J. Austral. Math. Soc. 25 (Series A) (1978), 103–117

MAL'CEV CONDITIONS, SPECTRA AND KRONECKER PRODUCT

WALTER D. NEUMANN

(Received 10 May 1976; revised 25 October 1976)
Communicated by H. Lausch

Abstract

It is shown that every possible spectrum of a Mal'cev definable class of varieties which should occur does occur. It follows that there are continuum many Mal'cev definable classes, a result also obtained by Taylor (1975) and Baldwin and Berman (1976).

Several specific Mal'cev classes are discussed, including some arising from spectrum conditions, from conditions on the fundamental groups of pointed topological algebras, and from automorphism group and endomorphism semigroup conditions.

This paper continues the author's study of Mal'cev conditions (Neumann, 1974). It is an expanded version of his talk at the Special Session on Varieties at the American Mathematical Society meeting at San Antonio in January 1976.

The spectrum of a Mal'cev definable class of varieties is defined as the set of cardinalities of finite algebras of varieties in the class. This is always a submonoid of the natural numbers N and we show that every submonoid occurs this way. In particular there are $2^{\aleph_0}$ Mal'cev definable classes, solving a problem of Taylor (1973) and Neumann (1974). This was also shown independently by Taylor (1975b) and Baldwin and Berman (1976).

Most of the remainder of the paper gives some examples of Mal'cev definable classes which the author finds particularly interesting. "Varieties of varieties" are also briefly discussed.

The spectrum $\mathscr{spec}(\mathfrak{V})$ of a variety $\mathfrak{V}$ is just the set of cardinalities of its finite algebras. The condition that $\mathscr{spec}(\mathfrak{V})$ be contained in some preassigned submonoid S of N is shown to be a weak or strong Mal'cev condition according as $\mathbf{N}-S$ is infinite or finite. This improves a result of Taylor (1973). Among other things, Taylor gave there an explicit Mal'cev condition for $S=\mathbf{N}-\{2\}$ in the form of a countable set of strong Mal'cev conditions. By our result, just one of his strong Mal'cev conditions suffices, but we do not know which one.

Using Kronecker product of varieties, a further class of Mal'cev definable classes is defined, which includes classes coming from conditions on the fundamental group of topological algebras (very closely related to conditions considered by

https://doi.org/10.1017/S1446788700038970 Published online by Cambridge University Press

Taylor, 1975a) as well as conditions on the automorphism groups and endomorphism monoids of algebras.

A tool for the result on fundamental groups which is of some interest for itself is a calculation, for any variety $\mathfrak{A}$, of the fundamental group of the free pointed topological $\mathfrak{A}$-algebra over a bouquet of circles to be the "free group in the variety $\mathfrak{A}$".

If countably presented varieties are permitted in defining Mal'cev conditions and strong Mal'cev conditions, it turns out that many conditions that were weak Mal'cev conditions become strong Mal'cev conditions. This is true of conditions expressed in terms of finite algebras, for instance the spectrum conditions mentioned above as well as many conditions discussed by Taylor (1973). It seems that this concept, which I call "(strong) ω-Mal'cev condition", is a good substitute for the rather unnatural concept of "weak Mal'cev condition".

CORRECTION. In the appendix of Neumann (1974) the algebraic structure of the product $\mathfrak{B}_1 \times \mathfrak{B}_2$ was incorrectly described. The set of laws should have been described as the disjoint union $\Sigma_1' \cup \Sigma_2' \cup \Sigma$, where Σ is as described there and Σ_1' and Σ_2' are the sets of laws (notation as in Neumann, loc. cit.).

$$\Sigma_1' = \{w_F(\chi).x = w_{F'}(\chi).x \mid (w_F(\chi) = w_{F'}(\chi)) \in \Sigma_1\},$$
$$\Sigma_2' = \{x.w_G(\chi) = x.w_{G'}(\chi) \mid (w_G(\chi) = w_{G'}(\chi)) \in \Sigma_2\}.$$

A discussion of product of varieties with excellent bibliography can be found in Taylor (1975c).

A further correction, for which I am grateful to the referee, is that Neumann (1974) consistently used the term *ideal* instead of *dual ideal* or *filter* in a lattice. This is corrected in the present paper.

1. Introduction and notation

We assume the reader knows what a Mal'cev condition is, see for instance Taylor (1973) or Neumann (1974). Our notation is the notation of Neumann (1974) with minor changes; we review it briefly.

$\mathcal{V}ar^f$ denotes the category of varieties of algebras and set-preserving functors of varieties. The superscript f stresses that we only allow finitary operations. Given $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in $\mathcal{V}ar^f$, we write

$$\mathfrak{B}_1 \geqslant \mathfrak{B}_2$$

if a set-preserving functor $\mathfrak{B}_1 \to \mathfrak{B}_2$ exists. Two varieties $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are called *Mal'cev indistinguishable*, in symbols

$$\mathfrak{B}_1 \simeq \mathfrak{B}_2,$$

if $\mathfrak{B}_1 \geqslant \mathfrak{B}_2$ and $\mathfrak{B}_2 \geqslant \mathfrak{B}_1$. The class of equivalence classes

$$L = \mathcal{V}ar^f / \simeq$$

is a complete lattice with respect to the order relation $\geqslant$. Meet and join in L are induced by product and sum of varieties (these terms refer to categorical product and sum in the dual category $(\mathscr{V}ar^f)^*$), that is

$$\bigvee_{i\in I}[\mathfrak{V}_i] = \left[\coprod_{i\in I}\mathfrak{V}_i\right],$$

$$\bigwedge_{i\in I}[\mathfrak{V}_i] = \left[\prod_{i\in I}{}^f\mathfrak{V}_i\right],$$

the superscript in $\prod^f$ again stressing that this product is in $\mathscr{V}ar^f$ (sum, unlike product, is independent of which (reasonably chosen) category of varieties we do it in, so no superscript is required for it).

Let $M\subset L$ be the sublattice represented by finitely presented varieties. It is not complete. The following is shown by Neumann (1974).

PROPOSITION 1.1. *A subclass $\mathscr{K}\subset\mathscr{V}ar^f$ is definable respectively by (i) a strong Mal'cev condition, (ii) a Mal'cev condition or (iii) a weak Mal'cev condition, if and only if $\mathscr{K}$ is closed with respect to Mal'cev indistinguishability and defines a subclass $K\subset L$ which is respectively: (i) a principal filter in L generated by an element of M, (ii) a filter in L generated by a subset of M, (iii) a countable intersection of filters as in (ii).*

COROLLARY 1.2. *A filter $K\subset L$ is definable by a Mal'cev condition if and only if every $[\mathfrak{V}]\in K$ is greater than or equal to some $[\mathfrak{V}_0]\in K$ with $\mathfrak{V}_0$ finitely presented (i.e. $[\mathfrak{V}_0]\in M$). It is defined by a strong Mal'cev condition if and only if it is in addition closed under countable (and hence arbitrary) meets.*

2. Some preliminaries on products of varieties

If $\mathfrak{n}$ is a regular cardinal number, let $\mathscr{V}ar^{\mathfrak{n}}$ denote the category of varieties of algebras permitting operations of rank $\leqslant\mathfrak{n}$, and set-preserving functors of such varieties. As pointed out in Neumann (1970), the dual category $(\mathscr{V}ar^{\mathfrak{n}})^*$ is itself isomorphic to the category of a variety $\mathfrak{C}^{\mathfrak{n}}\in\mathscr{V}ar^{\mathfrak{n}}$, the variety of $\mathfrak{n}$-clones. In particular it is a complete category. Product and sum of varieties means categorical product and sum in $(\mathscr{V}ar^{\mathfrak{n}})^*$, corresponding to product and sum of $\mathfrak{n}$-clones, and will be denoted

$$\prod_{i\in I}{}^{\mathfrak{n}}\mathfrak{V}_i \quad\text{and}\quad \coprod_{i\in I}\mathfrak{V}_i$$

respectively.

LEMMA 2.1. *If $\mathfrak{V}_i\in\mathscr{V}ar^{\mathfrak{n}}$, $i\in I$, and $|I|\leqslant\mathfrak{n}$, then as a category the product $\prod^{\mathfrak{n}}_{i\in I}\mathfrak{V}_i$ is the cartesian product of the underlying categories of the $\mathfrak{V}_i$ with the obvious underlying set functor (that is, it is the category of all cartesian products $\times_{i\in I}A_i$ with $A_i\in\mathfrak{V}_i$ for each $i\in I$).*

https://doi.org/10.1017/S1446788700038970 Published online by Cambridge University Press

The proof of this appears to be well known. The proof for finite products (see Neumann (1974) for a proof using the language of clones, or Taylor (1975c) for further references) goes through with no essential change, so we omit the details.

LEMMA 2.2. *If $\mathfrak{V}_i \in \mathscr{V}\!\mathit{ar}^f$, $i \in I$, and $\mathfrak{n}$ is infinite, then the product $\Pi^f_{i\in I}\mathfrak{V}_i$ is the reduct of $\Pi^{\mathfrak{n}}_{i\in I}\mathfrak{V}_i$ to the set of all its finitary algebraic operations. The corresponding statement holds also for $\Pi^{\mathfrak{m}}_{i\in I}\mathfrak{V}_i$ if $\mathfrak{m} \leqslant \mathfrak{n}$ and each $\mathfrak{V}_i$ is in $\mathscr{V}\!\mathit{ar}^{\mathfrak{m}}$.*

PROOF. The proof in Neumann (1974) for $\mathfrak{n} = \aleph_0$ works for any $\mathfrak{n}$, using $\mathfrak{n}$-clones instead of $\aleph_0$-clones.

Finally we need a presumably well-known lemma.

LEMMA 2.3. *Let $\mathfrak{m} \leqslant \mathfrak{n}$ be any (finite or infinite) cardinal numbers, $\mathfrak{n}$ regular, and let $\mathfrak{V} \in \mathscr{V}\!\mathit{ar}^{\mathfrak{n}}$. Let $\mathfrak{V}^{(\mathfrak{m})}$ be the reduct of $\mathfrak{V}$ to the set of all its algebraic operations of rank less than $\mathfrak{m}$. Then the natural forget functor $\mathfrak{V} \to \mathfrak{V}^{(\mathfrak{m})}$ is an equivalence on the subcategories of algebras of cardinality less than $\mathfrak{m}$.*

PROOF. Let $A \in \mathfrak{V}^{(\mathfrak{m})}$ have cardinality less than $\mathfrak{m}$. If we attempt to evaluate a $\mathfrak{V}$-operation on A it automatically interprets itself as a $\mathfrak{V}$-algebraic operation of rank $<\mathfrak{m}$ and is thus defined. The laws of $\mathfrak{V}$ interpret themselves similarly as laws in $<\mathfrak{m}$ variables, which thus hold in this $\mathfrak{V}$-structure on A.

These lemmata together imply the following.

PROPOSITION 2.4. *If $\mathfrak{V}_i \in \mathscr{V}\!\mathit{ar}^f$, $i \in I$, then the category of finite algebras in $\Pi^f_{i\in I}\mathfrak{V}_i$ is just the category of all cartesian products $\times_{i\in I} A_i$ of finite algebras $A_i \in \mathfrak{V}_i$, all but finitely many of which are trivial. A similar statement holds for algebras in $\Pi^{\mathfrak{m}}\mathfrak{V}_i$ of cardinality $\leqslant \mathfrak{m}$ if each $\mathfrak{V}_i$ is in $\mathscr{V}\!\mathit{ar}^{\mathfrak{m}}$.*

From this proposition follows that a subclass of $\mathscr{V}\!\mathit{ar}^f$ which is closed under finite products, if defined by a suitable sort of condition on finite algebras of its members, will actually be closed under arbitrary products of varieties. Thus if the subclass is definable by a Mal'cev condition, then it is actually definable by a strong Mal'cev condition by Corollary 1.2. We shall use this remark repeatedly in the following.

3. Spectra of Mal'cev conditions

Recall that the spectrum of a variety $\mathfrak{V}$ is the set

$$\mathit{spec}(\mathfrak{V}) = \{|A| \mid A \in \mathfrak{V}, |A| \text{ finite}\}$$

of cardinalities of finite $\mathfrak{V}$-algebras. (Define $0 \in \mathit{spec}(\mathfrak{V})$ if $\mathfrak{V}$ has no nullary algebraic operations.) This is a submonoid of the multiplicative monoid

https://doi.org/10.1017/S1446788700038970 Published online by Cambridge University Press

$\mathbf{N} = \{0, 1, 2, \ldots\}$. Let $\mathfrak{S}(\mathbf{N})$ denote the lattice of submonoids of $\mathbf{N}$. Observe that the join of two submonoids S_1 and S_2 is the product

$$S_1 S_2 = \{s_1 s_2 \mid s_1 \in S_1,\ s_2 \in S_2\}.$$

LEMMA 3.1. *Spectrum defines an antihomomorphism of complete lattices*

$$\mathit{spec}\colon L \to \mathfrak{S}(\mathbf{N}).$$

PROOF. Mal'cev indistinguishable varieties have equal spectra, so $\mathit{spec}\colon L \to \mathfrak{S}(\mathbf{N})$ is defined. That this map sends meet in L to join in $\mathfrak{S}(\mathbf{N})$ follows from the fact that meet in L is given by product of varieties, together with Proposition 2.4. That join in L goes to meet in $\mathfrak{S}(\mathbf{N})$ follows from the fact that an algebra of the sum $\amalg \mathfrak{V}_i$ of varieties $\mathfrak{V}_i$, $i \in I$, is just a set with a $\mathfrak{V}_i$-structure for each $i \in I$.

Now if $K \subseteq L$ is any filter, define its spectrum as

$$\mathit{spec}(K) = \prod_{[\mathfrak{V}] \in K} \mathit{spec}(\mathfrak{V}).$$

The above lemma implies the following one.

LEMMA 3.2. *Spectrum defines a homomorphism of complete lattices from the lattice $F(L)$ of filters in L to $\mathfrak{S}(\mathbf{N})$.*

The proof is trivial. In fact this lemma is more elementary than the preceding one, since it does not use completeness of the antihomomorphism of Lemma 3.1 and hence does not involve Proposition 2.4.

We are interested in which submonoids $S \subseteq \mathbf{N}$ occur as spectra of strong Mal'cev conditions, Mal'cev conditions, and others. For strong Mal'cev conditions this is equivalent to asking which S occur as spectra of finitely presented varieties, an as yet unsolved and apparently difficult problem. For Mal'cev conditions the answer turns out to be easy.

THEOREM 3.3. *Every submonoid $S \subseteq \mathbf{N}$ occurs as $\mathit{spec}(K)$ for some Mal'cev definable class K.*

PROOF. For $n \in \mathbf{N}$ let $\langle n \rangle$ denote the submonoid

$$\langle n \rangle = \{n^i \mid i \in \mathbf{N}\} \subseteq \mathbf{N}.$$

For any $n \in \mathbf{N}$, choose a finitely presented variety $\mathfrak{V}_n$ with $\mathit{spec}(\mathfrak{V}_n) = \langle n \rangle$ (these exist, for example if A is any primal algebra with n elements then $QSP(A)$ is suitable; finite presentedness in this case follows by a recent result of K. Baker that a finite algebra in a variety with distributive congruences has finitely based laws. (See, for instance, Taylor (1975c) for quite complete references on spectra.) For any submonoid $S \subseteq \mathbf{N}$ with $0 \notin S$, let K be the filter in L generated by all $\mathfrak{V}_n$ with $n \in S$. Then $\mathit{spec}(K) = S$. If $0 \in S$, we replace the variety $\mathfrak{V}_n$ used above by a variety whose spectrum is $\langle n \rangle \cup \{0\}$, for instance the nilpotent reduct of the $\mathfrak{V}_n$ described above, and repeat the proof.

COROLLARY 3.4. *There are* $2^{\aleph_0}$ *Mal'cev definable classes.*

PROOF. There are certainly at most this many, by Proposition 1.1, since M is countable, so there are exactly this many, since $|\mathfrak{S}(\mathbf{N})| = 2^{\aleph_0}$.

This corollary has been shown independently by Taylor (1975b) and Baldwin and Berman (1976). It answers a question raised in Taylor (1973) and Neumann (1974).

4. Mal'cev conditions of spectra

If $S \subseteq \mathbf{N}$ is a submonoid, let $\mathscr{K}(S)$ be the class of varieties $\mathfrak{V}$ with $\mathit{spec}(\mathfrak{V}) \subseteq S$. W. Taylor (1973) showed that $\mathscr{K}(S)$ is definable by a weak Mal'cev condition and is actually definable by a Mal'cev condition if $\mathbf{N}-S$ is finite. This result can be improved as follows.

THEOREM 4.1. *The following conditions are equivalent:*
(i) $\mathbf{N}-S$ *is finite,*
(ii) $\mathscr{K}(S)$ *is definable by a Mal'cev condition,*
(iii) $\mathscr{K}(S)$ *is definable by a strong Mal'cev condition.*

Taylor (loc. cit.) gave an explicit Mal'cev condition in the case $S = \mathbf{N}-\{2\}$ by means of an explicit doubly indexed set of finitely presented varieties. It follows from the above result that just one of his varieties suffices. Which one? More generally one can ask:

PROBLEM 4.2. *Find an explicit nice strong Mal'cev condition for* $\mathscr{K}(S)$ *when* $\mathbf{N}-S$ *is finite.*

PROOF OF THEOREM. We shall show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Observe first that $\mathscr{K}(S)$ is closed with respect to Mal'cev indistinguishability, so we can consider it as a subclass of L. As such, it is clearly a filter in L, and is closed under arbitrary meets by Proposition 2.4. Thus (ii) $\Rightarrow$ (iii) follows from Corollary 1.2.

The proof that (i) $\Rightarrow$ (ii) is essentially as in Taylor (1973). Suppose $\mathfrak{V} \in \mathscr{K}(S)$ and $\mathbf{N}-S$ is finite. Let Σ be a set of equational laws defining $\mathfrak{V}$. The statement "the structure A has exactly n elements" can easily be expressed by a first-order sentence e_n say. Then $\Sigma \cup \{\bigvee_{n \in \mathbf{N}-S} e_n\}$ is contradictory, so by compactness, some finite subset already is. This finite subset must have the form $\Sigma_0 \cup \{\bigvee_{n \in \mathbf{N}-S} e_n\}$. The equations Σ_0 involve only a finite set of operations of $\mathfrak{V}$. Let $\mathfrak{V}_0$ be the variety defined by this set of operations and the laws Σ_0. Then $\mathfrak{V}_0 \in \mathscr{K}(S)$ and clearly $\mathfrak{V} \geqslant \mathfrak{V}_0$, so by Corollary 1.2, $\mathscr{K}(S)$ is a Mal'cev class.

Finally to show (iii) $\Rightarrow$ (i), suppose $\mathscr{K}(S)$ is definable by a strong Mal'cev condition. Write S as the intersection of a collection S_i, $i \in I$, of submonoids of $\mathbf{N}$ with $\mathbf{N}-S_i$ finite for each i. Let $\mathfrak{V}_i$ be a finitely presented variety with

$\mathit{spec}(\mathfrak{V}_i) = S_i$ (exists, since (i) ⇒ (iii) already shown) and let $\mathfrak{V} = \amalg_{i \in I} \mathfrak{V}_i$. Then $\mathit{spec}(\mathfrak{V}) = S$, so by Corollary 1.2, there exists a finitely presented $\mathfrak{V}_0 \leqslant \mathfrak{V}$ with $\mathit{spec}(\mathfrak{V}_0) \subseteq S$. Since a finite generating set for the laws of $\mathfrak{V}_0$ will only involve the operations of finitely many of the $\mathfrak{V}_i$, $\mathfrak{V}_0 \leqslant \amalg_{i \in J} \mathfrak{V}_i$ for some finite subset $J \subseteq I$. Hence

$$\mathit{spec}(\mathfrak{V}_0) \supseteq \mathit{spec}\left(\coprod_{i \in J} \mathfrak{V}_i\right) = \bigcap_{i \in J} S_i,$$

so $\bigcap_{i \in J} S_i \subseteq S$, so $\mathbf{N} - S \subseteq \bigcup_{i \in J} (\mathbf{N} - S_i)$ is finite.

5. A class of Mal'cev conditions

In this section I shall discuss some further examples of Mal'cev conditions which I find very interesting. The idea was suggested partly by work of Taylor (1975a) and we discuss the connection with his work later in this section. We first recall a concept due first I think to P. Freyd (1966), see also Lawvere (1968).

If $\mathfrak{V}$ and $\mathfrak{W}$ are two varieties, then the "variety of $\mathfrak{V}$-algebras in $\mathfrak{W}$" or "Kronecker product of $\mathfrak{V}$ and $\mathfrak{W}$" is denoted $\mathfrak{V} \otimes \mathfrak{W}$, and can be defined as follows: if $\mathfrak{V}$ is defined by a set F of operations and a set Σ of equational laws in these operations and $\mathfrak{W}$ is defined by operations G and laws Γ, then $\mathfrak{V} \otimes \mathfrak{W}$ is defined by the disjoint union $F \cup G$ of operations and the disjoint union $\Sigma \cup \Gamma \cup \Delta$ of laws, where Δ consists of all laws of the form ($f \in F$, $g \in G$):

$$f(g(x_{11}, \ldots, x_{1n}), \ldots, g(x_{m1}, \ldots, x_{mn})) = g(f(x_{11}, \ldots, x_{m1}), \ldots, f(x_{1n}, \ldots, x_{mn})).$$

In other words, a $\mathfrak{V} \otimes \mathfrak{W}$-algebra is a set with a $\mathfrak{V}$-structure and a $\mathfrak{W}$-structure such that the $\mathfrak{W}$-operations are homomorphisms with respect to the $\mathfrak{V}$-structure (and vice versa; this follows automatically).

The following examples are easily checked by explicit calculation from the above laws. The first is well known, and is due to H. Hopf. If one has never calculated such examples before, it is an amusing exercise to do so.

EXAMPLES 5.1.
(1) (Groups) ⊗ (Unitary groupoids) = (Abelian groups),
(2) (Groups) ⊗ (Groups) = (Abelian groups),
(3) (Semilattices) ⊗ (Semilattices) = (Semilattices),
(4) (Semilattices) ⊗ (Lattices) = Trivial variety,
(5) (Groups) ⊗ (Semilattices) = ($\mathbf{Z}[\frac{1}{2}]$-modules),
(6) (Groups) ⊗ (Lattices) = Trivial variety,
(7) (R-modules) ⊗ (S-modules) = ($R \otimes S$-modules), (R and S rings),

LEMMA 5.2. *If* $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$, *are varieties then* $\mathfrak{U} \otimes (\mathfrak{V} \times \mathfrak{W}) \cong (\mathfrak{U} \otimes \mathfrak{V}) \times (\mathfrak{U} \otimes \mathfrak{W})$.

PROOF. Let C be a $\mathfrak{U} \otimes (\mathfrak{V} \times \mathfrak{W})$-algebra. That is, C has a $(\mathfrak{V} \times \mathfrak{W})$-structure and a $\mathfrak{U}$-structure, and all the $\mathfrak{U}$-operations on C are $(\mathfrak{V} \times \mathfrak{W})$-homomorphisms.

https://doi.org/10.1017/S1446788700038970 Published online by Cambridge University Press

The $(\mathfrak{V}\times\mathfrak{W})$-structure on C is equivalent to a splitting $C=A\times B$ of C into a $\mathfrak{V}$-algebra and a $\mathfrak{W}$-algebra. The $\mathfrak{U}$-operations, being $(\mathfrak{V}\times\mathfrak{W})$-homomorphisms, preserve this splitting, that is they induce $\mathfrak{U}$-structures on A and B such that $C=A\times B$ as a $\mathfrak{U}$-algebra. Thus A and B are in a natural way a $\mathfrak{U}\otimes\mathfrak{V}$ and $\mathfrak{U}\otimes\mathfrak{W}$ algebra respectively and C is a $(\mathfrak{U}\otimes\mathfrak{V})\times(\mathfrak{U}\otimes\mathfrak{W})$-algebra. Conversely it is clear that any $(\mathfrak{U}\otimes\mathfrak{V})\times(\mathfrak{U}\otimes\mathfrak{W})$-algebra is in a natural way a $\mathfrak{U}\otimes(\mathfrak{V}\times\mathfrak{W})$-algebra and that these correspondences give an equivalence of varieties. It is also not hard to write down this equivalence directly in terms of the algebraic structures of the two sides of the equation.

Now let φ denote the natural forget functor

$$\varphi\colon \mathfrak{V}\otimes\mathfrak{W}\to\mathfrak{V}.$$

We are interested in questions of the type "for which varieties $\mathfrak{W}$ is it true that a group in $\mathfrak{W}$ is abelian" (that is, φ (Groups $\otimes\,\mathfrak{W})\subseteq$ Abelian groups).

THEOREM 5.3. *Let $\mathfrak{V}$ be a variety and let $\mathfrak{V}_1\subseteq\mathfrak{V}$ be a subclass definable in $\mathfrak{V}$ by a single first-order sentence σ in the language of $\mathfrak{V}$ and such that $\mathfrak{V}_1$ is closed under finite products (for example, a subvariety defined by finitely many additional equational laws). Then the class of varieties $\mathfrak{W}$ such that every $\mathfrak{V}$-algebra in $\mathfrak{W}$ is in $\mathfrak{V}_1$:*

$$\mathscr{R}(\mathfrak{V},\mathfrak{V}_1)=\{\mathfrak{W}\mid\varphi(\mathfrak{V}\otimes\mathfrak{W})\subseteq\mathfrak{V}_1\},$$

is Mal'cev definable.

PROOF. Suppose $\mathfrak{W}\geqslant\mathfrak{W}'$, that is, a set-preserving functor $f\colon\mathfrak{W}\to\mathfrak{W}'$ exists. This induces a commuting square

$$\begin{array}{ccc}\mathfrak{V}\otimes\mathfrak{W} & \xrightarrow{1\otimes f} & \mathfrak{V}\otimes\mathfrak{W}'\\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\varphi}\\ \mathfrak{V} & = & \mathfrak{V}\end{array}$$

so $\varphi(\mathfrak{V}\otimes\mathfrak{W})\subseteq\varphi(\mathfrak{V}\otimes\mathfrak{W}')$. Thus $\mathfrak{W}'\in\mathscr{R}(\mathfrak{V},\mathfrak{V}_1)\Rightarrow\mathfrak{W}\in\mathscr{R}(\mathfrak{V},\mathfrak{V}_1)$. In particular $\mathscr{R}(\mathfrak{V},\mathfrak{V}_1)$ is closed under Mal'cev indistinguishability and if it is also closed under product, then it represents a filter in L. But it follows from the above lemma that $\varphi(\mathfrak{V}\otimes(\mathfrak{W}\times\mathfrak{W}'))=\{A\times B\in\mathfrak{V}\mid A\in\varphi(\mathfrak{V}\otimes\mathfrak{W}),B\in\varphi(\mathfrak{V}\otimes\mathfrak{W}')\}$, so the fact that $\mathfrak{V}_1$ is closed under finite products implies that $\mathscr{R}(\mathfrak{V},\mathfrak{V}_1)$ is product closed. It remains by Corollary 1.2 to show that if $\mathfrak{W}\in\mathscr{R}(\mathfrak{V},\mathfrak{V}_1)$, then there exists a finitely presented $\mathfrak{W}_0\leqslant\mathfrak{W}$ with $\mathfrak{W}_0\in\mathscr{R}(\mathfrak{V},\mathfrak{V}_1)$. But this follows by applying the compactness theorem to the contradictory set of statements

$$\Sigma\cup\Gamma\cup\Delta\cup\{\neg\sigma\},$$

where $\Sigma\cup\Gamma\cup\Delta$ are the laws for $\mathfrak{W}\otimes\mathfrak{V}$ described earlier, and σ is the sentence which defines $\mathfrak{V}_1$ in $\mathfrak{V}$.

One reason for the importance of this sort of Mal'cev condition is the following theorem. This theorem is closely related to a result of Taylor (1975a). First some notation.

Let $\mathfrak{S}^0$ be the variety of *pointed sets*, that is $\mathfrak{S}^0$ is defined by a single nullary operation and no laws. For any variety $\mathfrak{A}$, denote by $\mathfrak{A}^0$ the variety $\mathfrak{A} \otimes \mathfrak{S}^0$ of *well-pointed* $\mathfrak{A}$*-algebras* ($\mathfrak{A}$-algebras together with a chosen trivial subalgebra). Similarly Top^0 denotes the category of *well-pointed topological* spaces and $\mathfrak{A} \otimes Top^0$ denotes the category of *well-pointed topological* $\mathfrak{A}$*-algebras*. We will always denote the base point by e.

THEOREM 5.4. *Let* $\mathfrak{G}$ = (Groups) *and* $\mathfrak{G}_1$ *be a subclass of* $\mathfrak{G}$ *closed under quotients and finite products (for example, a subvariety). If* $\mathscr{R}$ *is the class of all varieties* $\mathfrak{A}$ *such that every well-pointed topological* $\mathfrak{A}$*-algebra* A *has fundamental group* $\pi_1(A, e) \in \mathfrak{G}_1$, *then* $\mathscr{R}$ *equals the class* $\mathscr{R}(\mathfrak{G}, \mathfrak{G}_1)$ *of Theorem* 5.3.

W. Taylor (loc. cit.) in fact describes complete conditions for the fundamental groups of the components of a topological $\mathfrak{A}$-algebra to be in a given subvariety $\mathfrak{G}_1 \subseteq \mathfrak{G}$ of groups, in the non-pointed case. This is considerably more complicated, since there is no natural choice of base point for calculating fundamental groups; $\pi_1: Top \to \mathfrak{G}$ is not a well-defined functor. He gives some explicit conditions of this type.

Since our theorem gives no control over the fundamental groups of the components of A not containing e, it is actually not directly implied by Taylor's results.

PROBLEM 5.5. *Give nice explicit Mal'cev conditions for some of the classes given by the above theorem (for example,* $\mathfrak{G}_1$ = (Trivial variety), $\mathfrak{G}_1$ = (Abelian groups), $\mathfrak{G}_1$ = (Underlying Abelian groups of $\mathbf{Z}[\frac{1}{2}]$-modules), *and so on).*

The proof of the above theorem is quite close to Taylor's methods but we sketch it anyway, since it is quite interesting. First note that if $A \in \mathfrak{A} \otimes Top^0$ then $\pi_1(A, e)$ inherits an $\mathfrak{A}$-structure which commutes with the group structure, so

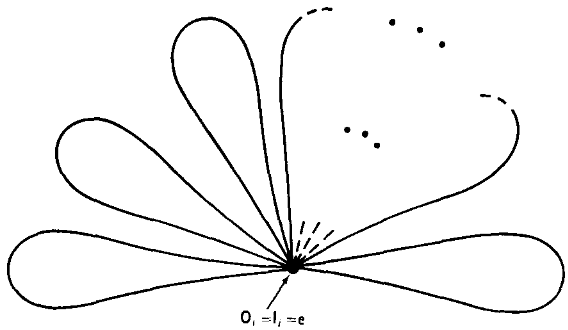
$$\pi_1(A, e) \in \varphi(\mathfrak{G} \otimes \mathfrak{A}).$$

Hence certainly

$$\mathscr{R}(\mathfrak{G}, \mathfrak{G}_1) \subseteq \mathscr{R}.$$

To see the reverse inclusion we consider the metric space X obtained by identifying all the endpoints in the disjoint union of a collection $[0,1]_i$, $i \in I$, of unit intervals to the base point e (so X is the "wedge" or "bouquet" of circles). Let $\mathfrak{A}^0 = \mathfrak{A} \otimes \mathfrak{S}^0$ be the variety of well-pointed $\mathfrak{A}$-algebras and let $F(X)$ be the free $\mathfrak{A}^0$-algebra on the set X. Then $F(X)$ can be given a topology as a topological $\mathfrak{A}^0$-algebra (Świerczkowski, 1964; see also Taylor, 1975a, for an exposition of this). Observe that $F(X)$ can be built up as a cell complex as follows: for each unary algebraic $\mathfrak{A}^0$-operation f and each $i \in I$, the set $f([0,1]_i)$ is a 1-cell in $F(X)$.

https://doi.org/10.1017/S1446788700038970 Published online by Cambridge University Press



X together with these 1-cells gives the 1-skeleton $F(X)^{(1)}$ of $F(X)$. Now if g is any binary $\mathfrak{A}^0$-algebraic operation and $i, j \in I$, $i \neq j$, then g defines a map of $[0,1]_i \times [0,1]_j$ to $F(X)$ which is injective on the interior and maps the boundary $(\{0,1\}_i \times [0,1]_j) \cup ([0,1]_i \times \{0,1\}_j)$ into $F(X)^{(1)}$ (since $g(e,-)$ and $g(-,e)$ are unary operations). This thus gives a 2-cell attached to $F(X)^{(1)}$ in $F(X)$. If $i = j$ we must consider the two 2-cells

$$\{(t_1, t_2) \in [0,1]_i \times [0,1]_i \mid t_1 \leqslant t_2\}$$

and

$$\{(t_1, t_2) \in [0,1]_i \times [0,1]_i \mid t_1 \geqslant t_2\}$$

which are both attached to $F(X)^{(1)}$ in $F(X)$ by g. Adding all these 2-cells to $F(X)^{(1)}$ for all possible g and i and j gives the 2-skeleton $F(X)^{(2)}$ of $F(X)$. Next, to each ternary algebraic $\mathfrak{A}^0$-operation h and each triple i, j, k, of elements of I, we get 1, 2 or 6 3-cells in $F(X)$ according as no two, precisely two, or all three of i, j, k, coincide. For example, if $i = j = k$, then these six 3-cells in $F(X)$ are the images under h of

$$\{(t_1, t_2, t_3) \in [0,1]_i \times [0,1]_i \times [0,1]_i \mid t_p \leqslant t_q \leqslant t_r\}$$

as p, q, r run through all six permutations of $1, 2, 3$. Similarly the 4-ary algebraic $\mathfrak{A}^0$-operations give us the 4-cells of $F(X)$ and so on. The union of all these cells is $F(X)$, giving a cell decomposition of $F(X)$ with just one 0-cell e (note that e is the only nullary operation, since it is a subalgebra). The fundamental group $\pi_1(F(X), e)$ can be calculated from this cell decomposition in the usual way: the 1-cells give the generators of $\pi_1(F(X), e)$ and the 2-cells give the relations. Thus as a $\mathfrak{G} \otimes \mathfrak{A}^0$-algebra, $\pi_1(F(X), e)$ has one generator represented by the loop $[0,1]_i$) for each $i \in I$. The relations of $\pi_1(F(X), e)$, which come from the 2-cells of $F(X)$, all have the form $g(x,e).g(e,y) = g(e,y).g(x,e)$ or $g(x,x) = g(e,x).g(x,e)$ for some binary algebraic $\mathfrak{A}^0$-operation g. But these are $\mathfrak{G} \otimes \mathfrak{A}^0$-laws, so $\pi_1(F(X), e)$ has no relations which are not $\mathfrak{G} \otimes \mathfrak{A}^0$-laws, so it is the free $\mathfrak{G} \otimes \mathfrak{A}^0$-algebra on the basis I.

Note that $\mathfrak{G} \otimes \mathfrak{A}^0 = (\mathfrak{G} \otimes \mathfrak{A})^0 = \mathfrak{G}^0 \otimes \mathfrak{A} = \mathfrak{G} \otimes \mathfrak{A}$, since $\mathfrak{G}$ is already a well-pointed variety, so we have shown

LEMMA 5.6. *$\pi_1(F(X), e)$ is the free $\mathfrak{G} \otimes \mathfrak{A}$-algebra on the basis I.*

It now follows that if $\mathfrak{A} \in \mathscr{R}$, then $\varphi(F) \in \mathfrak{G}_1$ for any free $\mathfrak{G} \otimes \mathfrak{A}$-algebra F, so certainly $\varphi(A) \in \mathfrak{G}_1$ for any $\mathfrak{G} \otimes \mathfrak{A}$-algebra A. Hence $\mathfrak{A} \in \mathscr{R}(\mathfrak{G}, \mathfrak{G}_1)$, completing the proof of 5.5.

The above suggests a possible direction of generalization of Theorem 5.3. Let $\mathfrak{X}$ be any category, $P: \mathfrak{X} \to \mathfrak{B}$ a product-preserving functor to a variety $\mathfrak{B}$, and $\mathfrak{B}_1 \subseteq \mathfrak{B}$ a product closed subclass of $\mathfrak{B}$. Then

$$\mathscr{R}(P: \mathfrak{X} \to \mathfrak{B}, \mathfrak{B}_1) = \{\mathfrak{A} \mid P(\varphi(\mathfrak{X} \otimes \mathfrak{A}) \in \mathfrak{B}_1\}$$

is always a filter in L and under suitable conditions on P, $\mathfrak{B}$, $\mathfrak{B}_1$, it will be a Mal'cev definable class. It also contains $\mathscr{R}(\mathfrak{B}, \mathfrak{B}_1)$ and under suitable conditions will equal it. We have just dealt with the case $(P: \mathfrak{X} \to \mathfrak{B}) = (\pi_1: Top^0 \to \mathfrak{G})$. One can ask: what are suitable conditions in general above?

A more interesting generalization of Theorem 5.3 than what we have just indicated is given in the following section.

6. More Mal'cev conditions; automorphism and endomorphism conditions

By looking just at finite algebras we can alter Theorem 5.3 as follows.

THEOREM 6.1. *Let $\mathfrak{B}$ and $\mathfrak{B}_1$ be as in Theorem 5.3 and let S be a set of natural numbers which contains all divisors of each of its non-zero members. Then*

$$\mathscr{R}_S(\mathfrak{B}, \mathfrak{B}_1) = \{\mathfrak{W} \mid A \in \varphi(\mathfrak{B} \otimes \mathfrak{W}) \text{ and } |A| \in S \Rightarrow A \in \mathfrak{B}_1\}$$

is a strong Mal'cev definable class if S is finite and a weak Mal'cev definable class (defined by a principal filter in L) if S is not finite.

PROOF. The proof that $\mathscr{R}_S(\mathfrak{B}, \mathfrak{B}_1)$ is a Mal'cev definable class if S is finite is as in Theorem 5.3. If S is infinite then $\mathscr{R}_S(\mathfrak{B}, \mathfrak{B}_1)$ is the intersection of a sequence of classes $\mathscr{R}_{S_i}(\mathfrak{B}, \mathfrak{B}_1)$ with $S_i \subseteq S$ finite, so it is a weak Mal'cev class. Finally $\mathscr{R}_S(\mathfrak{B}, \mathfrak{B}_1)$ is closed under arbitrary products of varieties because of Proposition 2.4, so it is a principal filter in L (note that we use here that the filter in L given by a weak Mal'cev definable class is generated by a set in L; this follows from 1.1). Hence if it is a Mal'cev class, then it is actually strong, by Corollary 1.2.

We obtain some interesting examples for this theorem as follows. Let G be a group and $G\mathfrak{S}$ the variety of G-sets considered as a variety with $|G|-1$ non-trivial unary operations. Then an algebra in $G\mathfrak{S} \otimes \mathfrak{A}$ is an $\mathfrak{A}$-algebra A together with a homomorphism $G \to \mathrm{Aut}(A)$. If H is a quotient of G, then $H\mathfrak{S}$ is a subvariety of

$G\mathfrak{S}$. Using $\mathfrak{V} = G\mathfrak{S}$ and $\mathfrak{V}_1 = H\mathfrak{S}$ in the above theorem we get, to pick some random examples:

EXAMPLES 6.2. *The following are definable by strong Mal'cev conditions on* $\mathfrak{A}$.

(1) *No algebra of* $\mathfrak{A}$ *with less than* 20 *elements has the alternating group* A_5 *on* 5 *symbols as a subgroup of its automorphism group* (*use* $G = A_5$ *and* $H = \{1\}$).

(2) *Every automorphism group of an* $\mathfrak{A}$*-algebra with less than* 20 *elements is solvable* (*use* G = *free group on* 19 *elements and* H = *free soluble group on* 19 *symbols of solubility length c, where c is the maximum solubility length of a soluble group of permutations of* 19 *symbols*).

The following is a weak Mal'cev condition.

(3) *The automorphism group of any* $A \in \mathfrak{A}$ *with* $|A|$ *prime is Abelian* (*use* G = *free group on two generators and H its Abelian quotient*).

We could clearly continue writing down such examples ad infinitum. Replacing the group G by a monoid and H by a quotient monoid, we get Mal'cev conditions out of similar conditions on the endomorphism monoids of algebras. Note that a condition of the following form for instance: "the automorphism group of an algebra with n elements (n prime) has solubility length n", though not directly dealt with by the above theorem, is a countable conjunction of strong Mal'cev conditions by the above theorem, so it is a weak Mal'cev condition.

7. Other classes of varieties

The main claim of this section is that the concept of weak Mal'cev condition is not the "right" concept. It has so far defied intrinsic characterization, and the attempts to do so by Taylor (1973), Neumann (1974) and Baldwin and Berman (1976) have only served to stress the unnaturalness of the concept. It seems probable that the following substitute is sufficient for all "natural" purposes.

DEFINITION. Strong ω-Mal'cev condition and ω-Mal'cev condition are defined just as strong Mal'cev condition and Mal'cev condition but using countably presented varieties in place of finitely presented ones. Thus an ω-Mal'cev definable class (strong ω-Mal'cev definable class) of varieties is a filter in L generated by a set of countably presented varieties (a single countably presented variety).

We leave to the reader the formulation of Theorem 1.1 and Corollary 1.2 for ω-Mal'cev conditions as well as the application of these to verify the following examples.

EXAMPLES 7.1. *The following conditions are strong* ω*-Mal'cev conditions:*

(1) $\mathit{spec}(\mathfrak{V}) \subseteq S$ (*S a submonoid of* $\mathbf{N}$).

(2) *The conditions of Theorem* 6.1.

https://doi.org/10.1017/S1446788700038970 Published online by Cambridge University Press

(3) $\mathfrak{V}$ *is Lagrangian, that is,* $A \subseteq B \in \mathfrak{V}$ *and* B *finite implies* $|A|$ *divides* $|B|$ (*this is a weak Mal'cev condition by Taylor*, 1963, *Theorem* 5.7).

(4) *Each of the weak Mal'cev conditions of Taylor* (*loc. cit.*, *Theorems* 5.9, 5.19, 5.21).

REMARK. A Mal'cev condition which is not strong cannot be equivalent to a strong ω-Mal'cev condition, but a weak Mal'cev condition can be (as the above examples show). Note also that a weak Mal'cev condition is always equivalent to a ω-Mal'cev condition—what is interesting is if it is equivalent to a strong one (or at the least a countably generated one; that is, the filter in L is countably generated).

PROBLEM 7.2. *Which congruence conditions* (*see Wille*, 1970) *are determined by strong* ω*-Mal'cev conditions*?

The literature on Mal'cev definable classes amply demonstrates the ubiquity of such classes in nature and thus certainly justifies their study. But of course not every interesting class of varieties is a Mal'cev definable class. I would like to mention briefly one example of a type of class of varieties that is perhaps at present more remarkable for its sparsity than the contrary.

DEFINITION. A class of varieties (allowing now operations of countably infinite rank) is a *variety of varieties* if it is closed under formation of product varieties, subvarieties and images of pure forget functors.

In other words, using the equivalence of the dual category $\mathscr{Var}^*$ with the category of the variety $\mathfrak{C}$ of $\aleph_0$-clones (Neumann, 1970), a variety of varieties is just a class of varieties whose class of $\aleph_0$-clones is a subvariety of $\mathfrak{C}$. Hence a class of varieties is a variety of varieties if and only if it can be defined equationally, that is, by a set of universal equational laws on operations.

EXAMPLES. A variety is *commutative* if every algebraic $\mathfrak{V}$-operation is a $\mathfrak{V}$-homomorphism (for example, Abelian groups).

A variety is *idempotent* if every algebraic $\mathfrak{V}$-operation is idempotent, that is, satisfies $f(x, x, \ldots, x) = x$ (for example, lattices).

Commutative varieties form a variety of varieties and so do idempotent varieties. One can find further examples, but it seems remarkably difficult to find examples not rather closely related to the above two.

PROBLEM. Why is this? Is the lattice of varieties of varieties maybe quite sparse? What is the smallest variety of varieties containing the variety of groups for example?

8. On the meaning of "nice"

In this paper we have several times asked that "nice" explicit Mal'cev conditions be supplied. Since the variety which generates a strong Mal'cev (or ω-Mal'cev) definable class is only defined up to Mal'cev indistinguishability, there is a lot of freedom of choice in picking a single explicit such variety. One criterion of niceness is the purely subjective one; Mal'cev's original Mal'cev condition, namely the variety

$$\mathfrak{M} = \langle p \text{ ternary} \mid p(x,x,y) = p(y,x,x) = y \rangle$$

generating the strong Mal'cev class of varieties with permuting congruences, is clearly "nice" by this criterion.

Here is a less subjective, but also probably less generally applicable criterion: call a variety $\mathfrak{V}$ *canonical* if every set-preserving functor $\mathfrak{V} \to \mathfrak{V}$ is an equivalence (for example, groups, lattices, semilattices). This will be our concept of "niceness".

LEMMA 8.1. *A Mal'cev indistinguishability class contains at most one canonical variety up to equivalence.*

The proof is trivial.

Thus if a strong Mal'cev or ω-Mal'cev class is generated by a canonical variety, then this variety is unique up to equivalence (and is also in some sense "minimal"), so it has a right to be called nice.

I do not know which strong Mal'cev and ω-Mal'cev classes mentioned in this paper have canonical generators. The following proposition, whose proof we leave to the reader, gives a general criterion, which however rarely seems applicable. It is the only result of this type that I know.

PROPOSITION 8.2. *The Mal'cev indistinguishability class of a locally finite variety contains a canonical representative.*

Note that Mal'cev's variety $\mathfrak{M}$ above, though subjectively nice, is *not* canonical. The operation q defined by

$$q(x,y,z) = p(p(x,y,z), y, p(p(x,y,z), p(z,y,x), y))$$

defines an $\mathfrak{M}$-structure on any $\mathfrak{M}$-algebra which gives a set-preserving functor $\mathfrak{M} \to \mathfrak{M}$ which is not an equivalence.

References

J. T. Baldwin and J. Berman (1976), "A model theoretic approach to Malcev conditions", Preprint, Univ. of Illinois at Chicago Circle.

P. Freyd (1966), "Algebra-valued functors in general and tensor products in particular", *Colloq. Math.* 14, 89–106.

F. W. Lawvere (1968), *Some Algebraic Problems in the Context of Functorial Semantics of Algebraic Theories*, Reports of the Midwest Category Seminar II, Lecture Notes in Math. 61 (Springer-Verlag, Berlin, 1968), pp. 41–61.

W. D. Neumann (1970), "Representing varieties of algebras by algebras", *J. Austral. Math. Soc.* **11**, 1–8.
W. D. Neumann (1974), "On Malcev conditions", *J. Austral. Math. Soc.* **17**, 376–384.
S. Świerczkowski (1964), "Topologies in free algebras", *Proc. London Math. Soc.* (3) **14**, 566–576.
W. Taylor (1973), "Characterizing Mal'cev conditions", *Alg. Univ.* **3**, 351–397.
W. Taylor (1975a), "Laws implying homotopy laws", Preprint.
W. Taylor (1975b), "Continuum many Mal'cev conditions", *Alg. Univ.* **5**, 335–337.
W. Taylor (1975c), "The fine spectrum of a variety", *Alg. Univ.* **5**, 263–303.
R. Wille (1970), *Kongruenzklassengeometrien*, Lecture Notes in Math. 113 (Springer-Verlag, Berlin, 1970).

Department of Mathematics
University of Maryland
College Park, Md 20746
USA