EXTENSIONS OF TOPOLOGICAL SPACES

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(received January 11, 1963)

Introduction. The undertaking of constructing spaces which contain a given space as a subspace is by no means new: the extension of the complex number plane to the complex number sphere by the addition of the one point at infinity, the extension of the real line by adjoining the two infinities ∞ and $-\infty$, and the construction of the space of real numbers from that of the rationals by means of Cauchy sequences or Dedekind cuts are 19th Century examples of this very thing. However, only the advent of general topology made it possible to raise the general question of space extensions. It appears that the first study of problems in this area was carried out by Alexandroff and Urysohn in the early twenties [1]. Another mile stone in the history of the subject was the 1929 paper by Tychonoff in which the product theorem for compact spaces is proved and used to identify the completely regular Hausdorff spaces as precisely those spaces which can be imbedded in a compact Hausdorff space [33]. During the same period, work on certain specific extension problems was done by Freudenthal [17] and Zippin [35]. However, the first large body of systematic theory, used for the investigation of a wide range of extension problems, was presented by Stone [31] in 1937. There, one also finds the remark that "one of the interesting and difficult problems of general topology is the study of all extensions of a given space", and it appears that Stone's own work must have convinced many others of the truth of this observation, for since that time there has been a steady succession of papers in this field. But apart from that, the study of extension spaces clearly has a very particular attraction for some mathematicians. The fundamental question

* Invited Address delivered before the Canadian Mathematical Congress at its annual meeting in June 1962.

Canad. Math. Bull. vol. 7, no. 1, January 1964

https://doi.org/10.4153/CMB-1964-001-5 Published online by Cambridge University Press

as to how an object of a specific kind can be imbedded into other such objects possesses a certain philosophical charm, since it sounds like asking "What possibilities in the unknown are determined by the known?". Of course, looking at the existing, widely different extension theories in this manner does not help very much with a single line of proof, but in one's irrational motivations one might nevertheless be influenced by such considerations, even if one otherwise abhors formulations which smack of metaphysics.

This paper is neither intended to present substantial new results nor to provide an encyclopaedic survey, but, rather, to give a unified account of certain aspects of the subject, for the choice of which there is no other excuse or justification but that of personal taste and preference. Regarding the topics omitted here, the most significant ones are probably the relation of uniformities, proximity functions (or relations) and algebras, lattices and vector spaces of continuous functions to extension spaces. However, it is easy to see in which way these items fit into the general framework presented below, since the trace filters (see Section 4) of the extensions resulting from these are essentially known. Another type of space extension not discussed here is that based on a generalization of Wallman's method [10]; for this, the trace filters have not been determined, but no doubt this will not be difficult to do.

1. Basic definitions. A topology on a set E is a collection O of subsets of E closed under arbitrary unions and finite intersections. If O is a topology on E, the pair (E, O) is the space with E as its set of points and O as its topology. However, more often than not, we shall talk about a space E with topology O (or O(E), if this seems preferable).

If E is a space and O its topology, we put $O(x) = \{V \mid x \in V \in O\}$; this is the collection of all open neighbourhoods of x in the topology O. O(x) is a filter in the lattice O, and we shall call it the neighbourhood filter of x. The correspondance $x \to O(x)$, $x \in E$, is in general not one-to-one. However, it will be no significant restriction to consider only this case, i.e., all spaces are assumed to be T_o.

Let E and E' be spaces with the respective topologies O and O', and $\varphi : E \to E'$ a mapping. The pair (E', φ) will

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be called an extension space of E iff $\varphi(E)$ is dense in E', φ is one-to-one, and $O = \{\varphi^{-1}(V) \mid V \in O'\}$. This means that φ is a homeomorphism of E with the dense subspace $\varphi(E)$ of E'. If, in particular, (E', φ) is an extension space of E such that $E \subseteq E'$ and φ maps E identically, then the reference to φ will be omitted and E' will itself be called an extension space of E.

Let (E', φ) be an extension space of the space E. Then, each point $u \in E'$ determines the proper filter $T(u) = \varphi^{-1}O'(u) = \{\varphi^{-1}(V) \mid V \in O'(u)\}$ in the lattice O, called the trace filter of u on E. Of course, if E' itself is an extension of E then $T(u) = \{V \cap E \mid V \in O'(u)\}$. The family $(T(u))_{u \in E'}$ will be called the filter trace of the extension space on E. If $E' \supseteq E$ then the filter trace of E' on E extends the family $(O(x))_{x \in E}$ of neighbourhood filters of E to a family of filters in O with larger indexing set since $T(x) = O(x)$ for $x \in E$.

An extension space (E', φ) of the space E may satisfy certain separation conditions with respect to E: If $u \neq v$ implies $T(u) \neq T(v)$ for $u, v \in E'$ the extension is called relatively T_o; if $T(u) \not\subseteq T(v)$ and $T(v) \not\subseteq T(u)$, under the same conditions, then the extension is called relatively T_1. Of course, relatively T_1 implies relatively T_o, and if E' is Hausdorff then it certainly is relatively T_1; however, E' may well be T_1 without even being relatively T_o.

A number of relations between the extension spaces of a given space E are of interest; these are:

(1) Isomorphism: Two extension spaces (E_1, φ_1) and (E_2, φ_2) of E are called isomorphic iff there exists a homeomorphism $\varphi : E_1 \to E_2$ such that $\varphi \circ \varphi_1 = \varphi_2$; the mapping φ is called an isomorphism from the first to the second extension. If both, φ_1 and φ_2, are identity mappings (i.e. E_1, E_2 extensions of E), the condition on φ says that it maps E identically. Clearly, the relation of isomorphism is an equivalence in the class of all extensions of E.

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(2) Projective order: An extension (E_1, φ_1) of E is called projectively larger than an extension (E_2, φ_2) of E iff there exists a continuous mapping $\varphi : E_1 \to E_2$ (onto) such that $\varphi \circ \varphi_1 = \varphi_2$. This relation is reflexive and transitive, i. e. strictly speaking a partial quasi-order.

(3) Injective order: An extension (E_1, φ_1) of E is called injectively larger than an extension (E_2, φ_2) of E iff there exists a continuous mapping $\varphi : E_2 \to E_1$ which is a homeomorphism from E_2 to the subspace $\varphi(E_2)$ of E_1 such that $\varphi \circ \varphi_2 = \varphi_1$. This relation is also, strictly speaking, a partial quasi-order. The equivalence relations associated with the second and third of these relations, i. e. (E_1, φ_1) projectively (injectively) larger than (E_2, φ_2) and conversely, coincide with the relation of isomorphism on suitable classes of extensions, i. e. for relatively T_1-extensions in the former and relatively T_o-extensions in the latter case.

It follows readily from the definitions that any extension (E', φ) of a space E is isomorphic to one containing E: assuming, without loss of generality, that $E' \cap E = \emptyset$ one takes the set $E'' = (E' - \varphi(E)) \cup E$ and defines the topology O'' on E'' in the obvious manner. Hence, one may always restrict oneself to extension spaces of a space E which contain E; this will be done unless the context makes it preferable not to do so.

2. All extensións of a given space. Consider any family $(F(u))u \in E'$ of filters in O which extends the family of neighbourhood filters of the space E, i. e. $E' \supseteq E$ and $F(x) = O(x)$ for each $x \in E$. Then, there exists two "natural" topologies on the set E' such that the resulting spaces are extensions of E whose filter trace on E is just the given family $(F(u))u \in E'$. The first of these spaces, called the strict extension of E with filter trace $(F(u))u \in E'$, has its topology O'_o generated by the sets $V^* = \{u | V \in F(u)\}$, $V \in O$; in the second space, with

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topology O'_1, here referred to as the simple extension of E with filter trace $(F(u))u \in E'$, each $u \in E'$ has as its basic neighbourhoods the sets $V \cup \{u\}$, $V \in F(u)$.

These remarks show that any extension of the family of neighbourhood filters of a space E is the filter trace of a suitable extension of E. Moreover, any topology O' on E' such that $O'_o \subseteq O' \subseteq O'_1$ (same notation as above) makes E' into an extension space of E with the given family of filters as its filter trace, and conversely any topology O' with this latter property satisfies the inequalities $O'_o \subseteq O' \subseteq O'_1$, since each set V^*, $V \in O$, is readily seen to be the largest $U \in O'$ with $U \cap E = V$, whereas, on the other hand, each $U \in O'$ is equal to $\bigcup (U \cap E) \cup \{u\}$ $(u \in U-E)$ and hence belongs to O'_1. In this manner one has obtained a complete description, up to isomorphism, of all possible extension spaces of a given space.

The preceding considerations lead to the following two particular types of extensions of a space E: an extension $E' \supseteq E$ will be called strict (simple) iff it is the strict (simple) extension of E with respect to its filter trace, and these terms will be used analogously for extension spaces (E', φ) of E in general.

If one considers relatively T_o-extensions only, one can in a sense replace the family of trace filters of an extension $E' \supseteq E$ in the above arguments by the set of these trace filters: the mapping $u \to T(u)$ is then one-to-one, and E' thus becomes isomorphic to an extension space of E whose points are filters in O, E being mapped into it by $x \to O(x)$, and the isomorphism given by $u \to T(u)$. It follows from this that the class of all relatively T_o-extensions of a space E contains a subset such that any extension of E is isomorphic to some member of this set.

As a consequence of this one has the following remark regarding the existence, for a given space E, of projective maxima in certain classes of extensions of E: let K be a class of Hausdorff spaces such that

(i) the product of any family in K belongs to K,

(ii) the closed subspaces of any member of K belong to K, and

(iii) the dense continuous image of any $X \in K$ in any $Y \in K$ is equal to Y.

Then if the class K(E) of all extensions of E which belong to K is non-void, it has a projective maximum. To see this one takes a representative set R for all extensions X of E in K(E) and considers in $P = \Pi X (X \in R)$ the closure E^* of the set $C \subseteq P$ of all constant families with values in E. Since E is clearly homeomorphic to C under the mapping $\varphi : E \to C$ which maps each $a \in E$ to the constant family with value a, (E^*, φ) is an extension space of E with $E^* \in K$ by (i) and (ii). It now follows from (iii) that for each $X \in R$ the restriction π_X to E^* of the natural projection of P onto X maps E^* continuously onto X such that $\pi_X \circ \varphi$ maps E identically. This establishes that (E^*, φ) has the desired property. It may be added that, on general grounds, (E^*, φ) is actually uniquely determined, up to isomorphism, as the projective maximum of K(E).

We remark that the above three hypotheses on the class K hold for the class of all compact Hausdorff spaces and for the class of all zero-dimensional such spaces. Also, variants of the above argument can be used in other cases.

In the following we shall consider relatively T_0-extensions only. As far as strict extensions are concerned, it may be observed that a strict T_0-extension is relatively T_0 anyway, and hence for these this is no new restriction.

3. The strict extensions of a given space. If an extension space $E' \supseteq E$ is either strict or simple then it is entirely determined by its trace filters and hence by data given within E only; this, of course, makes them particularly attractive. Of the two kinds, the strict extensions are by far the more important (and, perhaps, the more interesting); this section is devoted to some general remarks about them.

Let E be a space, O its topology and Φ the set of all proper filters $F \subseteq O$. In Φ, consider the subsets $\Phi_V = \{F \mid V \in F\}$. One readily sees that $\Phi_V \cap \Phi_W = \Phi_{V \cap W}$ for any $V, W \in O$ and $\Phi = \bigcup \Phi_V (V \in O)$; it follows from this that the sets Φ_V, $V \in O$, form the basis of a topology on Φ which will be called its natural topology. Φ together with this topology will be referred to as the space Φ, or the filter space of O. The closure operator Γ in this space is given by the identity

$$\Gamma\Sigma = \{F \mid F \subseteq \bigcup \Sigma\} \qquad\qquad (\Sigma \subseteq \Phi)$$

and we remark in passing that the analogous closure operator on the set of ideals of a commutative ring is of importance in connection with the components of an ideal.

As a first observation concerning the space Φ we have that (Φ, ν) with $\nu : x \rightarrow O(x)$ is a strict extension space of E. That ν is a homeomorphism from E onto the subspace $\nu(E) = \{O(x) \mid x \in E\}$ of Φ follows from the identity $\nu(V) = \Phi_V \cap \nu(E)$ for all $V \in O$, and that $\nu(E)$ is dense in Φ from $O(x) \in \Phi_V$ for $x \in V$. The strictness results from the fact that the trace filter on $\nu(E)$ of any $F \in \Phi$ is precisely $\nu(F)$ which implies (notation as in Section 2) that $\nu(V)^* = \Phi_V$ for each $V \in O$. However, much more significant is the fact that (Φ, ν) is the injective maximum of the class of all strict extensions of E. In order to see this, let $E' \supseteq E$ be any strict extension of E and $\{T(u) \mid u \in E'\}$ its set of trace filters. Now, the topology of E' is generated by the sets $V^* = \{u \mid V \in T(u)\}$, $V \in O$, and that of the subspace $\{T(u) \mid u \in E'\}$ of Φ by the sets $\Phi_V \cap \{T(u) \mid u \in E'\} = \{T(u) \mid V \in T(u)\}$, $V \in O$. This immediately shows that $\varphi : u \rightarrow T(u)$ is a homeomorphism of E' onto the subspace $\{T(u) \mid u \in E'\}$ of Φ; since $\varphi(x) = T(x) = O(x)$ for $x \in E$, this proves the assertion.

Since, conversely, any (Σ, ν) with a subspace Σ of Φ containing the neighbourhood filters of E is a strict extension space of E one can say, worded more suggestively but a little loosely, that the strict extensions of a space E are precisely the subspaces of Φ which contain all neighbourhood filters of the points of E.

In view of all that has been said so far concerning strict extensions, it is of some interest to know conditions which will ensure that a given extension $E' \supseteq E$ of a space E is strict. The most general condition known of this kind, which goes back to Stone [31], is that any semi-regular extension of a space is strict, where semi-regularity means that the regular open sets, i.e. the interiors of the closed sets, generate the topology. On the other hand, one can prove (as remarked by Banaschewski [9]) that a Hausdorff space with First Axiom of Countability which is a strict extension of any of its dense subspaces is semi-regular. We do not know what happens in the absence of the countability condition.

4. Some particular extension operators. The manner in which new types of space extensions usually make their appearance is that some construction procedure is described which yields a specific extension for each space of a certain type. In abstract terms, this amounts to the definition of a mapping ω from a class K of spaces into the class of all spaces such that ωE is an extension space for each $E \in K$, or, more generally, a mapping ω which assigns to each $E \in K$ an extension space $(\omega E, \varphi_\omega)$ in the general sense. A mapping ω of this type will be called an extension operator on the class K.

If an extension operator ω on a class K of spaces is strict, i.e. ωE is a strict extension of E for each $E \in K$, then ω is already determined by the set $\Omega(E)$ of trace filters of ωE on E. The effect of the correspondance $E \to \Omega(E)$ is simply to single out a subset of the set $\Phi(E)$ of all proper filters in the topology O(E) of E which contains all O(x), $x \in E$; conversely, any such correspondance on a class K of space does, of course, give rise to a uniquely determined strict extension operator on K.

In actual practice, this determination, for certain spaces E, of certain sets of filters in O(E) can usually be described in one of the following two ways:

I. Let Ω_o be a mapping defined on the class T of all spaces such that $\Omega_o(E)$ is a set of filters in O(E) for each $E \in T$. Then, $\Omega(E) = N(E) \cup \Omega_o(E)$ with N(E) the sets of all

O(x), $x \in E$, is taken to determine the desired strict extension operator ω on all of T. In this case, one often considers ω only on a suitable subclass K of T for which the ωE have particularly interesting properties.

II. Let Ω be a mapping defined on the class T such that, again, $\Omega(E)$ is a set of filters in O(E) for each $E \in T$. Then, the restriction of Ω to the subclass K of T of all those spaces E for which $\Omega(E)$ contains N(E) gives a strict extension operator on K, K being the largest class of spaces for which this is the case.

In the following, the first three examples come under I, the others under II, and they will be described accordingly.

(LC) In any space E, let W(E) be the filter consisting of the open sets V with compact complement, and put $\Omega_o(E) = \{W(E)\}$ if this filter is proper and $\Omega_o(E) = \emptyset$ otherwise. The resulting extension operator α is that of Alexandroff which extends any non-compact space to a compact one by the addition of one point [2]. Of particular interest is the fact that if E is locally compact Hausdorff then this extension is compact Hausdorff.

(H) For any space E, let $\Omega_o(E)$ be the set of all non-convergent maximal filters $M \subseteq O(E)$, and denote by μ the extension operator on T resulting as in I. Here, it is the class H of all Hausdorff spaces on which μ has interesting properties: for $E \in H$, μE is Hausdorff-complete i.e. has no proper Hausdorff extensions, and is characterized as essentially the only projectively maximal such Hausdorff extensions of E. The operator μ has, to our knowledge, not been considered before; however, it is closely related to Katětov's extension operator κ on H [24]: κE is the simple extension of E with the same trace filters as μE.

(SR) For any space E, a filter $F \subseteq O(E)$ will be called semi-regular iff it has a basis consisting of regular open sets. Let $\Omega_o(E)$ now be the set of all maximal non-convergent semi-regular filters, and denote by σ the resulting extension operator. The subclass of T on which σ has a particularly interesting effect is the class SR of semi-regular Hausdorff spaces. For $E \in SR$, σE is a Hausdorff-minimal extension, and essentially the only projectively maximal such extension of E. The extension operator σ on SR was introduced by Katětov [25] and studied in some detail by Banaschewski [9].

(CR) For any two open sets U and V of a space E, let $U < V$ mean that there exists a continuous mapping $f : E \to [0, 1]$ which is 0 on U and 1 on the complement of V. The relation $<$ on O(E) will be called the relation of completely regular inclusion, and a filter $F \subseteq O(E)$ is called completely regular (Alexandroff, Bourbaki) iff for each $U \in F$ there exists a $V \in F$ with $V < U$. It can then be proved that the space $B(E) \subseteq \Phi(E)$ of all maximal completely regular filters is compact Hausdorff. Moreover, $B(E) \supseteq N(E)$ iff E is a completely regular Hausdorff space. Thus, one has arrived at an extension operator on this class of spaces which extends every such space to a compact Hausdorff space, and it turns out that for each admissable E the resulting extension is βE, the Stone-Čech compactification, characterized as the projective maximum in the class of all compact Hausdorff extensions of E (whose existence is assured on general grounds). The extension operator β was in actual fact introduced by Tychonoff [33], but with an alternative characterization for βE. It was discussed in greater detail for the first time by Stone [31] and Čech [13]. However, the present description in terms of its trace filter was only given in 1939 by Alexandroff [3].

(RC) Let a filter $F \subseteq O(E)$, E any space, be called rim-compact iff it has a basis consisting of sets with compact boundary, and regular iff for each $U \in F$ there exists a $V \in F$ with $\Gamma V \subseteq U$. Then, the space $P(E) \subseteq \Phi(E)$ of all maximal rim-compact regular filters in O(E) is Hausdorff, and $P(E) \supseteq N(E)$ iff E is Hausdorff and O(E) is generated by the open sets with compact boundary, i.e. E is rim-compact. Moreover, in this case P(E) is compact. Thus, one has an extension operator ρ on the class of all rim-compact spaces. The extension ρE of such E is a compact Hausdorff space whose topology is generated by the open sets with boundary in E, and ρE is the projective maximum of this class of extensions. The operator ρ was introduced by Freudenthal [17] for a more restricted class of spaces, and was studied in full generality by Morita [26].

(Z) On any space E, let a filter $F \subseteq O(E)$ be called zero-dimensional iff it is generated by open-closed sets. Then, the space Z(E) of all such filters in O(E) is a zero-

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dimensional compact Hausdorff space, and can alternatively be described as the ultrafilter space of the Boolean lattice of all open closed sets in E. $Z(E) \supseteq N(E)$ holds iff E is zero-dimensional Hausdorff, and on the class Z of these spaces one thus has a compact Hausdorff extension operator ζ. For $E \in Z$, ζE is the projective maximum in the class of all zero-dimensional compact Hausdorff extensions of E. The operator ζ was apparently first studied by Banaschewski [4]. A noteworthy property of ζE is that it turns out to be the component space of βE.

5. Methods of generating extension operators. Some of the above examples have led, by abstraction, to certain methods for the construction of further extension operators. We first give a few instances of these, and then make some general remarks concerning the overall pattern which prevails in this context.

(L) Let K be a fixed class of spaces, and on each space E let W(E) be the filter consisting of all those open $V \subseteq E$ such that its complement $CV \subseteq U_1 \cup \ldots \cup U_n$ with open U_i for which $\Gamma U_i \in K$. Then, if one is fortunate in one's choice of K, the one-point extension of E by means of W(E) will belong to K if each point in E has an open neighbourhood whose closure belongs to K. This is obviously of the type discussed in the previous section, with K as the class of all compact spaces. It works similarly for the class of all countably compact spaces, and for the class of all Hausdorff-complete Hausdorff spaces (Obreanu [27]). With certain modifications, this method can also be applied when K is the class of normal Hausdorff spaces (Čech [13]).

(B) Another method which follows the general pattern I of the previous section, this time an abstraction from the examples (H) and (SR), is the following: given any basis B for the topology of a space E, let a filter $F \subseteq O(E)$ be called a B-filter iff $F \cap B$ generates F. Then, by considering the set $M_o(B)$ of all maximal non-convergent B-filters one obtains a strict extension of E. Of course, rather little can be said about this unless one imposes suitable conditions on B. Extensions of this kind have been considered in detail by Shanin [29].

(R) If one examines Alexandroff's description of the

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Stone-Čech compactification one can see that the relevant properties of the space of maximal completely regular filters are the outcome of certain conditions which the relation of completely regular inclusion satisfies. Put in terms of an unspecified binary relation ρ on the topology O of a space E these conditions are, for U, V and W in O:

R1. If $U\rho V$ then $\Gamma U \subseteq V$

R2. If $U\rho V$ and $U' \subseteq U$, $V \subseteq V'$ then $U'\rho V'$

R3. If $U\rho V$ and $U\rho W$ then $U\rho(V \cap W)$

R4. If $U\rho V$ and $W\rho V$ then $(U \cup W)\rho V$

R5. If $U\rho V$ then $C\Gamma V \rho C\Gamma V$

R6. If $U\rho V$ then $U\rho W$, $W\rho V$ for suitable W.

A relation ρ of this kind will be called <u>a relation of completely regular inclusion.</u>

A filter $F \subseteq O$ is called a ρ-filter iff for each $U \in F$ there exists a $V \in F$ such that $V\rho U$. Then the space $\Omega(\rho)$ of all maximal ρ-filters is a compact Hausdorff space, and $\Omega(\rho) \supseteq N(E)$ holds iff

R7. For each $U \in O$, $U = \bigcup V(V\rho U)$.

In this case, we shall call ρ <u>basic.</u> Thus, every basic relation of completely regular inclusion on $O(E)$ determines a compact Hausdorff extension of E. Clearly, then there exist such relations on $O(E)$ iff E is completely regular Hausdorff. As an added feature, one can prove that every compact Hausdorff extension of such a space E is obtainable in this fashion: if K is such an extension of E then the relation $\rho = \{(U,V) \mid \Gamma_K U \subseteq V^*\}$ where Γ_K denotes closure in K and $*$ has the usual meaning, is a basic relation of completely regular inclusion on $O(E)$ such that $\Omega(\rho)$ is isomorphic to K as an extension of E.

These relations ρ were first considered by Freudenthal [18] and have recently been studied in great detail by Taylor [32].

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(N) Let E be any space. A collection $N \subseteq O(E)$ is called a normal system in O iff

N1. N is $\cap$-closed

N2. For each $U \in N$, $U = I\Gamma U$ and $ICU \in N$

N3. If $\Gamma U \subseteq V$ for $U, V \in N$ then there exists a $W \in N$ with $\Gamma U \subseteq W$, $\Gamma W \subseteq V$

N4. For each $U \in N$, $U = \bigcup V(\Gamma V \subseteq U, V \in N)$.

The filters in O which are of interest in relation to a normal system N in O are the regular N-filters, i.e. the filters $F \subseteq O$ for which $F \cap N$ is a basis and $U \in F$ implies the existence of a $V \in F$ with $\Gamma V \subseteq U$. The space $M(N)$ of all maximal regular N-filters is then compact Hausdorff and $M(N) \supseteq N(E)$ iff N generates the topology $O(E)$. Hence, any generating normal system in $O(E)$ determines a compact Hausdorff extension of E.

The extensions by means of normal systems generalize an earlier type introduced by Fan and Gottesman [16] who assumed a stronger condition than N3, and who obtained their method by generalizing earlier work of Freudenthal [19]. Normal systems were introduced and studied extensively by Banaschewski [8].

Looking at the last three of these examples, and bearing in mind several others which we have not mentioned, one discerns the following pattern: What is, in actual fact, being considered are not extension operators on classes of mere spaces, but of composite objects which consist of (i) a topological space and (ii) some further structure given to the space by the addition of a new entity, e.g. a particular basis or a relation on the topology. Other examples of this kind are metric spaces, uniform spaces and proximity spaces where the additional structure is a metric, a uniformity or a proximity function respectively. In all these cases, the role of the additional entity on the space can be considered as that of singling out, usually by some extremality condition, a suitable set of filters in the lattice of open sets. This is obvious for all the examples mentioned except, perhaps, for the case of the completion of a uniform space; however, if one considers that type of extension closely one readily sees that its trace filters are precisely those filters in the topology which are minimal Cauchy filters with respect to the given uniformity.

6. Problems concerning extension operators. Many of the topics usually studied in connection with particular extension operators can be placed under one of the following headings:

(1) Characterization

(2) Functorial properties

(3) Comparison

(4) Internal properties

(5) Recovery

(6) Effect on subspaces

(7) Behaviour with respect to topological operations.

To illustrate this classification of problems we shall give a number of examples.

(1) For most of the extensions discussed in Section 4 we have already mentioned a characterization in terms of properties different from the defining conditions which always referred to the trace filters. In some cases, further such characterizations can be given. For instance: for any locally compact Hausdorff space E, αE is the projectively smallest compact Hausdorff extension of E (Samuel [28]). βE is that compact Hausdorff extension of the completely regular space E in which any two closed subsets $A, B \subseteq E$ with $A < CB$ have disjoint closures (Smirnov [30]). ζE is that zero-dimensional compact Hausdorff extension of the zero-dimensional Hausdorff space E in which any two disjoint open-closed subsets of E have disjoint closures. A similar characterization in terms of disjoint closures can be given for the extension operator ρ.

(2) Some extension operators turn out to be the "object

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part" of a covariant functor on a suitable category, and naturally it is of interest to know when this occurs. If it does, the usual effect of the functor on a mapping in the category is to extend it continuously to the extension spaces concerned. This is the case in all the following examples, and it is therefore sufficient to name the category for the extension operator concerned:

α: locally compact Hausdorff spaces and proper continuous mappings.

β: completely regular Hausdorff spaces and continuous mappings.

ρ: rim-compact spaces and proper continuous mappings.

ζ: zero-dimensional Hausdorff spaces and continuous mappings.

In each of these cases, the proof is based on the fundamental theorem concerning the extension of a continuous mapping from a dense subspace of a space into a regular space to a continuous mapping of the whole space (Bourbaki [12], Ch. I, § 6, Thm. 1).

(3) Given two extension operators ω_1 and ω_2 on the classes K_1 and K_2 respectively, it is natural to ask where on $K_1 \cap K_2$ ω_1 and ω_2 coincide, especially if $K_1 \supseteq K_2$. This problem has been considered in a number of cases. Thus $\beta E = \zeta E$ holds exactly for those zero-dimensional Hausdorff spaces E for which each set $\{x | f(x) < 0\}$, f any continuous real function on E, is the countable union of open-closed sets (Heider [22]), and there are examples for which $\beta E \neq \zeta E$ (Dowker [15]). On the other hand, a necessary and sufficient condition that $\alpha E = \beta E$ for a locally compact Hausdorff space E is that of any two closed $A, B \subseteq E$ with $A < CB$ at least one be compact (Doss [14]). A similar result for locally compact Hausdorff E, is that $\alpha E = \rho E$ if each compact subset of E is contained in some compact subset of E whose complement is connected (Banaschewski [11]). An extreme situation prevails for μ and β: there are no non-compact completely regular Hausdorff spaces with $\mu E = \beta E$ (Banaschewski [6]).

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(4) As a particular example of the study of internal properties of the spaces which occur under a specific extension operator we mention the following concerning the operator β on the class of completely regular Hausdorff spaces: βE is locally connected iff E is locally connected and every continuous real function on E is bounded (i.e. E is pseudo-compact). The necessity of this condition was obtained, in a slightly different form, by Banaschewski [5]; the sufficiency is due to Henriksen and Isbell [23], who gave several further results concerning the points of βE which have arbitrarily small connected neighbourhoods. As another result in the same context one has that for connected, locally connected and pseudo-compact E, βE is simply connected (in the covering space sense) iff E is. This example illustrates a general type of problem which comes under the present heading, namely the question as to what properties of a space E are inherited by ωE for a specific applicable extension operator ω (Wallace [34]).

(5) The problem referred to is the following: given an extension operator ω on a class of spaces K, find conditions for $E, E' \in K$ which will ensure that any homeomorphism $f: \omega E \to \omega E'$ induce a homeomorphism $E \to E'$. More specifically, a condition of this kind might be derived from a characterization, for certain E, of E as a subspace of ωE which amounts to recovering the original space from its larger extension.

The first condition of this type appears to have been given by Čech [13] when he proved that for a completely regular Hausdorff space E satisfying the First Axiom of Countability, E consists of exactly those points of βE whose neighbourhood filters have a countable basis. Less restricted conditions for the characterization of E within βE (and Hewitt's extension υE) were given by Heider [21]. A general approach to this topic was developed by Banaschewski [7], the main result being as follows: let K be a class of spaces such that $E \in K$ implies $E - \{x\} \in K$ for each $x \in E$, and $\omega: K \to K$ an extension operator which satisfies the two conditions

(i) if $f: E \to E'$ and $g: E' \to E$ are dense imbeddings such that the composite $g \circ f$ maps E identically then there exists continuous extensions $f^{\omega}: \omega E \to \omega E'$ and $g^{\omega}: \omega E' \to \omega E$

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of f and g respectively; and

(ii) for each $a \in E$, there exists a dense imbedding $h_a : \omega E - \{a\} \to \omega(E - \{a\})$ which maps $E - \{a\}$ identically.

Now, for each $E \in K$, let $\omega_o E$ denote the subspace of E consisting of all those points $a \in B$ for which the identity mapping $E - \{a\} \to E$ cannot be extended to a homeomorphism $\omega(E - \{a\}) \to \omega E$. These subspaces $\omega_o E$ play a decisive rôle with respect to the question of recovery on account of the following two results: first, if $f: E \to E'$ is any homeomorphism $(E, E' \in K)$ then $f(\omega_o E) = \omega_o E'$, and, secondly, $\omega_o(\omega E) = \omega_o E$ for each $E \in K$. This means that $\omega_o E$ is a topological invariant of $E \in K$, and that any $E \in K$ with $\omega_o E = E$ is characterized within ωE by $E = \omega_o(\omega E)$; in particular, any homeomorphism $\omega E \to \omega E'$ for $E, E' \in K$ with $\omega_o E = E$ and $\omega_o E' = E'$ maps E onto E'.

The above conditions on K and ω are satisfied by a considerable number of extension operators and their domains, and a characterization of the condition $\omega_o E = E$ in terms of internal properties of E is not very difficult to obtain in these cases [7].

(6) If an extension operator ω is defined on a class K of spaces, and a subspace A of a space $E \in K$ also belongs to K, one can ask what the relation is between ωA and ωE. An example of the kind of answer one might find is the following generalization of a result of Čech [13] concerning β and completely regular spaces: for any subspace $A \subseteq E$, βA is isomorphic to the closure of A in βE iff every bounded continuous real function has a bounded continuous extension to all of E.

(7) If a class K of spaces is closed under, say, summation or multiplication of suitable families $(E_\alpha)_{\alpha \in I}$ in K, and ω is an extension operator defined on K, the comparison of $\omega(\Sigma E_\alpha)$ with $\Sigma \omega E_\alpha$ or $\omega(\Pi E_\alpha)$ with $\Pi \omega E_\alpha$ poses a natural problem. A fairly obvious result along these lines is that

$\omega(\Sigma E_\alpha)$ and $\Sigma\omega E_\alpha$ are isomorphic extensions of the sums ΣE_α with finitely many summands for a considerable number of extension operators; in fact, this holds for all cases mentioned in Section 4 except for α. For products and the operator β, one has (Glicksberg [20]): $\beta(\Pi X_\alpha)$ is isomorphic to $\Pi\beta X_\alpha$ as extension of ΠX_α iff either $\Pi_{\alpha\neq\alpha_o} X_\alpha$ is finite for some α_o or ΠX_α is pseudocompact.

7. Problems concerning methods of generating extension operators. Apart from the analogues, as far as they make sense, of the problems which arise for individual extension operators one encounters new types of questions in this context which are specific to it. A few typical examples of these are:

(1) Characterization of scope.

(2) Characterization of range.

(3) Comparison.

The following remarks will illustrate the topics covered by each of these headings.

(1) More explicitly stated, the question is: For what classes of spaces is the method under consideration applicable? The aim, therefore, is to determine what types of spaces possess the additional entity of structure on which the method is based. In the case of the relations of completely regular inclusion, for instance, the scope is simply the class of all completely regular spaces, as was remarked earlier. Another result of the same nature concerns the construction of extensions by means of countable normal bases: the spaces which possess such bases are precisely the rim-compact Hausdorff spaces satisfying the Second Axiom of Countability (Banaschewski [8]).

(2) Whether or not the scope of a method of generating extension operators is known, there is also the other question as to what type of extensions the operator thus generated gives rise to. For instance, in the case of the relations of completely regular inclusion, the resulting extensions are precisely all

compact Hausdorff extensions for each completely regular Hausdorff space. Similarly, a description of all extensions obtainable from countable normal bases can be given: for each admissable space E, they are all metric compact extensions $E^* \supseteq E$ such that $\dim (E^* - E) = 0$.

(3) The comparison problem occurring in the present context is somewhat different from that discussed in the previous section. It can best be illustrated by an example: Every compact Hausdorff extension of a completely regular Hausdorff space E is obtainable by means of a relation of completely regular inclusion on the topology O of E, and some such extensions of E are derived from normal systems generating O. Hence, each of these normal systems must determine a relation of completely regular inclusion on O, indirectly via the associated extension of E, and it should therefore be possible to give a direct description of the correspondance normal system → relation. Put in the most general terms, the question is: Which additional entity of structure (of the type that generate extensions) of one kind on a space E corresponds to which of another kind, whenever that question is defined? The reason why this question could be of interest in particular cases (e.g. the one just mentioned) is that a satisfactory answer might lead to a better understanding as to which extensions of a given space are obtainable by a specific method, i.e. result in a characterization of the range of that method. A discussion which falls under the present heading, but concerned with entities not dealt with here, occurs in [10].

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