(t)=\left[K_t:\mathbb{Q}\right]$ and $d^+(t)=\left[F_t:\mathbb{Q}\right]$, the rational polynomial $\mathcal{E}_p(X)$ equals the product of $\prod_{\theta\neq\mathbf{1}_{\mathbb{Q}}}\left(1-\theta^{-1}(p)X\right)$ with

$$\prod_{t=1}^{n}\prod_{\Psi}\left(1-\Psi(\mathfrak{p}_t)\beta_{\mathfrak{p}}(\mathbf{g}_{\rho_t})X^{d^+(t)}\right)^{d(t)}\times\prod_{t=1}^{n}\prod_{\Psi\neq\mathbf{1}_{F_t}\text{ or }t>n_0}\left(1-\Psi(\mathfrak{p}_t)\alpha_{\mathfrak{p}}(\mathbf{g}_{\rho_t})X^{d^+(t)}\right)^{d(t)}.$$

The result will follow, providing one can establish:

(i) the multiplier $\mathfrak{M}_p\left(L_n,\phi\right)$ evaluated at $\phi=\mathbf{1}$ returns the value $\sqrt{\mathrm{disc}(L_n)}$;

(ii) the derivative of $\alpha_{\mathfrak{p}_t}(\mathcal{F}_k)$ at $k=2$ is equal to $-\frac{1}{2}\times[F_t:\mathbb{Q}]\times\frac{\log_p(\mathbf{q}_{E,p})}{\mathrm{ord}_p(\mathbf{q}_{E,p})}$;

(iii) $\mathcal{E}_p(X)$ agrees with the polynomial $\mathcal{E}_p\left(\Sigma_{L_n/\mathbb{Q}},X\right)$ given in the Introduction.

Working in reverse order, to show (iii) observe if $t\le n_0$ then the factor $\left(1-X^{d^+(t)}\right)$ occurs with multiplicity $d(t)=\left[K_t:\mathbb{Q}\right]$ inside of the $\det\left(1-X\cdot\Sigma_{L_n/\mathbb{Q}}(\mathrm{Frob}_p^{-1})\right)$. In other words, the full product $\prod_{t=0}^{n_0}\left(1-X^{d^+(t)}\right)^{[K_t:\mathbb{Q}]}\cdot\mathcal{E}_p(X)$ coincides with

$$\prod_{\theta}\left(1-\theta^{-1}(p)X\right)\times\prod_{t=1}^{n}\prod_{\Psi}\left(\left(1-\Psi(\mathfrak{p}_t)\alpha_{\mathfrak{p}}(\mathbf{g}_{\rho_t})X^{d^+(t)}\right)\left(1-\Psi(\mathfrak{p}_t)\beta_{\mathfrak{p}}(\mathbf{g}_{\rho_t})X^{d^+(t)}\right)\right)^{d(t)}$$

which is precisely the characteristic polynomial of $\Sigma_{L_n/\mathbb{Q}}\left(\mathrm{Frob}_p^{-1}\right)$, so we are done.

REMARK. Because F_t is a totally real field, we can use the result in [**15**, Proposition 8.7] to find the derivative of the Hecke eigenvalue $\alpha_{\mathfrak{p}_t}(\mathcal{F},k)$ at the base weight $k=2$. The relevant formula one needs is

$$\frac{\mathrm{d}\alpha_{\mathfrak{p}_t}(\mathcal{F}_k)}{\mathrm{d}k}\bigg|_{k=2} \;=\; -\frac{1}{2}\times\left[\mathcal{K}_{F_t,\mathfrak{p}_t}:\mathbb{F}_p\right]\times\frac{\log_p(\mathbf{q}_{E,p})}{\mathrm{ord}_p(\mathbf{q}_{E,p})}\quad\text{where }\ \mathcal{K}_{F_t,\mathfrak{p}_t}=\mathcal{O}_{F_t}/\mathfrak{p}_t,$$

however, the residue class degree $[\mathcal{K}_{F_t,\mathfrak{p}_t}:\mathbb{F}_p]=[F_t:\mathbb{Q}]$ as the prime p is inert.

This leaves us with the demonstration of (i). Writing out $\mathfrak{M}_p\left(L_n,\mathbf{1}\right)$ in full yields

$$\prod_{\theta}\tau_{\mathbb{Q}}\left(\theta^{-1}\right)\times\prod_{t=1}^{n}\prod_{\Psi}\left(\varepsilon_{F_t}\left(\rho_t^{(\Psi)}\right)\right)^{(q-1)q^{t-1}} = \;\varepsilon_{\mathbb{Q}}\left(\oplus\,\theta^{-1}\right)\;\times\prod_{\rho\in\Sigma_{L_n/\mathbb{Q}},\ \dim(\rho)>1}\varepsilon_{\mathbb{Q}}\left(\rho\right)^{\dim(\rho)}$$

where we have utilised the Artin formalism: $\varepsilon_F\left(\rho\right)=\varepsilon_{\mathbb{Q}}\left(\mathrm{Ind}_F^{\mathbb{Q}}(\rho)\right)$ for $F/\mathbb{Q}$ normal. The above is precisely the ε-factor associated to the regular representation $\Sigma_{L_n/\mathbb{Q}}$ at $s=0$, which is widely known [**20**] to equal the square root of $\mathrm{disc}(L_n)$. □

A. Appendix relating $\mathrm{Hol}\left(\mathbf{g}_l\middle|V_{\mathfrak{p}}\times\widetilde{\mathbf{E}}_{k-l}\right)^{\mathrm{ord}}$ to known quantities. Recall in Section 3, we omitted the proof of a $\mathfrak{p}$-stabilisation identity involving $\mathrm{Hol}(\mathcal{G}_l\widetilde{\mathbf{E}})$. We guard the same conditions as Theorem 3.4, and now supply the missing proof.

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LEMMA A.1. *If* $\mathbf{g}_l \in \mathcal{S}_l(\mathfrak{c}\mathfrak{p}^{m-1}, \eta)$ *and* $\theta = \psi\eta^{-1}\omega_F^{2-k}\epsilon \neq \mathbf{1}$, *then*

$$\begin{aligned}&\mathrm{Hol}\left(\mathbf{g}_l \middle| V_{\mathfrak{p}} \times \widetilde{\mathbf{E}}_{k-l}\right)^{\mathrm{ord}}\\ &\quad = \mathrm{Hol}\left(\mathbf{g}_l \middle| V_{\mathfrak{p}} \times \mathbf{E}_{k-l}\right)^{\mathrm{ord}} - \theta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-l-1-r} \times \mathrm{Hol}\left(\mathbf{g}_l \times \mathbf{E}_{k-l}\right)^{\mathrm{ord}}\Big| V_{\mathfrak{p}}.\end{aligned}$$

Proof. The argument reduces to the following four observations (A.1.1–A.1.4), concerning ξ-expansions of the various HMF's occurring in the statement of A.1.

Let us treat the case $r > 0$ first.

(A.1.1) During the demonstration of Proposition 3.1, we established that

$$\begin{aligned}&\mathrm{Hol}\left(\mathbf{g}_l \middle| V_{\mathfrak{p}} \times \widetilde{\mathbf{E}}_{k-l}\left(r-(k-l-1), \theta; \mathfrak{c}\mathfrak{p}^n\right)\right)_\lambda \Big| U_{\mathfrak{p}}^{N!}\\ &\quad \equiv \sum_{0 \ll \xi \in \widetilde{t}_\lambda} \delta_\lambda\left(\mathfrak{p}^{N!}\xi\right) \mathbf{e}_F(\xi z) \mod \mathcal{N}(\mathfrak{p})^{N!}\end{aligned}$$

where the ξ-coefficients

$$\begin{aligned}\delta_\lambda\left(\mathfrak{p}^{N!}\xi\right) \ :=\ & \mathcal{N}(\widetilde{t}_\lambda)^{k/2-r-1} \sum_{\mathfrak{p}^{N!}\xi = \xi_1+\xi_2} C\left(\xi_1\widetilde{t}_\lambda^{-1}, \mathbf{g}_l \middle| V_{\mathfrak{p}}\right) \times (-1)^{(k-l-1-r)[F:\mathbb{Q}]}\\ & \times \mathcal{N}(\xi_2)^{k-l-1-r} \times \sum_{\substack{\widetilde{\xi}_2 = \widetilde{b}\times\widetilde{c},\ b\in\widetilde{t}_\lambda\\ c\in\mathcal{O}_F-\mathfrak{p}}} \mathrm{sign}\mathcal{N}(\widetilde{b})^{k-l-1} \times \mathcal{N}(\widetilde{b})^{2r-(k-l-1)} \times \theta^*(\widetilde{c})\\ \equiv\ & (-1)^{r[F:\mathbb{Q}]} \times \mathcal{N}(\widetilde{t}_\lambda)^{k/2-1} \times \sum_{\mathfrak{p}^{N!}\xi = \xi_1+\xi_2} C\left(\mathfrak{p}^{-1}\xi_1\widetilde{t}_\lambda^{-1}, \mathbf{g}_l\right)\\ & \times \sum_{\substack{\widetilde{\xi}_2 = \widetilde{b}\times\widetilde{c},\ b\in\widetilde{t}_\lambda\\ c\in\mathcal{O}_F-\mathfrak{p}}} \left((-1)^{[F:\mathbb{Q}]}\mathrm{sign}\mathcal{N}(\widetilde{b})\right)^{(k-l-1)} \mathcal{N}(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1})^r\\ & \times \theta^*(\widetilde{c})\mathcal{N}(\widetilde{c})^{k-l-1} \mod \mathcal{N}(\mathfrak{p})^{N!}.\end{aligned}$$

(A.1.2) An absolutely identical argument shows

$$\begin{aligned}&\mathrm{Hol}\left(\mathbf{g}_l \middle| V_{\mathfrak{p}} \times \mathbf{E}_{k-l}\left(r-(k-l-1), \theta; \mathfrak{c}\mathfrak{p}^n\right)\right)_\lambda \Big| U_{\mathfrak{p}}^{N!}\\ &\quad \equiv \sum_{0 \ll \xi \in \widetilde{t}_\lambda} \delta'_\lambda\left(\mathfrak{p}^{N!}\xi\right) \mathbf{e}_F(\xi z) \mod \mathcal{N}(\mathfrak{p})^{N!}\end{aligned}$$

with the modified ξ-coefficient satisfying

$$\begin{aligned}\delta'_\lambda\left(\mathfrak{p}^{N!}\xi\right) \ \equiv\ & (-1)^{r[F:\mathbb{Q}]} \times \mathcal{N}(\widetilde{t}_\lambda)^{k/2-1} \times \sum_{\mathfrak{p}^{N!}\xi = \xi_1+\xi_2} C\left(\mathfrak{p}^{-1}\xi_1\widetilde{t}_\lambda^{-1}, \mathbf{g}_l\right)\\ & \times \sum_{\substack{\widetilde{\xi}_2 = \widetilde{b}\times\widetilde{c},\ b\in\widetilde{t}_\lambda\\ c\in\mathcal{O}_F}} \left((-1)^{[F:\mathbb{Q}]}\mathrm{sign}\mathcal{N}(\widetilde{b})\right)^{(k-l-1)} \mathcal{N}(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1})^r \times \theta^*(\widetilde{c})\mathcal{N}(\widetilde{c})^{k-l-1} \mod \mathcal{N}(\mathfrak{p})^{N!}.\end{aligned}$$

(Note $\delta'_\lambda\left(\mathfrak{p}^{N!}\xi\right)$ differs from $\delta_\lambda\left(\mathfrak{p}^{N!}\xi\right)$ only in the range of its second summation.)

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(A.1.3) Thirdly, replacing '$\mathbf{g}_l|V_{\mathfrak{p}}$' with '$\mathbf{g}_l$' one deduces

$$\mathrm{Hol}\Big(\mathbf{g}_l \times \mathbf{E}_{k-l}\big(r-(k-l-1),\theta;\mathfrak{c}\mathfrak{p}^n\big)\Big)_\lambda\Big|U_{\mathfrak{p}}^{N!} \equiv \sum_{0\ll\xi\in\widetilde{t}_\lambda} \delta''_\lambda\left(\mathfrak{p}^{N!}\xi\right)\mathbf{e}_F(\xi z) \mod \mathcal{N}(\mathfrak{p})^{N!}$$

where this time the coefficients satisfy the congruence

$$\delta''_\lambda\left(\mathfrak{p}^{N!}\xi\right) \equiv (-1)^{r[F:\mathbb{Q}]} \times \mathcal{N}(\widetilde{t}_\lambda)^{k/2-1} \times \sum_{\mathfrak{p}^{N!}\xi=\xi_1+\xi_2} C\big(\xi_1\widetilde{t}_\lambda^{-1},\mathbf{g}_l\big)$$
$$\times \sum_{\substack{\widetilde{\xi}_2=\widetilde{b}\times\widetilde{c},\ b\in\widetilde{t}_\lambda\\ c\in\mathcal{O}_F}} \big((-1)^{[F:\mathbb{Q}]}\mathrm{sign}\mathcal{N}(\widetilde{b})\big)^{(k-l-1)}\mathcal{N}(\widetilde{c}^{-1}\widetilde{b}\widetilde{t}_\lambda^{-1})^r \times \theta^*(\widetilde{c})\mathcal{N}(\widetilde{c})^{k-l-1} \mod \mathcal{N}(\mathfrak{p})^{N!}.$$

(A.1.4) Lastly, it is an easy exercise for the reader to check

$$\delta'_\lambda\left(\mathfrak{p}^{N!}\xi\right) \equiv \delta_\lambda\left(\mathfrak{p}^{N!}\xi\right) + \theta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-l-1-r}\times\delta''_\lambda\left(\mathfrak{p}^{N!}\times\mathfrak{p}^{-1}\xi\right) \mod \mathcal{N}(\mathfrak{p})^{N!}$$

in which case, on the level of ξ-expansions:

$$\mathrm{Hol}\big(\mathbf{g}_l|V_{\mathfrak{p}}\times\mathbf{E}_{k-l}\big)_\lambda\Big|U_{\mathfrak{p}}^{N!} \equiv \mathrm{Hol}\big(\mathbf{g}_l|V_{\mathfrak{p}}\times\widetilde{\mathbf{E}}_{k-l}\big)_\lambda\Big|U_{\mathfrak{p}}^{N!}$$
$$+\,\theta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-l-1-r}\times\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)_\lambda\Big|U_{\mathfrak{p}}^{N!}\Big|V_{\mathfrak{p}} \mod \mathcal{N}(\mathfrak{p})^{N!}\cdot\mathcal{O}_{\mathbb{C}_p}[\![\xi]\!].$$

The result follows for $r>0$, upon allowing the exponent $N\to\infty$. □

REMARK. To treat the $r=0$ situation, let us start by introducing 'correction terms'

$$c(\xi,k,\theta)_\lambda := \mathcal{N}(\widetilde{t}_\lambda)^{k/2-1}\times 2^{-[F:\mathbb{Q}]}\cdot\zeta_F^{(\mathfrak{c})}\big(1-(k-l),\theta_{\mathfrak{p}}\big)\times C\big(\xi\widetilde{t}_\lambda^{-1},\mathbf{g}_l|V_{\mathfrak{p}}\big),$$
$$c'(\xi,k,\theta)_\lambda := \mathcal{N}(\widetilde{t}_\lambda)^{k/2-1}\times 2^{-[F:\mathbb{Q}]}\cdot\zeta_F^{(\mathfrak{c})}\big(1-(k-l),\theta\big)\times C\big(\xi\widetilde{t}_\lambda^{-1},\mathbf{g}_l|V_{\mathfrak{p}}\big)$$
and
$$c''(\xi,k,\theta)_\lambda := \mathcal{N}(\widetilde{t}_\lambda)^{k/2-1}\times 2^{-[F:\mathbb{Q}]}\cdot\zeta_F^{(\mathfrak{c})}\big(1-(k-l),\theta\big)\times C\big(\xi\widetilde{t}_\lambda^{-1},\mathbf{g}_l\big).$$

Note if $r=0$ then the first factor $\delta_\lambda\left(\mathfrak{p}^{N!}\xi\right)$ requires the addition of $c\big(\mathfrak{p}^{N!}\xi,k,\theta\big)_\lambda$, the second factor $\delta'_\lambda\left(\mathfrak{p}^{N!}\xi\right)$ requires the addition of the correction $c'\big(\mathfrak{p}^{N!}\xi,k,\theta\big)_\lambda$, and likewise the final factor $\delta''_\lambda\left(\mathfrak{p}^{N!}\xi\right)$ requires the addition of the $c''\big(\mathfrak{p}^{N!}\xi,k,\theta\big)_\lambda$. However, at every $N>0$, these three terms satisfy the equality

$$c'\big(\mathfrak{p}^{N!}\xi,k,\theta\big)_\lambda = c\big(\mathfrak{p}^{N!}\xi,k,\theta\big)_\lambda + \theta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-l-1}\times c''\big(\mathfrak{p}^{N!}\times\mathfrak{p}^{-1}\xi,k,\theta\big)_\lambda$$

so it again happens that

$$\mathrm{Hol}\big(\mathbf{g}_l|V_{\mathfrak{p}}\times\mathbf{E}_{k-l}\big)_\lambda\Big|U_{\mathfrak{p}}^{N!} \equiv \mathrm{Hol}\big(\mathbf{g}_l|V_{\mathfrak{p}}\times\widetilde{\mathbf{E}}_{k-l}\big)_\lambda\Big|U_{\mathfrak{p}}^{N!}$$
$$+\,\theta^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-l-1}\times\mathrm{Hol}\big(\mathbf{g}_l\times\mathbf{E}_{k-l}\big)_\lambda\Big|U_{\mathfrak{p}}^{N!}\Big|V_{\mathfrak{p}} \mod \mathcal{N}(\mathfrak{p})^{N!}\cdot\mathcal{O}_{\mathbb{C}_p}[\![\xi]\!].$$

Sending the exponent $N\to\infty$ as before, the case $r=0$ is also established.

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B. Appendix Writing the pseudo-eigenvalue $\varpi(\mathbf{g}_\rho)$ in terms of ε-factors. Before supplying the proof of Lemma B.1, let us first discuss some related concepts. For any Artin representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}(V, \mathbb{C})$, its global ε-factor over F can be decomposed as an infinite product

$$\varepsilon_F(\rho, s) \quad = \prod_{\text{all places } \nu} \varepsilon_{F_\nu}(\rho_\nu, \tau_\nu, \mathrm{d}x_\nu; s).$$

Each local factor depends on a normalisation of additive characters τ_ν, and of Haar measures $\mathrm{d}x_\nu$. We shall choose the Haar measure which assigns $\mathbb{Z}_p$ measure one, and the additive character $\tau : (\mathbb{Q}_p, +) \longrightarrow \mathbb{C}^\times$ given by $\tau(ap^{-m}) = \exp(2\pi ia/p^m)$ at every element $a \in \mathbb{Z}_p$.

The Artin L-function attached to ρ over the field F is given by an Euler product

$$L(\rho, s) \quad = \prod_{\text{finite places } \nu} \det\Big(1 - \mathcal{N}_{F/\mathbb{Q}}(\nu)^{-s} \cdot \mathrm{Frob}_\nu^{-1} \Big| V(\rho)^{I_\nu}\Big) \quad \text{for } \mathrm{Re}(s) \gg 0,$$

where Frob_ν is an arithmetic Frobenius element for ν, and I_ν is the inertia group. If one completes the L-function at infinity by multiplying it with the gamma factor $\Gamma_\infty(s) := \big((2\pi)^{-s} \times \Gamma(s)\big)^{[F:\mathbb{Q}]}$, the completed L-function (conjecturally) extends to a meromorphic function on the whole of $\mathbb{C}$, and satisfies an equation relating the value at s with the value at $1 - s$.

(Thus, $\varepsilon_F(\rho, 1/2)$ corresponds to the root number in this functional equation.)

LEMMA B.1. *If $\mathbf{g}_\rho$ is the weight one HMF associated to $\rho : G_F \to \mathrm{GL}_2(\mathbb{C})$, then*

$$\varepsilon_F(\rho) \;:=\; \varepsilon_F(\rho, s)\Big|_{s=0} \;=\; i^{[F:\mathbb{Q}]} \cdot D_F \cdot \sqrt{\mathcal{N}\big(\mathfrak{n}(\mathbf{g}_\rho)\big)} \;\times\; \varpi(\mathbf{g}_\rho).$$

Proof. Following Shimura, one defines $R(\mathbf{g}_\rho, s) := \mathcal{N}\big(\mathfrak{n}(\mathbf{g}_\rho)\mathfrak{d}^2\big)^{s/2}\, \Gamma_\infty(s) \times L(\mathbf{g}_\rho, s)$. Applying [**18**, equation (2.48)], on the level of primitive forms there is a symmetry

$$R(\mathbf{g}_\rho, s) \quad = \quad i^{[F:\mathbb{Q}]} \cdot R\big(\mathbf{g}_\rho \big| J_{\mathfrak{n}(\mathbf{g}_\rho)}, 1 - s\big)$$

and by the definition of $\varpi(\mathbf{g}_\rho)$, one has $R\big(\mathbf{g}_\rho \big| J_{\mathfrak{n}(\mathbf{g}_\rho)}, 1 - s\big) = \varpi(\mathbf{g}_\rho) \times R\big(\mathbf{g}_\rho^\#, 1 - s\big)$. However, $L(\rho, s) = L(\mathbf{g}_\rho, s)$ and $L(\rho^\vee, s) = L(\mathbf{g}_\rho^\#, s)$, thus the functional equation

$$\Gamma_\infty(s) \times L(\rho, s) \quad = \quad \varepsilon_F(\rho, s) \cdot \Gamma_\infty(1 - s) \times L(\rho^\vee, 1 - s)$$

can be rewritten as

$$R(\mathbf{g}_\rho, s) \quad = \quad \varepsilon_F(\rho, s) \cdot \mathcal{N}\big(\mathfrak{n}(\mathbf{g}_\rho)\mathfrak{d}^2\big)^{s - \frac{1}{2}} \times R(\mathbf{g}_\rho^\#, 1 - s).$$

Comparing this with the version $R(\mathbf{g}_\rho, s) = i^{[F:\mathbb{Q}]} \cdot \varpi(\mathbf{g}_\rho) \times R(\mathbf{g}_\rho^\#, 1 - s)$ above, one immediately deduces $i^{[F:\mathbb{Q}]} \cdot \varpi(\mathbf{g}_\rho) = \varepsilon_F(\rho, s) \cdot \mathcal{N}(\mathfrak{n}(\mathbf{g}_\rho)\mathfrak{d}^2)^{s-\frac{1}{2}}$ for all $s \in \mathbb{C}$; clearly putting $s = 0$ will then produce the desired equality. $\square$

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