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ROW CONVEX TABLEAUX AND BOTT–SAMELSON VARIETIES

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Abstract

By using row convex tableaux, we study the section rings of Bott–Samelson varieties of type A. We obtain flat deformations and standard monomial type bases of the section rings. In a separate section, we investigate a three-dimensional Bott–Samelson variety in detail and compute its Hilbert polynomial and toric degenerations.

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1. Introduction

Let $G = \mathrm{GL}_n(\mathbb{C})$ be the general linear group over the complex number field $\mathbb{C}$ and B be its Borel subgroup consisting of upper triangular matrices. For a word $\mathbf{i} = (i_1, \ldots, i_\ell)$ with $1 \leq i_j \leq n-1$, the *Bott–Samelson variety* $Z_\mathbf{i}$ can be defined as the quotient space

$$P_{i_1} \times P_{i_2} \times \cdots \times P_{i_\ell}/B^\ell.$$

Here, P_{i_j} is the minimal parabolic subgroup of G associated to the simple reflection

$$s_{i_j} = (i_j, i_j + 1)$$

and $(b_1, \ldots, b_\ell) \in B^\ell$ acts on the product of P_{i_j}s by

$$(p_1, \ldots, p_\ell) \cdot (b_1, \ldots, b_\ell) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{\ell-1}^{-1} p_\ell b_\ell).$$

The Bott–Samelson varieties are defined in [1, 2, 4] to desingularize the Schubert varieties in the flag manifold G/B, and then used to study the Chow ring of G/B. In representation theory, the Bott–Samelson varieties provide Demazure's character formula, which can be understood as a generalized Weyl character formula, through the section spaces of their line bundles.

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One can also realize the Bott–Samelson variety $Z_{\mathbf{i}}$ as a configuration variety in the product of the Grassmann varieties $\mathrm{Gr}(i,n)$ via the map

$$Z_{\mathbf{i}} \longrightarrow \mathrm{Gr}(i_1,n)\times\cdots\times\mathrm{Gr}(i_\ell,n)$$
$$(p_1,\ldots,p_\ell)\longmapsto(p_1E^{i_1},p_1p_2E^{i_2},\ldots,p_1\cdots p_\ell E^{i_\ell})$$

where E^i is the i-dimensional subspace of $\mathbb{C}^n$ spanned by the first i elementary basis elements $\{e_1,\ldots,e_i\}$. From such a realization, Lakshmibai and Magyar investigated generalized Demazure modules and described their standard monomial bases in terms of root operators [10, 11]. See also [9].

We note that there is a natural line bundle induced from the Plücker bundles on the factors $\mathrm{Gr}(i_j,n)$, and, as is the case for the Grassmann varieties and the flag varieties, we can investigate the Plücker coordinates in terms of minors over a matrix or Young tableaux and straightening relations among them.

In this paper, using the language of row convex tableaux introduced by Taylor [15], we study the section rings of the Bott–Samelson varieties and their explicit standard monomial type bases which are different from the ones given in [10, 11]. For $\mathbf{i}$ in (2.1) and $\mathbf{m}=(m_1,\ldots,m_\ell)\in\mathbb{Z}^\ell_{\geq 0}$, our main results are as follows.

Theorem 1.1. *Let $\mathsf{M}(\mathbf{i},d\mathbf{m})$ be the space spanned by tableaux of shape $(\mathbf{i},d\mathbf{m})$. The section ring of the Bott–Samelson variety with respect to the line bundle $L_{\mathbf{m}}$ is*

$$\mathcal{R}_{\mathbf{i},\mathbf{m}}\cong\bigoplus_{d\geq 0}\mathsf{M}(\mathbf{i},d\mathbf{m}),$$

and straight tableaux of shape $(\mathbf{i},d\mathbf{m})$ form a $\mathbb{C}$-basis of the space $\mathsf{M}(\mathbf{i},d\mathbf{m})$.

Then from SAGBI–Gröbner degeneration techniques (e.g. [12, 14]), we obtain a flat degeneration of the section ring.

Theorem 1.2. *The section ring $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ of the Bott–Samelson variety $Z_{\mathbf{i}}$ is a flat deformation of an affine semigroup ring.*

In the final section, we provide a detailed study of an example for the case of $\mathrm{GL}_3(\mathbb{C})$, including toric degenerations, the corresponding moment polytopes, and computations of the Hilbert polynomial.

Proposition 1.3. *The Hilbert polynomial of the Bott–Samelson variety Z is*

$$\mathrm{HP}_Z(s)=\frac{5s^3+11s^2+8s+2}{2}.$$

In [6], Grossberg and Karshon studied a family of complex structures on a Bott–Samelson manifold, such that the underlying real manifold remains the same, but the limit complex manifold admits a complete, full-dimensional torus action. (They call such varieties ‘Bott towers’. An algebraic version of their construction appeared in [13].) Our deformation is algebraic in nature, yet is different, as can be seen in examples and also from the fact that in the limit, the relationship between $Z_{\mathbf{i}}$ and G/B naturally extends to the whole flat family. The resulting toric variety does not seem to be smooth or simplicial in general.

This paper is organized as follows. In Section 2 we fix notation and some basic definitions which will be useful in the sequel. Then we describe the section ring of the Bott–Samelson variety in terms of row convex tableaux. In Section 3, by using the fact that straight tableaux form bases of the space of sections, we show that the section ring is a flat deformation of an affine semigroup ring. In Section 4 we further investigate straight tableaux and study their properties. In Section 5, for a three-dimensional Bott–Samelson variety, we compute its toric degenerations and Hilbert polynomial.

2. Row convex tableaux and the section rings

In this section, after introducing row convex tableaux and related notation, we describe the section ring of the Bott–Samelson variety associated with a reduced expression for the longest element of the symmetric group.

2.1. Row convex tableaux. A *shape* is a finite collection of pairs of positive integers. A *tableau* t of shape D is an assignment of positive integers to elements in D:

$$t : D \longrightarrow \mathbb{Z}_{>0}.$$

One can identify a shape D with a collection of cells arranged in rows and columns in such a way that there is a cell in the ith row and jth column if and only if $(i, j) \in D$. In this realization, a tableau of a shape D is a filling of cells in D with positive integers.

Definition 2.1. A *row convex shape* is a shape without gaps in any row. That is, if (r, i) and (r, k) are in a shape D, then $(r, j) \in D$ for all $i < j < k$. A *row convex tableau* is a filling of a row convex shape with positive integers.

Our construction does not depend on the order of rows. Therefore, we will assume that all the row convex shapes in this paper satisfy the following conditions: the higher rows end at least as far to the right as lower rows. Such shapes may be understood as a generalization of skew Young diagrams in the following sense (cf. [15]). For two Young diagrams

$$\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{Z}^\ell \text{ such that } \lambda_1 \geq \cdots \geq \lambda_\ell \geq 0,$$
$$\mu = (\mu_1, \dots, \mu_\ell) \in \mathbb{Z}^\ell \text{ such that } \mu_1 \geq \cdots \geq \mu_\ell \geq 0,$$

with $\lambda_i \geq \mu_i$ for all i, a skew Young diagram λ/μ is the set-theoretic difference of the Young diagrams of λ and μ. If we replace a Young diagram μ with a sequence of nonnegative integers $m = (m_1, \dots, m_\ell)$ with $\lambda_i \geq m_i$ for all i, then we can obtain a row convex shape λ/m by removing the first m_i boxes in the ith row of the Young diagram λ for all i.

2.2. Column sets. Let us consider the following reduced decomposition of the longest element w_0 in $\mathfrak{S}_n$:

$$\underline{w}_0 = \underline{w}_0^{(n)} = (s_1)(s_2 s_1)(s_3 s_2 s_1) \cdots (s_{n-1} s_{n-2} \cdots s_1)$$

where s_{i_j} is the simple reflection $(i_j, i_j + 1)$. Note that the length of $\underline{w}_0$ is $\ell = n(n-1)/2$. Once and for all, we fix the word

$$\mathbf{i} = (i_1, \ldots, i_\ell) = (1, 2, 1, 3, 2, 1, \ldots, n-1, n-2, \ldots, 1) \tag{2.1}$$

associated to the reduced expression $\underline{w}_0 = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of the longest element given above.

Definition 2.2. For the reduced word $\mathbf{i}$, the *column sets* are

$$C^{(k)} = s_{i_1} s_{i_2} \cdots s_{i_k}[i_k]$$

where $[i_k]$ is the set of positive integers not more than i_k for $1 \le k \le \ell$.

Column sets can be defined for any word, but for the reduced word $\mathbf{i}$, we can explicitly describe all the column sets. In particular, it is straightforward to prove that each column set contains consecutive integers.

Lemma 2.3. *For each k, if $a < c < b$ and both a and b are in $C^{(k)}$, then $c \in C^{(k)}$. To be more precise, for each j with $2 \le j \le n-1$, let $p_j = j(j-1)/2$. Then the column sets are*

$$C^{(p_j+t)} = \{t+1, t+2, \ldots, j+1\}$$

for $1 \le t \le j$, and $C^{(1)} = \{2\}$.

This shows in particular that if we stack $C^{(k+1)}$ on top of $C^{(k)}$, the column sets we defined form a row convex shape

$$D = \bigcup_{1 \le k \le \ell} \{(k, c) | c \in C^{(k)}\}$$

and its higher rows end at least as far to the right as lower rows.

Example 2.4. For $n = 3$, the reduced word is $\mathbf{i} = (121)$ and the column sets are

$$\begin{aligned} C^{(1)} &= s_1\{1\} = \{2\}, \\ C^{(2)} &= s_1 s_2\{1, 2\} = \{2, 3\}, \\ C^{(3)} &= s_1 s_2 s_1\{1\} = \{3\}. \end{aligned}$$

For $n = 4$, the reduced word is $\mathbf{i} = (121321)$ and we have three additional column sets,

$$\begin{aligned} C^{(4)} &= s_1 s_2 s_1 s_3\{1, 2, 3\} = \{2, 3, 4\}, \\ C^{(5)} &= s_1 s_2 s_1 s_3 s_2\{1, 2\} = \{3, 4\}, \\ C^{(6)} &= s_1 s_2 s_1 s_3 s_2 s_1\{1\} = \{4\}. \end{aligned}$$

Then the corresponding row convex shapes for $n = 3$ and $n = 4$ indicated by X are respectively

$$\begin{array}{|c|c|c|} \hline & & X \\ \hline & X & X \\ \hline & X & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|} \hline & & & X \\ \hline & & X & X \\ \hline & X & X & X \\ \hline & & X & \\ \hline & X & X & \\ \hline & X & & \\ \hline \end{array}$$

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2.3. Bott–Samelson varieties. We will use the realization of the Bott–Samelson variety as the variety of configurations of subspaces of $\mathbb{C}^n$. For various constructions of the Bott–Samelson varieties and their equivalences, we refer the readers to [11, Section 1].

For the word $\mathbf{i}$ in (2.1), let us write $\mathrm{Gr}(\mathbf{i})$ for

$$\mathrm{Gr}(i_1, n) \times \cdots \times \mathrm{Gr}(i_\ell, n)$$

where $\mathrm{Gr}(i_k, n)$ is the Grassmann variety of i_k-dimensional subspaces in $\mathbb{C}^n$.

Definition 2.5. The *Bott–Samelson variety* $Z_{\mathbf{i}}$ is the closure of the B-orbit of $\underline{x}_{\mathbf{i}}$ in $\mathrm{Gr}(\mathbf{i})$:

$$Z_{\mathbf{i}} = \overline{B \cdot \underline{x}_{\mathbf{i}}} \subset \mathrm{Gr}(\mathbf{i})$$

where $\underline{x}_{\mathbf{i}} = (x_{i_1}, \ldots, x_{i_\ell})$ is the point in $\mathrm{Gr}(\mathbf{i})$ whose kth coordinate x_{i_k} is the $|C^{(k)}|$-dimensional subspace of $\mathbb{C}^n$ spanned by the elementary basis elements e_j for all j in the column set $C^{(k)}$ for $1 \le k \le \ell$.

We observe that there is a natural line bundle induced from the Plücker bundles $\mathcal{O}(1)$ on the factors of $\mathrm{Gr}(\mathbf{i})$. That is, for $\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell_{\ge 0}$, we take the powers of the Plücker bundles to obtain an effective line bundle on $\mathrm{Gr}(\mathbf{i})$:

$$\mathcal{O}(\mathbf{m}) = \mathcal{O}^{\otimes m_1} \otimes \cdots \otimes \mathcal{O}^{\otimes m_\ell}.$$

We define the line bundle $L_{\mathbf{m}}$ on the Bott–Samelson variety $Z_{\mathbf{i}}$ as the restriction of $\mathcal{O}(\mathbf{m})$ to $Z_{\mathbf{i}} \subset \mathrm{Gr}(\mathbf{i})$,

$$L_{\mathbf{m}} = \mathcal{O}(\mathbf{m})_{|Z_{\mathbf{i}}},$$

and then study the section ring

$$\mathcal{R}_{\mathbf{i},\mathbf{m}} = \bigoplus_{d \ge 0} H^0(Z_{\mathbf{i}}, L^d_{\mathbf{m}}).$$

We will also discuss the relation between $L_{\mathbf{m}}$ and a line bundle on G/B in Section 4.

2.4. Minors and tableaux. Let $M_n = M_n(\mathbb{C})$ be the space of complex $n \times n$ matrices and $B_n = \overline{B}$ be the subspace consisting of upper triangular matrices:

$$B_n = \{(x_{ij}) \in M_n : x_{ij} = 0 \text{ for } i > j\}.$$

For $k \le n$, consider subsets $R = \{r_1, \ldots, r_k\}$ and $C = \{c_1, \ldots, c_k\}$ of $\{1, \ldots, n\}$ such that $r_1 < \cdots < r_k$ and $c_1 < \cdots < c_k$. Then we let $[R : C]$ denote the map from B_n to $\mathbb{C}$ by assigning to a matrix $b \in B_n$ the determinant of the $k \times k$ minor of b formed by taking rows R and columns C:

$$[R : C] = \det \begin{bmatrix} x_{r_1 c_1} & \cdots & x_{r_1 c_k} \\ \vdots & \ddots & \vdots \\ x_{r_k c_1} & \cdots & x_{r_k c_k} \end{bmatrix}$$

where $x_{rc} = 0$ if $r > c$.

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For subsets S and S' of $\{1,\dots,n\}$ of the same size, we can impose a partial ordering: $S \le S'$ if, for each k, the kth smallest element of S is less than or equal to the kth smallest element of S'. Then note that $[R : C]$ is nonzero only if $R \le C$. A map with this property is referred to as *flagged*. Since we consider only minors defined on B, from now on we continue to assume this property.

By using a Young diagram with a single row consisting of n boxes, we can record $[R : C]$ by filling in the c_ith box counting from left to right with r_i for each i. For example, for $n = 6$, if $R_1 = \{1,3,4\}$ and $C_1 = \{2,3,4\}$ then $[R_1 : C_1]$ can be drawn as

	1	3	4		

The product of k of these row tableaux $[R_i : C_i]$ can be encoded in a $k \times n$ rectangular array whose ith row counting from bottom to top is $[R_i : C_i]$ for $1 \le i \le k$. For example, if $R_2 = \{2,3,5\}, C_2 = \{3,4,5\}, R_3 = \{4,5\}$ and $C_3 = \{5,6\}$, then $\prod_{1\le i\le 3}[R_i : C_i]$ can be drawn as

				4	5
		2	3	5	
	1	3	4		

Next, for $\ell = n(n-1)/2$ and $\mathbf{m} = (m_1,\dots,m_\ell) \in \mathbb{Z}^\ell_{\ge 0}$, consider a collection

$$\bigcup_{1\le k\le \ell} \{[R^{(k)}_j : C^{(k)}] \mid 1 \le j \le m_k\}$$

where the $R^{(k)}_j$ are subsets of $\{1,2,\dots,n\}$ and the $C^{(k)}$ are the column sets with respect to $\mathbf{i}$ (Definition 2.2). Write $|\mathbf{m}|$ for $\sum_k m_k$. Then, by repeating $C^{(k)}$ m_k times for each k, the product t of the $[R^{(k)}_j : C^{(k)}]$ can be encoded in a $|\mathbf{m}| \times n$ rectangular array having $[R^{(i)}_j : C^{(i)}]$ as its $(m_1 + \cdots + m_{i-1} + j)$th row counting from bottom to top. In this way, we can identify tableaux and products of minors.

Definition 2.6. A *tableau* t of shape $(\mathbf{i}, \mathbf{m})$ is

$$\mathsf{t} = \Big(\prod_{1\le j\le m_1} [R^{(1)}_j : C^{(1)}]\Big)\cdot\Big(\prod_{1\le j\le m_2} [R^{(2)}_j : C^{(2)}]\Big)\cdot\ldots\cdot\Big(\prod_{1\le j\le m_\ell} [R^{(\ell)}_j : C^{(\ell)}]\Big). \tag{2.2}$$

Note that up to sign, we can always assume that the entries in each row of t are increasing from left to right. If such is the case, then t is called a *row standard tableau*.

2.5. Section ring. From the realization of $Z_{\mathbf{i}}$ as a configuration space in $Gr(\mathbf{i})$, we can obtain an explicit description of the space of sections $H^0(Z_{\mathbf{i}}, L_{\mathbf{m}})$ of the line bundle $L_{\mathbf{m}}$. In fact, such spaces can be described in a general setting. See [9, 10] and [11, Section 3].

Theorem 2.7. *For* $\mathbf{m} = (m_1,\dots,m_\ell) \in \mathbb{Z}^\ell_{\ge 0}$, *let* $\mathsf{M}(\mathbf{i},\mathbf{m})$ *be the space spanned by tableaux of shape* $(\mathbf{i},\mathbf{m})$. *Then*

$$\mathsf{M}(\mathbf{i},\mathbf{m}) \cong H^0(Z_{\mathbf{i}}, L_{\mathbf{m}}).$$

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PROOF. In the setting

$$Z_{\mathbf{i}} = \overline{B \cdot \underline{x}_{\mathbf{i}}} \subset \mathrm{Gr}(i_1, n) \times \cdots \times \mathrm{Gr}(i_\ell, n),$$

the sections of the line bundle $\mathcal{O}(1)$ over the Grassmannian $\mathrm{Gr}(i_k, n)$ can be identified with the maximal minors $\delta_j^{(k)}$ defined on the space X_k of $n \times i_k$ complex matrices. Therefore, the space of sections of $\mathcal{O}(\mathbf{m})$ over $\mathrm{Gr}(\mathbf{i})$ is spanned by the products

$$\prod_{j=1}^{m_1} \delta_j^{(1)} \cdot \prod_{j=1}^{m_2} \delta_j^{(2)} \cdot \cdots \cdot \prod_{j=1}^{m_\ell} \delta_j^{(\ell)}.$$

We can restrict these sections to $Z_{\mathbf{i}}$ to obtain the sections of $L_{\mathbf{m}}$ over $Z_{\mathbf{i}}$. We restrict it further to the dense orbit $B \cdot \underline{x}_{\mathbf{i}}$ of $Z_{\mathbf{i}}$, and then by using the orbit map

$$B \longrightarrow B \cdot \underline{x}_{\mathbf{i}} \subset Z_{\mathbf{i}}$$

we pull back the restriction to obtain functions ξ on $B_n = \overline{B}$.

Recall that x_{i_k} in $\underline{x}_{\mathbf{i}} = (x_{i_1}, \ldots, x_{i_\ell})$ is the $|C^{(k)}|$-dimensional subspace of $\mathbb{C}^n$ spanned by e_j for all $j \in C^{(k)}$. Therefore, the functions ξ derived from $\delta_j^{(k)}$ are the minors defined on B_n with the columns specified by the column set $C^{(k)}$. This shows that $H^0(Z_{\mathbf{i}}, L_{\mathbf{m}})$ is spanned by tableaux of shape $(\mathbf{i}, \mathbf{m})$ given in (2.2). □

Then we can consider the section ring $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ with respect to $L_{\mathbf{m}}$ as the $\mathbb{Z}_{\geq 0}$ graded algebra generated by tableaux of shape $(\mathbf{i}, \mathbf{m})$:

$$\mathcal{R}_{\mathbf{i},\mathbf{m}} = \bigoplus_{d \geq 0} \mathsf{M}(\mathbf{i}, d\mathbf{m})$$

where $d\mathbf{m} = (dm_1, \ldots, dm_\ell)$. We remark that the multiplicative structure of this ring can be described by the *straightening laws*, which are in our case essentially Grosshans–Rota–Stein syzygies given in [5]. We refer the readers to [15] for more details.

3. Flat deformations of the section rings

In this section we describe $\mathbb{C}$-bases of the section spaces and then prove that the section ring is a flat deformation of a semigroup ring.

3.1. Straight tableaux. For a Young diagram λ, it is well known that semistandard tableaux form a $\mathbb{C}$-basis of the space spanned by tableaux of shape λ (see [5, 12]). We now discuss an analogous result for row convex shape, which is given in [15] in a general setting of polynomial superalgebras.

DEFINITION 3.1. A row standard tableau $\mathfrak{t}$ of shape $(\mathbf{i}, \mathbf{m})$ given in (2.2) is called a *straight tableau* if it satisfies the following condition: for two cells (i, k) and (j, k) with $i > j$ in the same column, the entry in the upper cell (i, k) may be strictly larger than the entry in the lower cell (j, k) only if the cell $(i, k - 1)$ exists and contains an entry weakly larger than the one in the cell (j, k).

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For example, each of the first three tableaux below can be a part of a straight tableau while the last one cannot be, because in the last tableau 3 in the second column is less than 4 in the same column and 1 to the left of the 4 is less than 3:

$$
\begin{array}{|c|c|c|c|}\hline & & 1 & 2 \\ \hline 3 & 4 & 5 & 6 \\ \hline & & 5 & \\ \hline & 5 & 7 & \\ \hline \end{array}\,,\quad
\begin{array}{|c|c|c|c|}\hline & & 1 & 2 \\ \hline 3 & 4 & 5 & 6 \\ \hline & & 3 & \\ \hline & 6 & 7 & \\ \hline \end{array}\,,\quad
\begin{array}{|c|c|c|c|}\hline & & 2 & 5 \\ \hline 3 & 4 & 5 & 7 \\ \hline & & 6 & \\ \hline & 3 & 8 & \\ \hline \end{array}\,,\quad
\begin{array}{|c|c|c|c|}\hline & & 2 & 5 \\ \hline 1 & 4 & 5 & 7 \\ \hline & & 7 & \\ \hline & 3 & 8 & \\ \hline \end{array}
$$

A monomial order on the polynomial ring $\mathbb{C}[M_n]$ is called a *diagonal term order* if the leading monomial of a determinant of any minor defined on M_n is equal to the product of the diagonal elements. For a subring $\mathcal{R}$ of the polynomial ring we let $in(\mathcal{R})$ denote the algebra generated by the leading monomials $in(f)$ of all $f \in \mathcal{R}$ with respect to a given monomial order. Note that the collection of leading monomials forms a semigroup, therefore $in(\mathcal{R})$ is a semigroup algebra and $Spec(in(\mathcal{R}))$ is an affine toric variety in the sense of [14]. Recall that for a subring $\mathcal{R}$ of a polynomial ring, a set $\{f_i : i \in I\}$ of elements of $\mathcal{R}$ is called a SAGBI basis if $\{in(f_i) : i \in I\}$ generates the associated semigroup algebra $in(\mathcal{R})$.

Proposition 3.2. *Let D be a row convex shape.*

(1) [15, Theorem 6.2] *Straight tableaux of shape D form a $\mathbb{C}$-basis for the space spanned by all the tableaux of shape D.*

(2) [15, Theorem 7.8] *Straight tableaux of shape D form a SAGBI basis of the graded algebra $\mathcal{R} \subset \mathbb{C}[M_n]$ generated by all the tableaux of shape D with respect to any diagonal term order.*

From the fact that the shape $(\mathbf{i}, \mathbf{m})$ is row convex, it follows from the above proposition that straight tableaux form a $\mathbb{C}$-basis of the section ring $\mathcal{R}_{\mathbf{i},\mathbf{m}}$, and that the straight tableaux of shape $(\mathbf{i}, \mathbf{m})$ form a SAGBI basis for $\mathcal{R}_{\mathbf{i},\mathbf{m}}$. We will study more properties of straight tableaux in Section 4.

3.2. Flat deformation. We now study a flat deformation of the section ring $\mathcal{R}_{\mathbf{i},\mathbf{m}}$. The technique is basically the same as that for the Grassmannians and the flag varieties given in, for example, [8, 12, 14].

Theorem 3.3. *The section ring $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ of the Bott–Samelson variety $Z_{\mathbf{i}}$ can be flatly deformed into an affine semigroup ring.*

Proof. We show that there is a flat $\mathbb{C}[t]$ module $\mathcal{R}^t_{\mathbf{i},\mathbf{m}}$ whose general fiber is isomorphic to $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ and whose special fiber is isomorphic to the semigroup ring $in(\mathcal{R}_{\mathbf{i},\mathbf{m}})$. Lemma 2.3 shows that any tableau of shape $(\mathbf{i}, \mathbf{m})$ with the column sets $\{C^{(1)}_1, \ldots, C^{(\ell)}_\ell\}$ is a row convex tableau. Therefore, we can apply Proposition 3.2 to $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ to conclude that the set of straight tableaux of shape $(\mathbf{i}, \mathbf{m})$ forms a SAGBI basis for the ring $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ with respect to a diagonal term order. Then, from the existence of a finite SAGBI

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basis, by [3], there exists a $\mathbb{Z}_{\geq 0}$ filtration $\{F_\alpha\}$ on $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ such that the associated graded ring of the Rees algebra $\mathcal{R}^t_{\mathbf{i},\mathbf{m}}$ with respect to $\{F_\alpha\}$,

$$\mathcal{R}^t_{\mathbf{i},\mathbf{m}} = \bigoplus_{\alpha \geq 0} F_\alpha(\mathcal{R}_{\mathbf{i},\mathbf{m}})t^\alpha,$$

is isomorphic to $in(\mathcal{R}_{\mathbf{i},\mathbf{m}})$. Then, by the general property of the Rees algebra, $\mathcal{R}^t_{\mathbf{i},\mathbf{m}}$ is flat over $\mathbb{C}[t]$ with general fiber isomorphic to $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ and with special fiber isomorphic to the associated graded ring, which is $in(\mathcal{R}_{\mathbf{i},\mathbf{m}})$. □

4. Straight tableaux and the space of sections

In this section we study in more detail the $\mathbb{C}$-basis of the space $\mathsf{M}(\mathbf{i},\mathbf{m}) \cong H^0(Z_{\mathbf{i}}, L_{\mathbf{m}})$ given by straight tableaux in Proposition 3.2, and then its connection to the natural map from the Bott–Samelson variety to the flag variety.

4.1. Contra-tableaux. To simplify our notation, we retain the notation

$$\ell = n(n-1)/2 \quad \text{and} \quad p_j = j(j-1)/2$$

for $2 \leq j \leq n-1$. We also fix an arbitrary multiplicity $\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell_{\geq 0}$.

Definition 4.1. A *contra-tableau* is a filling of a skew Young diagram

$$(k, k, \ldots, k)/(\lambda_1, \lambda_2, \ldots)$$

with $k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ such that the entries in each column are weakly increasing from top to bottom and the entries in each row are strictly increasing from left to right.

For example, a contra-tableau of shape $(4,4,4,4,4)/(3,3,3,2,1)$ can be encoded in a rectangular array as follows:

			1
			1
			2
		1	2
	1	3	4

Recall that the usual semistandard tableaux can encode weight basis elements for irreducible polynomial representations of the general linear group. Similarly, one can use contra-tableaux to encode weight vectors of a contra-gradient representation of an irreducible polynomial representation of the general linear group. Here, our goal is to decompose a straight tableau into contra-tableaux.

First, we can decompose the shape $(\mathbf{i},\mathbf{m})$ into skew Young diagrams as follows. For $1 \leq j \leq n-1$, let us set $\mathbf{m}(j) = (m'_1, \ldots, m'_\ell)$ where $m'_i = m_i$ for $p_j < i \leq p_{j+1}$ and $m'_i = 0$ otherwise. Then $\mathbf{m} = \mathbf{m}(1) + \cdots + \mathbf{m}(n-1)$ in $\mathbb{Z}^\ell$.

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EXAMPLE 4.2. If $n = 4$ and $\mathbf{m} = (1, 1, \dots, 1) \in \mathbb{Z}^6_{\geq 0}$, then $(\mathbf{i}, \mathbf{m}(1))$, $(\mathbf{i}, \mathbf{m}(2))$, $(\mathbf{i}, \mathbf{m}(3))$ respectively correspond to the shapes:

$$
\begin{array}[b]{|c|c|c|c|}\hline & X & & \\ \hline \end{array}
\quad
\begin{array}[b]{|c|c|c|c|}\hline & & X & \\ \hline & X & X & \\ \hline \end{array}
\quad
\begin{array}[b]{|c|c|c|c|}\hline & & & X \\ \hline & & X & X \\ \hline & X & X & X \\ \hline \end{array}
$$

Note that this is equivalent to the decomposition of the shape $(\mathbf{i}, \mathbf{m})$ given in Example 2.4 into maximal possible Young diagrams.

If $\mathbf{m} = (1, 1, \dots, 1)$, then, from the second statement of Lemma 2.3, the shape $(\mathbf{i}, \mathbf{m}(j))$ is a skew Young diagram $(j + 1, j + 1, \dots, j + 1)/(j, j - 1, \dots, 1)$ of length j. By repeating the kth rows m_{p_j+k} times, we have a skew Young diagram of length $|\mathbf{m}(j)|$. Then from the definition of straight tableaux, it is straightforward to check that every straight tableau in a skew diagram is a contra-tableau. See also [15, Proposition 4.3].

LEMMA 4.3. *For each j, every straight tableau of shape $(\mathbf{i}, \mathbf{m}(j))$ is a contra-tableau.*

Note that this lemma shows that the basis of the space $\mathsf{M}(\mathbf{i}, \mathbf{m}(j)) \cong H^0(Z_{\mathbf{i}}, L_{\mathbf{m}(j)})$ is simply given by contra-tableaux, and as a consequence we can obtain a description of elements in the section ring $\mathcal{R}_{\mathbf{i},\mathbf{m}}$ as products of contra-tableaux. That is, we have a natural projection

$$\mathsf{M}(\mathbf{i}, \mathbf{m}(1)) \otimes \cdots \otimes \mathsf{M}(\mathbf{i}, \mathbf{m}(n - 1)) \to \mathsf{M}(\mathbf{i}, \mathbf{m}) \tag{4.1}$$

sending $\mathsf{t}_1 \otimes \cdots \otimes \mathsf{t}_{n-1}$ to the product $\mathsf{t}_1 \cdot \dots \cdot \mathsf{t}_{n-1} \in \mathsf{M}(\mathbf{i}, \mathbf{m})$ where t_j is a contra-tableau in $\mathsf{M}(\mathbf{i}, \mathbf{m}(j))$ for each j.

For example, if $n = 4$ and $\mathbf{m} = (1, 2, 1, 1, 1, 3)$, then the product map $\mathsf{t}_1 \otimes \mathsf{t}_2 \otimes \mathsf{t}_3 \to \mathsf{t}$ gives

$$
\begin{array}[b]{|c|c|c|c|}\hline & 1 & & \\ \hline \end{array}
\otimes
\begin{array}[b]{|c|c|c|c|}\hline & & 2 & \\ \hline & 1 & 3 & \\ \hline & 2 & 3 & \\ \hline \end{array}
\otimes
\begin{array}[b]{|c|c|c|c|}\hline & & & 1 \\ \hline & & & 1 \\ \hline & & & 2 \\ \hline & & 1 & 2 \\ \hline & 1 & 3 & 4 \\ \hline \end{array}
\to
\begin{array}[b]{|c|c|c|c|}\hline & & & 1 \\ \hline & & & 1 \\ \hline & & & 2 \\ \hline & & 1 & 2 \\ \hline & 1 & 3 & 4 \\ \hline & & 2 & \\ \hline & 1 & 3 & \\ \hline & 2 & 3 & \\ \hline & 1 & & \\ \hline \end{array}
$$

Note that the product is not a straight tableau, but it can be expressed by a linear combination of straight tableaux in $\mathsf{M}(\mathbf{i}, \mathbf{m}) \subset \mathcal{R}_{\mathbf{i},\mathbf{m}}$ by successive application of the straightening laws mentioned after Proposition 2.7.

4.2. Projection to G/B. We now discuss the natural map from the Bott–Samelson variety $Z_{\mathbf{i}}$ to the flag variety G/B in terms of our basis description.

The projection map (4.1) is compatible with the decomposition of a straight tableau of shape $(\mathbf{i}, \mathbf{m})$ into contra-tableaux. More precisely, a straight tableau t of shape $(\mathbf{i}, \mathbf{m})$

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can be factored into a product $\mathsf{t}_1 \cdot \ldots \cdot \mathsf{t}_{n-1}$ of straight tableaux t_j of shape $(\mathbf{i}, \mathbf{m}(j))$ for $1 \leq j \leq n-1$. Then, by Lemma 4.3, the t_j are contra-tableaux for all j.

In particular, for each $1 \leq j \leq n-2$, let us consider a straight tableau t_j^0 of shape $(\mathbf{i}, \mathbf{m}(j))$ such that, for each a and b such that $1 \leq b \leq m_j$ and $p_j + 1 \leq a \leq p_{j+1}$, the row indices and the column indices are equal, $R_b^{(a)} = C^{(a)}$, that is,

$$\mathsf{t}_1^0 = [C^{(2)} : C^{(2)}]^{m_1};$$
$$\mathsf{t}_j^0 = [C^{(p_j+1)} : C^{(p_j+1)}]^{m_{p_j+1}} \cdot [C^{(p_j+2)} : C^{(p_j+2)}]^{m_{p_j+2}} \cdot \ldots \cdot [C^{(p_{j+1})} : C^{(p_{j+1})}]^{m_{p_{j+1}}}$$

for $2 \leq j \leq n-2$. This is equivalent to saying that t_j^0 is obtained by filling in all the cells corresponding to the subshapes $(\mathbf{i}, \mathbf{m}(j))$ of the shape $(\mathbf{i}, \mathbf{m})$ with maximum possible numbers.

Then, for any contra-tableau t of shape $(\mathbf{i}, \mathbf{m}(n-1))$, we can find a straight tableau $\widehat{\mathsf{t}}$ of shape $(\mathbf{i}, \mathbf{m})$ such that

$$\widehat{\mathsf{t}} = (\mathsf{t}_1^0 \cdot \ldots \cdot \mathsf{t}_{n-2}^0) \cdot \mathsf{t}$$

and this provides the injection

$$\begin{aligned} H^0(G/B, L_\lambda) &\to \mathsf{M}(\mathbf{i}, \mathbf{m}) \\ \mathsf{t} &\mapsto (\mathsf{t}_1^0 \cdot \ldots \cdot \mathsf{t}_{n-2}^0) \cdot \mathsf{t} \end{aligned} \tag{4.2}$$

where $H^0(G/B, L_\lambda)$ is the section space of the line bundle L_λ on G/B and λ is the dominant weight determined by the shape $\mathbf{m}(n-1)$ as a Young diagram. For example,

$$\begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & 1 \\ \hline & & & 2 \\ \hline & & 1 & 2 \\ \hline & 1 & 3 & 4 \\ \hline \end{array} \;\to\; \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & 1 \\ \hline & & & 2 \\ \hline & & 1 & 2 \\ \hline & 1 & 3 & 4 \\ \hline & & 3 & \\ \hline & 2 & 3 & \\ \hline & 2 & 3 & \\ \hline & 2 & & \\ \hline \end{array}$$

Finally, by extending the map (4.2), we have the following proposition.

PROPOSITION 4.4. *There is a natural map from the section ring of the flag variety to the section ring of $Z_{\mathbf{i}}$:*

$$\bigoplus_{d \geq 0} H^0(G/B, L_\lambda^d) \longrightarrow \mathcal{R}_{\mathbf{i},\mathbf{m}} = \bigoplus_{d \geq 0} H^0(Z_{\mathbf{i}}, L_{\mathbf{m}}^d).$$

5. Three-dimensional example: toric degenerations

In this section we will consider explicit examples of toric degenerations of a three-dimensional Bott–Samelson variety, and compute the corresponding Hilbert polynomials.

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5.1. Three-dimensional Bott–Samelson variety. Let P_1 and P_2 be the following parabolic subgroups of $\mathrm{GL}_3(\mathbb{C})$:

$$P_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad P_2 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Denote by $\bar{P}_1$ and $\bar{P}_2$ their closures in the space M_3 of 3×3 matrices.

Let Z be the Bott–Samelson variety defined as in Section 1 with $n = 3$ and $\mathbf{i} = (121)$. That is,

$$Z = P_1 \times P_2 \times P_1 / B^3$$

with the action of B^3:

$$(p_1, p_2, p_3) \cdot (b_1, b_2, b_3) = (p_1 b_1, b_1^{-1} p_2 b_2, b_2^{-1} p_3 b_3).$$

It can also be viewed as an invariant theory quotient of the product of the closures $\bar{P}_1 \times \bar{P}_2 \times \bar{P}_1$ by the action of B^3 in the obvious way.

We will denote the elements of the first copy of P_1 by

$$p_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

the elements of P_2 by

$$p_2 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix},$$

and the elements of the second copy of P_1 by

$$p_3 = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}.$$

The same notation will be used for the elements of their closures in M_3.

5.2. Hilbert polynomial. Next, we will describe a Plücker-type embedding of Z into the product of three projective spaces:

$$\mathcal{H} := \mathrm{Proj}(s_1, s_2) \times \mathrm{Proj}(r_{23}, r_{13}, r_{12}) \times \mathrm{Proj}(q_1, q_2, q_3) \simeq \mathbb{CP}^1 \times \mathbb{CP}^2 \times \mathbb{CP}^2.$$

Let a point in Z be represented by three matrices (p_1, p_2, p_3) in the above form. Then we denote by s_i the 1×1 minor of the matrix p_1 with column 1 and row i. Therefore,

$$s_1 = a_{11} \quad \text{and} \quad s_2 = a_{21}.$$

(s_3 would be identically equal to zero, so we do not use it.) Next, we denote by r_{ij} the 2×2 minor of the matrix $p_1 p_2$ with columns $1, 2$ and rows i, j. Explicitly,

$$\begin{aligned} r_{12} &= a_{11} b_{11} (a_{22} b_{22} + a_{23} b_{32}) - a_{21} b_{11} (a_{12} b_{22} + a_{13} b_{32}), \\ r_{13} &= a_{11} a_{33} b_{11} b_{32}, \\ r_{23} &= a_{21} a_{33} b_{11} b_{32}. \end{aligned}$$

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Finally, we denote by q_i the 1×1 minor of the matrix $p_1p_2p_3$ with column 1 and row i:

$$\begin{aligned} q_1 &= a_{11}b_{11}c_{11} + (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32})c_{21}, \\ q_2 &= a_{21}b_{11}c_{11} + (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32})c_{21}, \\ q_3 &= a_{33}b_{32}c_{21}. \end{aligned}$$

Then Z can be viewed as a subvariety of $\mathcal{H}$, the product of three projective spaces, defined by the following two homogeneous equations (or Plücker relations):

$$s_1r_{23} - s_2r_{13} = 0 \quad \text{and} \quad q_1r_{23} - q_2r_{13} + q_3r_{12} = 0. \tag{5.1}$$

PROPOSITION 5.1. *The Hilbert polynomial of Z is given by*

$$\mathrm{HP}_Z(s) = \frac{5s^3 + 11s^2 + 8s + 2}{2}.$$

PROOF. Let, as before, $\mathcal{H} = \mathbb{CP}^1 \times \mathbb{CP}^2 \times \mathbb{CP}^2$ and let π_1, π_2 and π_3 stand for the projections onto the corresponding factors. Write

$$L = \pi_1^*(\mathcal{O}(1)), \quad M_1 = \pi_2^*(\mathcal{O}(1)), \quad M_2 = \pi_3^*(\mathcal{O}(1)).$$

We will also denote by the same letters L, M_1, M_2 the corresponding classes of divisors in the Chow ring of $\mathcal{H}$. Let X be the element of the Chow ring of $\mathcal{H}$ corresponding to Z, and let

$$D = n(L + M_1 + M_2).$$

For large enough integral values of s, the Hilbert polynomial $\mathrm{HP}_Z(s)$ coincides with $\dim(H^0(sD_{|Z}))$, which, due to vanishing, is the same as the Euler characteristic of $sD_{|Z}$.

The Riemann–Roch theorem for smooth Fano threefolds (see [7]) asserts that

$$\chi(nD_{|Z}) = \frac{D_{|Z}^3}{6}n^3 - \frac{D_{|Z}^2K_Z}{4}n^2 + \frac{D_{|Z}(K_Z^2 + c_2(Z))}{12}n + 1.$$

Now $X = (L + M_1)(M_1 + M_2)$, therefore by the adjunction formula we get $-K_Z = (L + M_1 + 2M_2)_{|Z}$ and hence $(L + M_1 + 2M_2)_{|Z}c_2(Z) = 24$. To find $(M_2)_{|Z}c_2(Z)$, we will use the same Riemann–Roch formula, but for M_2 note that $\dim(H^0(M_2)) = 3$. Finally, the intersection products satisfy

$$L^2 = 0, \quad M_1^3 = M_2^3 = 0, \quad LM_1^2M_2^2 = 1,$$

which leads to a straightforward computation of the required polynomial. □

5.3. Deformation. Ignoring the first component, one can consider the projection

$$\mathcal{H} \to \mathbb{CP}^2 \times \mathbb{CP}^2.$$

The image of Z under this projection is naturally the three-dimensional flag variety Fl_3, sitting inside $\mathbb{CP}^1 \times \mathbb{CP}^2$ as the zero set of the second Plücker relation in (5.1).

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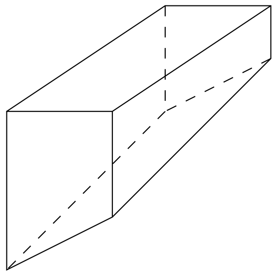

FIGURE 1. Moment polytope for D_3.

There are two naive ways to construct toric degenerations of Z. The first is to consider the family of varieties, parameterized by $\tau \in \mathbb{C}$, where the second equation is modified to

$$q_1 r_{23} - q_2 r_{13} + \tau q_3 r_{12} = 0.$$

One can easily observe that the special toric fiber of this family, corresponding to $\tau = 0$, is a reducible variety and has two irreducible components: one, denoted by $\mathcal{G}$, is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^2$, and corresponds to $r_{23} = r_{13} = 0$; and the other, denoted by D_3, a three-dimensional toric variety, which is actually nonsingular. Combinatorially, the moment polytope for D_3 is a cube, and is drawn schematically in Figure 1. (To simplify computations, we assumed that the members of the family are polarized by the invertible sheaf induced from $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(1)$ on $\mathcal{H}$.)

The fact that the special fiber of this flat family of varieties over $\mathbb{C}$ is reducible and is given by the union of two nonsingular components is quite amusing. The intersection of these two components is a smooth two-dimensional toric variety, denoted by K_2, known as the Hirzebruch surface of degree one.

One can compute the Hilbert polynomials for the chosen polarization, denoted by HP, of the irreducible components, which are known (see [12]) to be the same as the Ehrhart polynomials, denoted by EP of their moment polytopes, as well as their Ehrhart series, ES. Using the LatteE macchiato computer program (http://www.math.ucdavis.edu/~mkoeppe/latte/), we have obtained:

$$\mathrm{ES}(D_3) = \frac{3t^2 + 8t + 1}{(1-t)^4}, \quad \mathrm{EP}(D_3) = \mathrm{HP}(D_3) = 2s^3 + 5s^2 + 4s + 1 = (s+1)^2(2s+1),$$

$$\mathrm{ES}(\mathcal{G}) = \frac{1+2t}{(1-t)^4}, \quad \mathrm{EP}(\mathcal{G}) = \mathrm{HP}(\mathcal{G}) = \frac{s^3 + 4s^2 + 5s + 2}{2} = \frac{(s+1)^2(s+2)}{2},$$

$$\mathrm{ES}(K_2) = \frac{1+2t}{(1-t)^3}, \quad \mathrm{EP}(K_2) = \mathrm{HP}(K_2) = \frac{3s^2 + 5s + 2}{2} = \frac{(s+1)(3s+2)}{2}.$$

This allows us to check that

$$\begin{aligned}\mathrm{HP}(Z) &= \mathrm{HP}(D_3) + \mathrm{HP}(\mathcal{G}) - \mathrm{HP}(K_2)\\ &= \frac{5s^3 + 11s^2 + 8s + 2}{2} = \frac{(s+1)(5s^2 + 6s + 2)}{2}.\end{aligned}$$

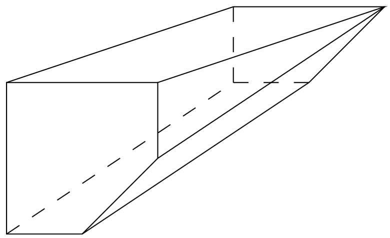

Figure 2. Moment polytope for Y_3.

This fact was also verified, independently, using the Singular software package (http://www.singular.uni-kl.de/), by representing Z as a subvariety in $\mathbb{CP}^{17}$ via Segre embedding, defined by the following 95 equations, where $[a_1 : \cdots : a_9 : b_1 : \cdots : b_9]$ are the homogeneous coordinates on $\mathbb{CP}^{17}$:

$$a_i b_j = a_j b_i \quad \text{for } 1 \le i < j \le 9,$$

and similarly,

$$\begin{aligned} a_k a_l &= a_m a_n, & a_k b_l &= a_m b_n, & a_k b_l &= b_m a_n, \\ b_k a_l &= a_m b_n, & b_k a_l &= b_m a_n, & b_k b_l &= b_m b_n, \end{aligned}$$

for the following nine choices of quadruples of indexes (k, l, m, n):

$$\begin{aligned} &(1,5,2,4), (1,6,3,4), (2,6,3,5), (1,8,2,7), (1,9,3,7), \\ &(2,9,3,8), (4,8,5,7), (4,9,6,7), \text{ and } (5,9,6,8), \end{aligned}$$

and the last five:

$$\begin{aligned} &a_1 + b_4 = 0, \quad a_2 + b_5 = 0, \quad a_3 + b_6 = 0, \\ &a_1 + a_5 + a_9 = 0, \quad \text{and} \quad b_1 + b_5 + b_9 = 0. \end{aligned}$$

5.4. Another deformation. The second way to obtain a flat toric degeneration of Z is to consider a different family of varieties inside $\mathcal{H}$, also parameterized by $\tau \in \mathbb{C}$ and given by the following two equations:

$$s_1 r_{23} - s_2 r_{13} = 0 \quad \text{and} \quad q_1 r_{23} - \tau q_2 r_{13} + q_3 r_{12} = 0.$$

One can see that the special fiber of this flat family, corresponding to $\tau = 0$, is a singular toric variety, denoted by Y_3, whose moment polytope is combinatorially is represented in Figure 2.

The Ehrhart series and the Ehrhart polynomial of the moment polytope of the special fiber corresponding to the same, previously chosen, polarization, is given by

$$\mathrm{ES}(Y_3) = \frac{5t^2 + 9t + 1}{(1-t)^4} \quad \text{and} \quad \mathrm{EP}(Y_3) = \frac{5s^3 + 11s^2 + 8s + 2}{2}.$$

Not surprisingly, we again see that $\mathrm{EP}(Y_3) = \mathrm{HP}(Z)$.

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