THE STEINER POINT OF A CONVEX POLYTOPE

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1. Introduction. Associated with each bounded convex set K in n-dimensional euclidean space E^n is a point $\mathbf{s}(K)$ known as its Steiner point. First considered by Steiner in 1840 (**6**, p. 99) in connection with an extremal problem for convex regions, this point has been found useful in some recent investigations (for example, **4**) because of the linearity property

$$\mathbf{s}(\lambda_1 K_1 + \lambda_2 K_2) = \lambda_1 \mathbf{s}(K_1) + \lambda_2 \mathbf{s}(K_2). \tag{1}$$

Addition on the left is vector addition of convex sets.

A definition of $\mathbf{s}(K)$ (which differs from that of Steiner, but is more convenient here) will be given in §2. We mention, by way of example, that if K is a line segment, then $\mathbf{s}(K)$ is its mid-point, and if K is a convex p-gon with internal angle θ_i at the vertex V_i $(i = 1, \ldots, p)$, then $\mathbf{s}(K)$ is the centroid of a system of masses, the mass $\pi - \theta_i$ being attached to the vertex V_i; see (**4**).

The purpose of this paper is to establish an interesting identity connecting the Steiner point of a convex polytope and the Steiner points of its faces:

(2) THEOREM. *Let P be any d-dimensional convex polytope in E^n and, for $0 \leqslant j \leqslant d-1$, let F_i^j $(i = 1, \ldots, f_j)$ be its j-faces. Then*

$$(1 + (-1)^{d-1})\mathbf{s}(P) = \sum_{i=1}^{f_0} \mathbf{s}(F_i^0) - \sum_{i=1}^{f_1} \mathbf{s}(F_i^1) + \ldots + (-1)^{d-1} \sum_{i=1}^{f_{d-1}} \mathbf{s}(F_i^{d-1}). \tag{3}$$

This relation resembles the well-known identity of Euler and Schläfli:

$$(1 + (-1)^{d-1}) = f_0 - f_1 + \ldots + (-1)^{d-1} f_{d-1},$$

but is of a different nature being a vector, as opposed to a scalar, identity. If we conventionally put $f_d = f_{-1} = 1$ and $\mathbf{s}(F_1^d) = \mathbf{s}(F_1^{-1}) = \mathbf{s}(P)$, then (3) can be written in the more symmetrical form

$$\sum_{j=-1}^{d} (-1)^j \sum_{i=1}^{f_j} \mathbf{s}(F_i^j) = \mathbf{0}.$$

The proof of the theorem will be given in §§2 and 3, and in §4 we shall show that, in the case of simple polytopes, there exist further linear relations between the Steiner points that are analogous to the Dehn–Sommerville relations between the numbers f_i.

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I am indebted to Professor C. A. Rogers for reading an early version of this paper and suggesting a number of improvements that have been incorporated.

2. External angles of polytopes. Let P be any d-dimensional convex polytope in E^n and let S^{n-1} denote the unit $(n-1)$-sphere centred on the origin $\mathbf{0}$. For any vertex F^0 of P, denote by $V(F^0, P)$ the subset of S^{n-1} consisting of all those unit vectors $\mathbf{u}$ which are normal to the supporting hyperplanes $L(\mathbf{u}, P)$ of P for which $L(\mathbf{u}, P) \cap P = F^0$. It is clear that $V(F^0, P)$ is a convex spherical polytope in S^{n-1} since it is the intersection of S^{n-1} with a convex polyhedral cone bounded by hyperplanes through $\mathbf{0}$ which are perpendicular to the edges of P that meet in F^0. The ratio of the $(n-1)$-content of $V(F^0, P)$ to the $(n-1)$-content of S^{n-1} is denoted by $\psi(F^0, P)$ and is, for obvious geometrical reasons, called the *external angle* of P at the vertex F^0. We now establish a simple formula for $\mathbf{s}(P)$ in terms of these external angles.

(4) LEMMA. *Let $\mathbf{v}_j$ be the position vector of the vertex F_j^0 of P $(j = 1, \ldots, f_0)$. Then*

$$\mathbf{s}(P) = \sum_{j=1}^{f_0} \mathbf{v}_j\, \psi(F_j^0, P). \tag{5}$$

Proof. The Steiner point of an arbitrary closed bounded convex set K in E^n is conveniently defined by (**4**, p. 11)

$$\mathbf{s}(K) = \frac{1}{\sigma_n} \int_{S^{n-1}} \mathbf{u} H(\mathbf{u}, K)\, d\omega, \tag{6}$$

where $\mathbf{u}$ is a variable unit vector, $H(\mathbf{u}, K)$ is the supporting function of K (**2**, p. 23), $d\omega$ is an element of surface area of the sphere S^{n-1}, and σ_n is the volume of the n-dimensional unit ball. From this definition, the linearity (1) of $\mathbf{s}(K)$ is immediate and so is the fact that $\mathbf{s}(K)$ is a continuous function of K.

If K is a strictly convex body that is sufficiently smooth for the supporting function to have a continuous gradient vector $\nabla H(\mathbf{u}, K)$ everywhere except possibly at $\mathbf{0}$, then for $\mathbf{u} \neq \mathbf{0}$, the point of contact $\xi(\mathbf{u}, K) = L(\mathbf{u}, K) \cap K$ is given by $\xi(\mathbf{u}, K) = \nabla H(\mathbf{u}, K)$ (**2**, p. 26). Following a suggestion of C. A. Rogers, we integrate this over the unit ball B^n to obtain

$$\int_{B^n} \xi(\mathbf{u}, K)\, d\mathbf{u} = \int_{B^n} \nabla H(\mathbf{u}, K)\, d\mathbf{u} = \int_{S^{n-1}} \mathbf{u} H(\mathbf{u}, K)\, d\omega,$$

the singularity at the origin being so mild that there is no difficulty in the necessary justification. Hence from (6) we deduce

$$\mathbf{s}(K) = \frac{1}{\sigma_n} \int_{B^n} \xi(\mathbf{u}, K)\, d\mathbf{u}.$$

Since $\xi(\mathbf{u}, K) = \xi(\lambda \mathbf{u}, K)$ for $\lambda > 0$, this can also be written

$$\mathbf{s}(K) = \frac{1}{\tau_n} \int_{S^{n-1}} \xi(\mathbf{u}, K)\, d\omega, \tag{7}$$

where $\tau_n = n\sigma_n$ is the $(n-1)$-dimensional surface area of S^{n-1}.

We shall now show that (7) holds generally, that is, without the conditions of smoothness, so long as the integral is interpreted in the sense of Lebesgue. Let $K_1, K_2, \ldots,$ be a sequence of convex bodies converging to any given closed bounded convex set K, where each K_i satisfies the smoothness conditions mentioned above. Then (7) holds for each K_i and, as $i \to \infty$, the left side approaches $\mathbf{s}(K)$ by continuity. For each u, the sequence of points $\{\xi(\mathbf{u}, K_i)\}$ is uniformly bounded and so some subsequence converges to a point p. Since $L(\mathbf{u}, K_i) \to L(\mathbf{u}, K)$, we deduce $p \in L(\mathbf{u}, K)$, and since $K_i \to K$, we deduce $p \in K$. Hence $p \in L(\mathbf{u}, K) \cap K$. Now $L(\mathbf{u}, K) \cap K$ is the single point $\xi(\mathbf{u}, K)$ except when $\mathbf{u}$ belongs to a set of measure zero on S^{n-1}; see **(1)**. Consequently the Lebesgue integral

$$\int_{S^{n-1}} \xi(\mathbf{u}, K)\, d\omega \tag{8}$$

exists, and by the bounded convergence theorem,

$$\lim_{i\to\infty} \int_{S^{n-1}} \xi(\mathbf{u}, K_i)\, d\omega = \int_{S^{n-1}} \xi(\mathbf{u}, K)\, d\omega.$$

We deduce that (7) holds for the set K.

Suppose, in particular, that $K = P$. In this case $\xi(\mathbf{u}, P)$ takes the constant value $\mathbf{v}_j$ on the region $V(F_j^0, P) \subset S^{n-1}$ and these open spherical polytopes (one corresponding to each vertex) cover S^{n-1} except for a set of measure zero. From (7), therefore, $\mathbf{s}(P)$ is equal to

$$\sum_{i=1}^{f_0} \mathbf{v}_j\, \psi(F_j^0, P)$$

and the lemma is proved.

If K is any closed bounded convex set in E^n, then it is easy to deduce from the definition (6) that the position of $\mathbf{s}(K)$ relative to K is independent of the value of n. In the case of a polytope this also follows simply from the lemma, for the value of $\psi(F^0, P)$ does not depend on n, Hence we may assume, without loss of generality, that the d-dimensional polytope P lies in E^d.

We remark that relation (5) may be used, instead of (6), to define the Steiner point of the polytope P, and in some ways it is more convenient to do so. Proceeding in this way has the disadvantage that relation (1) is no longer obvious. We digress briefly to show the geometrical significance of (1) when interpreted in terms of the external angles of polytopes and their vector sums.

Let P and Q be two polytopes in E^d, P having vertices F_i^0 with position vectors $\mathbf{v}_i$, and Q having vertices G_j^0 with position vectors $\mathbf{w}_j$. Writing $V_i^P = V(F_i^0, P)$, $V_j^Q = V(G_j^0, Q)$, and $V_{ij} = V_i^P \cap V_j^Q$, we see that $\{\bar{V}_i^P\}$, $\{\bar{V}_j^Q\}$, $\{\bar{V}_{ij}\}$ are three coverings of S^{d-1} by closed spherical polytopes, the third being a common refinement of the first two. Let $\tau_d\,\psi_i^P$, $\tau_d\,\psi_j^Q$, $\tau_d\,\psi_{ij}$

be the $(n-1)$-contents of V_i^P, V_j^Q, V_{ij} respectively, where τ_d is defined as in (7). Then, by (5),

$$\mathbf{s}(P) + \mathbf{s}(Q) = \sum_i \mathbf{v}_i \psi_i^P + \sum_j \mathbf{w}_j \psi_j^Q = \sum_{i,j} (\mathbf{v}_i + \mathbf{w}_j)\psi_{ij}.$$

If $\mathbf{u} \in V_{ij}$, then $L(\mathbf{u}, P) \cap P = F_i^0$ and $L(\mathbf{u}, Q) \cap Q = G_j^0$. We deduce that $\mathbf{v}_i + \mathbf{w}_j$ is the position vector of a vertex of $P + Q$ and ψ_{ij} is the external angle of $P + Q$ at this vertex. Hence

$$\sum_{i,j} (\mathbf{v}_i + \mathbf{w}_j)\psi_{ij} = \mathbf{s}(P + Q),$$

and the additivity of $\mathbf{s}(K)$ is proved. This, together with $\mathbf{s}(\lambda K) = \lambda \mathbf{s}(K)$ for any real λ, yields (1).

3. Proof of the theorem. For any vertex F_i^0 of P consider the region $V(F_i^0, P)$. This is an open spherical convex polytope in S^{d-1} and its $(d-1)$-content may be computed from the well-known formula of Sommerville (**5**, p. 157). In this way we obtain for $\psi(F_i^0, P)$ the expression

$$(1 + (-1)^{d-1})\psi(F_i^0, P) = \sum_{j=0}^{d-1} (-1)^{d-j-1}\alpha_i^j, \tag{9}$$

where α_i^j denotes the sum of the $(d-j-1)$-dimensional solid angles subtended by the polytope $V(F_i^0, P)$ at its j-faces. (In this formula, $\alpha_i^{d-2} = \frac{1}{2}m$, where m is the number of $(d-2)$-faces of $V(F_i^0, P)$ and, conventionally, $\alpha_i^{d-1} = 1$.) Now the $(d-2)$-faces of $V(F_i^0, P)$ lie in hyperplanes through the centre of S^{d-1} which are perpendicular to the edges of P meeting at F_i^0. Hence the solid angle at a j-face of $V(F_i^0, P)$ is bounded by the hyperplanes perpendicular to the edges of a $(d-j-1)$-face of P meeting at F_i^0, and so is equal to the external angle at F_i^0 of that face. Thus

$$\alpha_i = \sum_{k=1}^{f_{d-j-1}} \psi(F_i^0, F_k^{d-j-1}),$$

where $\psi(F_i^0, F_k^{d-j-1})$ is put equal to zero if F_i^0 is not a vertex of F_k^{d-j-1}. If we substitute these values of α_i^j in (9), multiply by $\mathbf{v}_i$, and sum for i from 1 to f_0, using (5), we obtain

$$\begin{aligned}(1 + (-1)^{d-1})\mathbf{s}(P) &= \sum_{i=1}^{f_0} \mathbf{v}_i\left(\sum_{j=0}^{d-1} (-1)^{d-j-1}\left(\sum_{k=1}^{f_{d-j-1}} \psi(F_i^0, F_k^{d-j-1})\right)\right)\\ &= \sum_{j=0}^{d-1} (-1)^{d-j-1}\left(\sum_{k=1}^{f_{d-j-1}} \left(\sum_{i=1}^{f_0} \mathbf{v}_i\, \psi(F_i^0, F_k^{d-j-1})\right)\right)\\ &= \sum_{j=0}^{d-1} (-1)^{d-j-1}\left(\sum_{k=1}^{f_{d-j-1}} \mathbf{s}(F_k^{d-j-1})\right).\end{aligned}$$

This is (3), and concludes the proof of the theorem.

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4. Simple polytopes. In §1 we remarked on the resemblance between relation (3) and the Euler–Schläfli identity. We now consider other ways in which linear relations between the Steiner points of a polytope and of its faces are analogous to those connecting the numbers f_i. In the first place it is not difficult to see that (3) is the only such relation that is true for *all* convex polytopes, just as the Euler–Schläfli identity is unique. On the other hand, there are further relations connecting the Steiner points of a *simple* convex polytope and its faces which are analogous to the Dehn–Sommerville relations. (The Dehn–Sommerville relations are usually stated for simplicial polytopes (**3**, §7.1), those for simple polytopes following by duality. There appears to be no corresponding duality for the relations between Steiner points of simple and simplicial polytopes.)

We recall that a simple d-polytope is one with the property that, for $j < r < d$, each j-face of P is incident with exactly $\binom{d-j}{r-j}$ r-faces. Applying (3) to an r-face F^r of P gives

$$(10) \qquad (1 + (-1)^{r-1})\mathbf{s}(F^r) = \textstyle\sum'\mathbf{s}(F_i^0) - \sum'\mathbf{s}(F_i^1) + \ldots + (-1)^{r-1}\sum'\mathbf{s}(F_i^{r-1}),$$

where $\sum'$ means summation over those suffixes i for which the face F_i^j $(j < r)$ is incident with F^r. Now when relation (10) is summed over all the r-faces of P, noticing that each $\mathbf{s}(F_i^j)$ occurs exactly $\binom{d-j}{r-j}$ times, we obtain,

$$(11) \quad (1 + (-1)^{r-1})\textstyle\sum\mathbf{s}(F_i^r) = \binom{d}{r}\sum \mathbf{s}(F_i^0) - \binom{d-1}{r-1}\sum \mathbf{s}(F_i^1) + \ldots + (-1)^{r-1}\binom{d-r+1}{1}\sum \mathbf{s}(F_i^{r-1}),$$

where $\sum$ means summation over all the faces of P whose dimension is indicated by the superscript of F. Putting $r = 1, \ldots, d$, we obtain d relations of type (11). These are not all linearly independent, as we shall now prove.

(12) THEOREM. *For a simple d-polytope there are exactly* $[\frac{1}{2}(d + 1)]$ *linearly independent relations of type* (11), *for example those corresponding to the values*

$$r = 1, 3, 5, \ldots, m$$

where m is the largest odd integer not exceeding d.

Proof. Rewrite equations (11) in the form

$$0 = (-1)^r\binom{d}{d-r}\textstyle\sum \mathbf{s}(F_i^0) + (-1)^{r-1}\binom{d-1}{d-r}\sum \mathbf{s}(F_i^1) + \ldots + (-1)^1\binom{d-r+1}{d-r}\sum \mathbf{s}(F_i^{r-1}) + (1 + (-1)^{r-1})\sum \mathbf{s}(F_i^r),$$

and denote the right side of this equation by (S_r^d) $(r = 1, \ldots, d)$.

If d is even, then it is simple to verify that

$$2(S_d^d) + \sum_{r=1}^{d-1} (S_r^d) = 0. \tag{13}$$

If r is even and $r < d$, then

$$\sum_{i=0}^{r-1} \binom{d-r+i}{1+i} (S_{r-i}^d) = 0. \tag{14}$$

To see this, we notice that for $0 \leqslant k \leqslant r-1$, the coefficient of $\sum \mathbf{s}(F_i^k)$ in the left side of (14), after slight simplification, is equal to

$$\sum_{i=-1}^{r-k} (-1)^i \binom{d-r+i}{1+i} \binom{d-k}{r-k-i}.$$

But this is the coefficient of x^{r-k+1} in the formal product of

$$(1+x)^{d-k} = \binom{d-k}{0} + \binom{d-k}{1} x + \ldots + \binom{d-k}{d-k} x^{d-k}$$

and

$$-(1+x)^{-d+r} = -\binom{d-r-1}{0} + \binom{d-r}{1} x - \binom{d-r+1}{2} x^2 + \ldots$$

and so is zero. Hence (14) is proved.

Relations (13) and (14) show that (S_r^d) is, for even r, linearly dependent on (S_j^d) $(j = 1, \ldots, r)$, and so the equations $(S_r^d) = 0$ (r even) are redundant. The remaining equations (those with odd r) are linearly independent since the matrix of coefficients is of triangular form, (S_r^d) containing no term $\sum \mathbf{s}(F_i^j)$ for $j > r$. This proves Theorem (12).

5. Remarks. It seems reasonable to conjecture that every linear relation connecting the Steiner points of a simple polytope and of its faces must be linearly dependent on the $[\frac{1}{2}(d+1)]$ relations given in (12), but this has yet to be proved. Another open question is whether there exist any other interesting classes of polytopes, besides the simple ones, for which there exist more than one linear relation between the Steiner points.

Theorems (3) and (12) can be generalized in a similar manner to that suggested by Grünbaum (**3**, §14.3) for the Gram relations between the angles of a polytope. Let μ be any centrally symmetric countably additive set function defined on the Borel sets of S^{n-1}, with $\mu(S^{n-1}) = 1$. Then define

$$\psi_\mu(F^0, P) = \mu(V(F^0, P))$$

and

$$\mathbf{s}_\mu(P) = \sum_j \mathbf{v}_j \psi_\mu(F_j^0, P);$$

cf. (5). This μ-Steiner point, as it may be called, has the additive property (1), satisfies (3), and, in the case of a simple polytope, also satisfies (11). These assertions are easily proved by modifying the arguments given above. However, unless μ is a constant multiple of the $(n-1)$-content, the μ-Steiner point is not congruence-invariant, that is

$$\mathbf{s}_\mu(TP) = T\mathbf{s}_\mu(P)$$

will not hold for every congruence transformation T. Hence the special case discussed above is geometrically the most interesting.

References

1. R. D. Anderson and V. Klee, *Continuous functions and upper semi-continuous collections*, Duke Math. J., *19* (1952), 349–357.
2. T. Bonnesen and W. Fenchel, *Konvexe Körper* (New York, 1948).
3. B. Grünbaum, *Convex polytopes* (to be published).
4. G. C. Shephard, *Approximation problems for convex polyhedra*, Mathematika, *11* (1964), 9–18.
5. D. M. Y. Sommerville, *An introduction to the geometry of n dimensions* (New York, 1958).
6. J. Steiner, *Gesammelte Werke*, 2 vols. (Berlin, 1881, 1882).

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