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Hyperbolic sets for twist maps

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Abstract. An example is given of an area-preserving monotone twist map such that a uniformly hyperbolic structure exists on the closure of its Birkhoff maximizing orbits.

This note provides a rigorous example of an area preserving monotone twist map f with the property that $Df|_{\bar{B}}$ has a uniformly hyperbolic structure, where $\bar{B}$ denotes the closure of the Birkhoff maximizing orbits. As shown by Mather [**8**] and by Aubry, La Daeron, and André [**3**], the set $\bar{B}$ associated with f contains invariant Cantor sets of all possible rotation numbers.

A result which would imply the hyperbolicity of these invariant Cantor sets was announced by Aubry in [**1**]. The heuristic justification given there is discussed further in [**2**]. Nevertheless, Katok raises the hyperbolicity question again in [**4**] and [**5**]. The construction below gives a rigorous answer based on an estimate first due to Aubry. Another proof that hyperbolic Cantor sets λ can exist in $\bar{B}$ was obtained independently by Michel Herman.

Consider the 'standard' one parameter family of area preserving monotone twist maps of the cylinder $\mathbb{T}^1 \times \mathbb{R}$. One lift to $\mathbb{R}^2$ of the map in this family corresponding to the parameter k has the form

$$f(x, y) = \left(x + y - \frac{k}{2\pi}\sin 2\pi x,\; y - \frac{k}{2\pi}\sin 2\pi x\right).$$

The function

$$h(x, x') = -\tfrac{1}{2}(x - x')^2 - \frac{k}{4\pi^2}\cos 2\pi x$$

generates f in the sense that $f(x, y) = (x', y')$ if and only if

$$y = \frac{\partial h}{\partial x}(x, x') \quad \text{and} \quad y' = -\frac{\partial h}{\partial x'}(x, x').$$

Thus, given a sequence $\{x_n\}$, there exists a sequence $\{y_n\}$ such that $(x_n, y_n) = f^n(x_0, y_0)$ if and only if $\{x_n\}$ satisfies

$$\frac{\partial h}{\partial x'}(x_{n-1}, x_n) + \frac{\partial h}{\partial x}(x_n, x_{n+1}) = 0, \tag{1}$$

in which case

$$y_n = \frac{\partial h}{\partial x}(x_n\, x_{n+1}). \tag{2}$$

Following Birkhoff, this generating function for f can be used to prove the existence of certain periodic orbits. Given integers p and $q > 0$, define the action $w: \mathbb{R}^q \to \mathbb{R}$ by

$$w(x_1, \ldots, x_q) = \sum_{n=1}^{q} h(x_{n-1}, x_n),$$

where $x_0 + p = x_q$. It can be shown that the maximum of w is achieved at a critical point, which therefore satisfies (1). If we associate second coordinates to this sequence using (2), the resulting union of q points in $\mathbb{R}^2$ is called a Birkhoff maximizing orbit of type (p, q) and its projection to $\mathbb{T}^1 \times \mathbb{R}$ is a periodic trajectory. Let B denote the union of all the orbits obtained in this way as p and q vary, and let $\bar{B}$ denote the closure of B in $\mathbb{R}^2$.

For details concerning the construction and properties of B and $\bar{B}$, see **[3]**, **[5]**, **[6]** and **[8]**. Although the last three references treat boundary preserving twist maps of an annulus, straightforward modifications allow the relevant results to apply to the standard family on the cylinder. See **[7]** in this regard.

LEMMA (Aubry, **[2]**). *Let f be a standard map with parameter $k > 2\sqrt{1+\pi^2}$. If (x_0, y_0) belongs to the set B associated with f then $-k \cos 2\pi x_0 \geq 2$.*

Proof. Suppose (x_0, y_0) belongs to a Birkhoff maximizing orbit of type (p, q). By choosing an appropriate lift f, we can assume without loss of generality that $0 < p/q \leq 1$. Let $(x_n, y_n) = f^n(x_0, y_0)$. Then $0 < x_{n+1} - x_n \leq 1$ and, from (1), we have

$$(x_{n+1} - x_n) - (x_n - x_{n-1}) + \frac{k}{2\pi} \sin 2\pi x_n = 0.$$

This condition, which is independent of the lift chosen, implies $|\sin 2\pi x_n| \leq 2\pi/k$, and hence, for $k \geq 2\pi$,

$$|\cos 2\pi x_n| \geq \sqrt{1 - \frac{4\pi^2}{k^2}}. \tag{3}$$

Now from the definition of B as sequences which maximize the function w, we also have

$$\frac{\partial^2 w}{\partial x_n^2}(x_1, \ldots, x_q) = -2 + k \cos 2\pi x_n \leq 0. \tag{4}$$

For $k > 2\sqrt{1+\pi^2}$, (3) and (4) together ensure that

$$-k \cos 2\pi x_n \geq k\sqrt{1 - \frac{4\pi^2}{k^2}} > 2,$$

which completes the proof. □

THEOREM. *Let f be a standard map with parameter $k > 2\sqrt{1+\pi^2}$. Then the set $\bar{B}$ associated with f has a uniform hyperbolic structure.*

Proof. Using the characterization of hyperbolicity given by Newhouse and Palis [**10**], it suffices to find a cone C in $\mathbb{R}^2$ and a positive integer m such that for each (x, y) in $\bar{B}$, the derivative of f at (x, y), $D_{(x,y)}f$, maps C into itself and such that $D_{(x,y)}f^m$ expands C and $D_{(x,y)}f^{-m}$ expands $\mathbb{R}^2 \setminus C$ (see also [**9**]).

From the lemma, we have that for (x, y) in $\bar{B}$,

$$D_{(x,y)}f = \begin{bmatrix} 1 - k\cos 2\pi x & 1 \\ -k\cos 2\pi x & 1 \end{bmatrix} = \begin{bmatrix} 1+\varepsilon_0 & 1 \\ \varepsilon_0 & 1 \end{bmatrix}$$

for some $\varepsilon > 2$. Let $C = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 v_2 \geq 0\}$. Then $D_{(x,y)}f$ not only maps each v in C back into C, but also expands its Euclidean norm in the sense that

$$\|D_{(x,y)}f[v]\| \geq \sqrt{2}\|v\|$$

for v in C. Similarly, we find

$$D_{(x,y)}f = \begin{bmatrix} 1 & -1 \\ -\varepsilon & 1+\varepsilon \end{bmatrix},$$

for some $\varepsilon > 2$, and so

$$\|D_{(x,y)}f[v]\| \geq \sqrt{5}\|v\|,$$

whenever v belongs to $\mathbb{R}^2 \setminus C$. This completes the proof. □

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