Can. J. Math., Vol. XXX, No. 4, 1978, pp. 851–855

ON THE HOMOLOGY OF THE GENERAL LINEAR GROUPS OVER $Z/4$

VICTOR SNAITH

1. Introduction. Let p be a prime. The algebraic K-theory of Z/p^2 is unknown. However it is easy to show that $K_i(Z/p^2)$ is finite if $i > 0$ and that it differs only in its p-torsion from $K_i(Z/p)$ which was computed in [**2**]. To proceed further one surely needs the mod p (co-)homology of GLZ/p^2. There is an exact sequence

$$(1.1) \quad 1 \rightarrow M_nZ/p \xrightarrow{i_n} GL_nZ/p^2 \xrightarrow{j_n} GL_nZ/p \rightarrow 1.$$

In (1.1) j_n is reduction mod $p, M_nZ/p$ is the additive group of $n \times n$ matrices with entries in Z/p and if ϕ is the canonical inclusion of Z/p into Z/p^2 then $i_n(A) = I + \phi(A)$. Since $BGLZ/p$ is a p-local homology point [**2**] one might expect the following to be true:

1.2. CONJECTURE. *Let $k_n:GL_nZ/p^2 \rightarrow GLZ/p^2$ be the canonical inclusion and let i_n be as in* (1.1). *Then*

$$\text{im } (k_n \circ i_n)_* = \text{im } (k_n)_* \subset H_*(BGLZ/p^2; Z/p)$$

where $(-)_*$ denotes the induced map in mod p *homology.*

In this note I will prove the following general results, which are applied below to verify Conjecture 1.2 when $p = 2$ and $n = 1$ or 2.

THEOREM A. *The image of $H_*(T_nZ/4) \rightarrow H_*(GL_\infty Z/4)$ lies in the image of $H_*(M_\infty Z/2) \rightarrow H_*(GL_\infty Z/4)$ for any $n \geqq 1$.*

THEOREM B. *The homomorphism $\tilde{H}_*(U_nZ/4) \rightarrow \tilde{H}_*(GL_\infty Z/4)$ is zero for any $n \geqq 1$.*

In Theorems A and B—and throughout the rest of this paper—$H_*(G)$ means the mod 2 singular homology of the classifying space of G. A similar convention for $H^*(G)$ is used. Also for any ring A, D_nA, U_nA, T_nA and R_nA are the following subgroups of the general linear group, GL_nA. D_nA is the diagonal subgroup. T_nA is the upper triangular subgroup and

$$U_nA = \{(a_{ij}) \in T_nA | a_{kk} = 1 \text{ for } 1 \leqq k \leqq n\}.$$

$R_nA = \{(a_{ij}) \in T_nA | a_{kk} = 1 \text{ for } 2 \leqq k \leqq n,\ a_{ij} = 0 \text{ if } 2 \leqq i < j \leqq n\}$ is the first-row subgroup of T_nA.

1.3. PROPOSITION. *Conjecture* 1.2 *is true when $p = 2$ and $n = 1$ or* 2.

Received May 16, 1977 and in revised form, September 16, 1977.

Proof. When $n = 1$ $GL_1Z/p^2 \cong M_1Z/p \times Z/(p-1)$ and there is nothing to prove.

When $n = 2$ a Sylow 2-subgroup, H, of $GL_2Z/4$ consists of matrices of the form

$$\begin{bmatrix} 1+2a & b \\ 2c & 1+2d \end{bmatrix}$$

with a, b, c and d being arbitrary elements of $Z/4$. It is easy to show that H is a semi-direct product of the form $(Z/2)^4 \ltimes Z/2$ where $Z/2 \to \text{Aut}\ ((Z/2)^4)$, given by conjugation, sends the generator to the involution $\tau(a_1, a_2, a_3, a_4) = (a_3, a_4, a_1, a_2)$. The $(Z/2)^4$ in H is just $M_2Z/2$. It is well-known that the mod 2 cohomology of H is detected by the two subgroups $M_2Z/2$ and $G \times Z/2$ where $G \subset M_2Z/2$ is the subgroup of matrices fixed under the $Z/2$-action. However, in this case, $G \times Z/2 \subset T_2Z/4$. Hence we have an injection

$$H^*(GL_2Z/4) \to H^*(M_2Z/2) \oplus H^*(T_2Z/4)$$

and by Theorem A we can detect $\text{im}\ (k_2)^* \subset H^*(GL_2Z/4)$ faithfully in $H^*(M_2Z/2) \oplus H^*(D_2Z/4)$. But $D_2Z/4 \subset M_2Z/2$ so

$$\text{im}\ (k_2)^* \xrightarrow{i_2^*} \text{im}\ (k_2 \circ i_2)^* \subset H^*(M_2Z/2)$$

is an injection. The proof is completed by a simple computation using the duality between H^* and H_*.

Theorem A is proved in § 3.5 and Theorem B in § 3.6. The ideas in the proof are due to Quillen (c.f. [**1**, § 4] and [**2**, § 11]). In § 2 are gathered together the exact sequences and the rings which will be needed later. The analogous results are also true when $Z/4$ is replaced by Z/p^2 for any prime p.

2. If $\alpha \in D_nA$ has entry t_i in the (i, i)-th place then $\alpha(a_{ij})\alpha^{-1} = (t_it_j^{-1}a_{ij})$ for any $(a_{ij}) \in GL_nA$. Hence $R_nA \lhd T_nA$, $U_nA \lhd T_nA$ and every element in $A^* = \{(a_{ij}) \in D_nV | a_{jj} = 1, j \geqq 2\}$ commutes with every element of $T_{n-1}A \cong \{(a_{ij}) \in T_nV | a_{1j} = 0 \text{ for } 2 \leqq j \leqq n\}$. Note that

$$R_nA \cap U_nA \cong \bigoplus_1^{n-1} A$$

where the right hand group has the natural additive structure.

We have exact sequences

$$(2.1) \quad 1 \to R_nA \to T_nA \to T_{n-1}A \to 1$$

and

$$(2.2) \quad 1 \to \left(\bigoplus_1^{n-1} A\right) \to R_nA \to A^* \to 1$$

where A^* is the group of units.

https://doi.org/10.4153/CJM-1978-073-x Published online by Cambridge University Press

Consider now the Galois fields $GF(2^d)$. Choose an increasing sequence of *odd* integers $1 = d_1 < d_2 < d_3 < \cdots$ such that $2^{d_i} - 1$ is prime. This is possible by a result of Dirichlet. Set $k_i = GF(2^{d_i})$ so that $k_1 = Z/2$ and

$$k_i \cong Z/2[X]/p_i(X)$$

for some $p_i(X) \in Z/2[X]$.

Set

$$A_i = \frac{Z/4[X]}{q_i(X)}$$

where q_i reduces mod 2 to p_i. The additive group of A_i is just $\bigoplus_1^{d_i} Z/4$ while A_i^* is isomorphic to $(\bigoplus_1^{d_i} Z/2)k_i^*$. Reduction mod 2 gives an epimorphism $\pi_i: A_i \to k_i$.

3. First we need a well-known result from Galois theory. Let $\bar{k}$ be the algebraic closure of the field k.

3.1. Lemma. *There is a ring isomorphism*

$$\phi: k_i \bigotimes_{k_1} \bar{k}_1 \overset{\cong}{\Rightarrow} \bigoplus_1^{d_i} \bar{k}_1 \quad (i \geqq 1)$$

given by $\phi(x \otimes y) = (y, x^2y, x^4y, \ldots, x^{2^{d_i-1}}y)$.

3.2. Proposition. *In dimensions $j < d_i$ the natural inclusion induces isomorphisms*

$$H^j(R_nA_i) \cong H^j(A_i^*) \quad \textit{and} \quad H_j(R_nA_i) \cong H_j(A_i^*)$$

for all $n, i \geqq 1$.

Proof. From (2.2) we obtain a spectral sequence

$$E_2^{p,q} = H^p\left(A_i^*; H^q\left(\bigoplus_1^{n-1} A_i\right)\right) \bigotimes_{k_1} \bar{k}_1 \Rightarrow H^{p+q}(R_nA_i) \bigotimes_{k_1} \bar{k}_i.$$

Now $H^*(A_i) \cong \Lambda(A_i^\#) \otimes S(A_i^\#)$, from the discussion in § 2, where $A_i^\# = \mathrm{Hom}_{k_1}(A_i, k_1) \cong k_i^\#$. The generators of the exterior algebra $\Lambda(A_i^\#)$ have dimension one while those of the symmetric algebra $S(A_i^\#)$ have dimension two. Hence

$$(3.3) \quad H^*\left(\bigoplus_1^{n-1} A_i\right) \bigotimes_{k_1} \bar{k}_1 \cong \Lambda\left(\bigoplus_1^{n-1} k_i^\#\right) \otimes S\left(\bigoplus_1^{n-1} k_i^\#\right) \bigotimes_{k_1} \bar{k}_1 \cong \Lambda(V) \otimes S(V)$$

where, by Lemma 3.1,

$$V \cong \bigoplus_1^{n-1} \left(\bigoplus_1^{d_i} \bar{k}_1^\#\right).$$

https://doi.org/10.4153/CJM-1978-073-x Published online by Cambridge University Press

The action of

$$A_i^* = \left(\bigoplus_1^{d_i} Z/2\right) \times k_i^*$$

(see § 2) factors through projection onto k_i^*. k_i^* acts on each factor $(\bigoplus_1^{d_i} \bar{k}_1^{\#})$ by the dual of multiplication, since this is what conjugation does on the first row (see § 2). By Lemma 3.1 this action transforms to an action on each factor $\bigoplus_1^{d_i} \bar{k}_i$ of V given by

$$\lambda(x_1, \ldots, x_{d_i}) = (x_1, \lambda^{-2}x_2, \lambda^{-4}x_3, \ldots, \lambda^{-2^{d_i-1}}x_{d_i})$$
$$(\lambda, x_1, x_2 \ldots \in \bar{k}_1 \cong \bar{k}_1^{\#}).$$

Hence we have a Kunneth isomorphism.

$$H^*\left(A_i^*; H^*\left(\bigoplus_1^{n-1} A_i\right)\right) \otimes_{k1} \bar{k}_i \cong H^*\left(\bigoplus_1^{d_i} Z/2\right) \otimes_{k1} H^*\left(k_i^*; H^*\left(\bigoplus_1^{n-1} A_i\right)\right) \otimes_{k1} \bar{k}_1$$

The first factor is $H^*(A_i^*)$, since $|k_i^*|$ is odd, and the second factor is $\mathrm{Hom}_{k_i*}(\bar{k}_1, \Lambda(V) \otimes S(V))$.

We conclude the proof with an argument from [**1**, § 4].

There are no non-trivial k_i^*-invariants in $\Lambda(V) \otimes S(V)$ in dimensions $<d_i$. For the eigenvalues of multiplication by a generator $\lambda \in k_i^*$ in dimension n will be of the form $(\lambda^{-1})^s$ where $s = e_0 + 2e_1 + 4e_2 + \ldots + 2^{d_i-1}e_{d_i-1}$ satisfying $n = l + 2m$ and $\sum_t e_t = l + m$, $e_i \geqq 0$. For an invariant subspace we must have $s \equiv o(2^{d_i} - 1)$. Consider the set of positive integers $e_1', e_2', \ldots, e'_{d_i-1}$ such that $\sum_t e_t' 2^t \equiv 0(2^{d_i} - 1)$ and $\sum e_t'$ is minimal. Then $e_t' = 1$ for all t, since if $e_t' \geqq 2$ replace (e_t', e_{t+1}') by $(e_t' - 2, e_{4+1}' + 1)$, so $\sum_t e_t' 2^t$ is the dyadic expansion of $2^{d_i} - 1$ and $d_i = \sum_t e_t' \leqq \sum_t e_t = l + m \leqq l + 2m = n$.

Hence in each total dimension $<d_i$ $E_2^{*,*}$ is isomorphic to $H^*(A_i^*) \otimes_{k_1} \bar{k}_1$ in that dimension. From the spectral sequence when $r < d_i$,

$$\dim_{k_1} H^r(A_i^*) \geqq \dim_{k_1} H^r(R_n A_i).$$

But the inclusion $A_i^* \to R_n A_i$ is split, so by dimension-counting this inclusion induces an isomorphism in cohomology (and hence in homology).

3.4. Proposition. *In dimensions $j < d_i$ the natural inclusion induces isomorphisms*

$$H^j(D_n A_i) \cong H^j(T_n A_i) \quad \text{and} \quad H_j(D_n A_i) \cong H_j(T_n A_i)$$

for all n, $i \geqq 1$.

Proof. We use induction on n. The case $n = 1$ is obvious. From (2.1) we have a spectral sequence

$$E_2^{p,q} = H^p(T_{n-1}A_i; H^q(R_n A_i)) \Rightarrow H^{p+q}(T_n A_i).$$

In dimensions $p + q < d_i$ $E_2^{p,q}$ is isomorphic, by Proposition 3.2, to

$$H^p(T_{n-1}A_i; H^q(A_i^*)) \cong H^q(A_i^*) \otimes H^p(T_{n-1}A_i).$$

This last isomorphism follows from the conjugation action of $T_{n-1}A_i$ being trivial on A_i^* (see § 2). From the multiplicative properties of the spectral sequence it is easy to see that in total degree $<d_i$,

$$\begin{aligned}\dim_{k_1} H^*(T_nA_i) &= \dim_{k_1}(H^*(A_i^*) \otimes H^*(T_{n-1}A_i)) \\ &= \dim_{k_1}(H^*(A_i^*) \otimes H^*(D_{n-1}A_i)) \\ &= \dim_{k_1}(H^*(D_nA_i).\end{aligned}$$

Since $D_nA_i \to T_nA_i$ is split, the result follows by dimension counting.

The following proof is based on an argument of [**2**, § 11].

3.5. *Proof of Theorem* A. Suppose we have proved the result in dimensions $<m$. $H_*(GL_\infty A_i)$ and $H_*(D_\infty A_i)$ are Hopf algebra with diagonal ψ, induced by juxtaposition of matrices.

Suppose $x \in H_m(T_nZ/4)$ maps to $y \in H_*(GL_\infty Z/4)$ with $y \not\equiv 0$ (mod $H_*(M_\infty Z/2)$). Then, by induction,

$$\psi(y) = y \otimes 1 + 1 \otimes y \quad (\text{mod } H_*(M_\infty Z/2)^{\otimes 2}).$$

Consider the diagram $(m < d_i)$

$$\begin{array}{ccccc} & & H_m(T_nZ/4) & \to & H_m(GL_\infty Z/4) \\ & & \downarrow\alpha' & & \downarrow\alpha \\ H_m(D_nA_i) & \cong & H_m(T_nA_i) & \to & H_m(GL_\infty A_i) \\ \downarrow\beta'' & & \downarrow\beta' & & \downarrow\beta \\ H_m(M_{nd_i}Z/2) & \to & H_m(GL_{nd_i}Z/4) & \to & H_m(GL_\infty Z/4) \end{array}$$

in which α, α' are induced by $(- \otimes_{Z/4} A_i)$ while β, β', β'' are induced by the forgetful map. Then

$$\beta(\alpha(y)) \equiv d_iy \equiv y \quad (\text{mod } H_m(M_\infty Z/2))$$

because y is primitive mod $H_*(M_\infty Z/2)$. However $\beta(\alpha(y))$ is the image of $\beta'(\alpha'(x))$ which lies in the image of $H_*(M_{nd_i}Z/2)$.

3.6. *Proof of Theorem* B. The proof is entirely analogous to that of Theorem A. Throughout we replace R_nA_i by its subgroup of matrices (a_{ij}) with $a_{11} \in k_i^* \subset A_i^*$, $D_nA_i = \bigoplus_1^n A_i^*$ by its subgroups $C_nA_i = \bigoplus_1^n k_i^*$. The proof then shows that im $(H_*(U_nZ/4) \to H_*(GL_\infty Z/4))$ is contained in im $(H_*(C_\infty Z/4) \to H_*(GL_\infty Z/4))$. However $\tilde{H}_*(C_nZ/4) = 0$.

References

1. E. M. Friedlander, *Computations of K-theories of finite fields*, Topology *15* (1976), 87–109.

2. D. G. Quillen, *On the cohomology and K-theory of the general linear group over a finite field*, Annals of Math *96* (1972), 552–586.

The University of Western Ontario,
London, Ontario