5

Trees and the Composition of Generating Functions

5.1 The Exponential Formula

If $F(x)$ and $G(x)$ are formal power series with $G(0) = 0$, then we have seen (after Proposition 1.1.9) that the composition $F(G(x))$ is a well-defined formal power series. In this chapter we will investigate the combinatorial ramifications of power series composition. In this section we will be concerned with the case where $F(x)$ and $G(x)$ are exponential generating functions, and especially the case $F(x) = e^x$.

Let us first consider the combinatorial significance of the product $F(x)G(x)$ of two exponential generating functions

$$F(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!},$$

$$G(x) = \sum_{n \geq 0} g(n) \frac{x^n}{n!}.$$

Throughout this chapter K denotes a field of characteristic 0 (such as $\mathbb{C}$ with some indeterminates adjoined). We also denote by $E_f(x)$ the exponential generating function of the function $f : \mathbb{N} \to K$, that is,

$$E_f(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!}.$$

5.1.1 Proposition. *Given functions $f, g : \mathbb{N} \to K$, define a new function $h : \mathbb{N} \to K$ by the rule*

$$h(\#X) = \sum_{(S,T)} f(\#S) g(\#T), \tag{5.1}$$

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where X is a finite set, and where (S,T) ranges over all weak ordered partitions of X into two blocks, that is, $S \cap T = \emptyset$ and $S \cup T = X$. Then

$$E_h(x) = E_f(x)E_g(x). \tag{5.2}$$

Proof. Let $\#X = n$. There are $\binom{n}{k}$ pairs (S,T) with $\#S = k$ and $\#T = n-k$, so

$$h(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)g(n-k).$$

From this equation (5.2) follows. □

One could also prove Proposition 5.1.1 by using Theorem 3.18.4[1] applied to the binomial poset $\mathbb{B}$ of Example 3.18.3.

We have stated Proposition 5.1.1 in terms of a certain relationship (5.1) among functions f, g and h, but it is important to understand its combinatorial significance. Suppose we have two types of structures, say α and β, which can be put on a finite set X. We assume that the allowed structures depend only on the cardinality of X. A new "combined" type of structure, denoted $\alpha \cup \beta$, can be put on X by placing structures of type α and β on subsets S and T, respectively, of X such that $S \cup T = X, S \cap T = \emptyset$. If $f(k)$ (respectively $g(k)$) are the number of possible structures on a k-set of type α (respectively, β), then the right-hand side of (5.1) counts the number of structures of type $\alpha \cup \beta$ on X. More generally, we can assign a weight $w(\Gamma)$ to any structure Γ of type α or β. A combined structure of type $\alpha \cup \beta$ is defined to have weight equal to the product of the weights of each part. If $f(k)$ and $g(k)$ denote the sum of the weights of all structures on a k-set of types α and β, respectively, then the right-hand side of (5.1) counts the sum of the weights of all structures of type $\alpha \cup \beta$ on X.

5.1.2 Example. Given an n-element set X, let $h(n)$ be the number of ways to split X into two subsets S and T with $S \cup T = X, S \cap T = \emptyset$; and then to linearly order the elements of S and to choose a subset of T. There are $f(k) = k!$ ways to linearly order a k-element set, and $g(k) = 2^k$ ways to choose a subset of a k-element set. Hence

$$\begin{aligned} \sum_{n \geq 0} h(n)\frac{x^n}{n!} &= \left(\sum_{n\geq 0} n!\frac{x^n}{n!}\right)\left(\sum_{n\geq 0} 2^n\frac{x^n}{n!}\right) \\ &= \frac{e^{2x}}{1-x}. \end{aligned}$$

[1] All references to Volume 1 are to the *second edition*.

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Proposition 5.1.1 can be iterated to yield the following result.

5.1.3 Proposition. *Fix $k \in \mathbb{P}$ and functions $f_1, f_2, \dots, f_k : \mathbb{N} \to K$. Define a new function $h : \mathbb{N} \to K$ by*

$$h(\#S) = \sum f_1(\#T_1) f_2(\#T_2) \cdots f_k(\#T_k),$$

where $(T_1, \dots, T_k)$ ranges over all weak ordered partitions of S into k blocks, that is, $T_1, \dots, T_k$ are subsets of S satisfying: (i) $T_i \cap T_j = \emptyset$ if $i \neq j$, and (ii) $T_1 \cup \cdots \cup T_k = S$. Then

$$E_h(x) = \prod_{i=1}^{k} E_{f_i}(x).$$

We are now able to give the main result of this section, which explains the combinatorial significance of the composition of exponential generating functions.

5.1.4 Theorem (the Compositional Formula). *Given functions $f : \mathbb{P} \to K$ and $g : \mathbb{N} \to K$ with $g(0) = 1$, define a new function $h : \mathbb{N} \to K$ by*

$$\begin{aligned} h(\#S) &= \sum_{\pi = \{B_1, \dots, B_k\} \in \Pi(S)} f(\#B_1) f(\#B_2) \cdots f(\#B_k) g(k), \quad \#S > 0, \\ h(0) &= 1, \end{aligned}$$

where the sum ranges over all partitions (as defined in Section 1.9) $\pi = \{B_1, \dots, B_k\}$ of the finite set S. Then

$$E_h(x) = E_g(E_f(x)).$$

(Here $E_f(x) = \sum_{n \geq 1} f(n) \frac{x^n}{n!}$, since f is only defined on positive integers.)

Proof. Suppose $\#S = n$, and let $h_k(n)$ denote the right-hand side of (5.3) for fixed k. Since $B_1, \dots, B_k$ are nonempty they are all distinct, so there are $k!$ ways of linearly ordering them. Thus by Proposition 5.1.3,

$$E_{h_k}(x) = \frac{g(k)}{k!} E_f(x)^k. \tag{5.3}$$

Summing (5.3) over all $k \geq 1$ yields the desired result. □

Theorem 5.1.4 has the following combinatorial significance. Many structures on a set, such as graphs or posets, may be regarded as disjoint unions of their connected components. In addition, some additional structure may be placed on the components themselves, for example, the components could be linearly ordered. If there are $f(j)$ connected structures on a j-set and $g(k)$ ways

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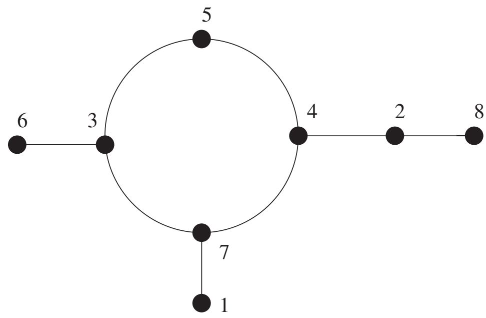

Figure 5.1 A circular arrangement of lines

to place an additional structure on k components, then $h(n)$ is the total number of structures on an n-set. There is an obvious generalization to weighted structures, such as was discussed after Proposition 5.1.1.

The following example should help to elucidate the combinatorial meaning of Theorem 5.1.4; more substantial applications are given in Section 5.2.

5.1.5 Example. Let $h(n)$ be the number of ways for n persons to form into nonempty lines, and then to arrange these lines in a circular order. Figure 5.1 shows one such arrangement of nine persons. There are $f(j) = j!$ ways to linearly order j persons, and $g(k) = (k-1)!$ ways to circularly order $k \geq 1$ lines. Thus

$$E_f(x) = \sum_{n\geq 1} n!\frac{x^n}{n!} = \frac{x}{1-x},$$

$$E_g(x) = 1 + \sum_{n\geq 1}(n-1)!\frac{x^n}{n!} = 1 + \log(1-x)^{-1},$$

so

$$\begin{aligned} E_h(x) &= E_g(E_f(x)) \\ &= 1 + \log\left(1 - \frac{x}{1-x}\right)^{-1} \\ &= 1 + \log(1-2x)^{-1} - \log(1-x)^{-1} \\ &= 1 + \sum_{n\geq 1}(2^n - 1)(n-1)!\frac{x^n}{n!}, \end{aligned}$$

whence $h(n) = (2^n - 1)(n-1)!$. Naturally such a simple answer demands a simple combinatorial proof. Namely, arrange the n persons in a circle in $(n-1)!$

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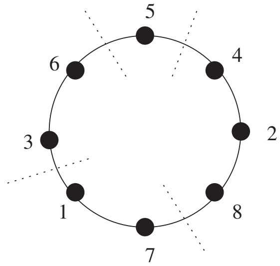

Figure 5.2 An equivalent form of Figure 5.1

ways. In each of the n spaces between two persons, either do or do not draw a bar, except that at least one bar must be drawn. There are thus $2^n - 1$ choices for the bars. Between two consecutive bars (or a bar and itself if there is only one bar) read the persons in clockwise order to obtain their order in line. See Figure 5.2 for this method of representing Figure 5.1.

The most common use of Theorem 5.1.4 is the case where $g(k)=1$ for all k. In combinatorial terms, a structure is put together from "connected" components, but no additional structure is placed on the components themselves.

5.1.6 Corollary (the Exponential Formula). *Given a function $f : \mathbb{P} \to K$, define a new function $h : \mathbb{N} \to K$ by*

$$\begin{aligned} h(\#S) &= \sum_{\pi=\{B_1,\ldots,B_k\}\in\Pi(S)} f(\#B_1)f(\#B_2)\cdots f(\#B_k), \qquad \#S > 0, \\ h(0) &= 1. \end{aligned} \tag{5.4}$$

Then

$$E_h(x) = \exp E_f(x). \tag{5.5}$$

Let us say a brief word about the computational aspects of equation (5.5). If the function $f(n)$ is given, then one can use (5.4) to compute $h(n)$. However, there is a much more efficient way to compute $h(n)$ from $f(n)$ (and conversely).

5.1.7 Proposition. *Let $f : \mathbb{P} \to K$ and $h : \mathbb{N} \to K$ be related by $E_h(x) =$ $\exp E_f(x)$ (so in particular $h(0) = 1$). Then we have for $n \geq 0$ the recurrences*

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$$h(n+1) = \sum_{k=0}^{n} \binom{n}{k} h(k) f(n+1-k), \tag{5.6}$$

$$f(n+1) = h(n+1) - \sum_{k=1}^{n} \binom{n}{k} h(k) f(n+1-k). \tag{5.7}$$

Proof. Differentiate $E_h(x) = \exp E_f(x)$ to obtain

$$E_h'(x) = E_f'(x) E_h(x). \tag{5.8}$$

Now equate coefficients of $x^n/n!$ on both sides of (5.8) to obtain (5.6). (It is also easy to give a combinatorial proof of (5.6).) Equation (5.7) is just a rearrangement of (5.6). □

The compositional and exponential formulas are concerned with structures on a set S obtained by choosing a partition of S and then imposing some "connected" structure on each block. In some situations it is more natural to choose a *permutation* of S and then impose a "connected" structure on each cycle. These two situations are clearly equivalent, since a permutation is nothing more than a partition with a cyclic ordering of each block. However, permutations arise often enough to warrant a separate statement. Recall that $\mathfrak{S}(S)$ denotes the set (or group) of all permutations of the set S.

5.1.8 Corollary (the Compositional Formula, permutation version). *Given functions $f : \mathbb{P} \to K$ and $g : \mathbb{N} \to K$ with $g(0) = 1$, define a new function $h : \mathbb{P} \to K$ by*

$$\begin{aligned} h(\#S) &= \sum_{\pi \in \mathfrak{S}(S)} f(\#C_1) f(\#C_2) \cdots f(\#C_k) g(k), \quad \#S > 0, \\ h(0) &= 1, \end{aligned} \tag{5.9}$$

where $C_1, C_2, \ldots, C_k$ are the cycles in the disjoint cycle decomposition of π. Then

$$E_h(x) = E_g\left(\sum_{n \geq 1} f(n) \frac{x^n}{n} \right).$$

Proof. Since there are $(j-1)!$ ways to cyclically order a j-set, equation (5.9) may be written

$$h(\#S) = \sum_{\pi = \{B_1, \ldots, B_k\} \in \Pi(S)} [(\#B_1 - 1)! f(\#B_1)] \cdots [(\#B_k - 1)! f(\#B_k)] g(k),$$

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so by Theorem 5.1.4,

$$\begin{aligned} E_h(x) &= E_g\left(\sum_{n\geq 1}(n-1)!f(n)\frac{x^n}{n!}\right) \\ &= E_g\left(\sum_{n\geq 1}f(n)\frac{x^n}{n}\right) \end{aligned}$$

□

5.1.9 Corollary (the Exponential Formula, permutation version). *Given a function $f:\mathbb{P}\to K$, define a new function $h:\mathbb{N}\to K$ by*

$$\begin{aligned} h(\#S) &= \sum_{\pi\in\mathfrak{S}(S)} f(\#C_1)f(\#C_2)\cdots f(\#C_k), \quad \#S>0, \\ h(0) &= 1, \end{aligned}$$

where the notation is the same as in Corollary 5.1.8. Then

$$E_h(x) = \exp\sum_{n\geq 1}f(n)\frac{x^n}{n}.$$

In Chapter 3.18 (see Example 3.18.3(b)) we related addition and multiplication of exponential generating functions to the incidence algebra of the lattice of finite subsets of $\mathbb{N}$. There is a similar relation between *composition* of exponential generating functions and the incidence algebra of the lattice Π_n of partitions of $[n]$ (or any n-set). More precisely, we need to consider simultaneously all Π_n for $n\in\mathbb{P}$. Recall from Section 3.10 that if $\sigma\leq\pi$ in Π_n, then we have a natural decomposition

$$[\sigma,\pi]\cong\Pi_1^{a_1}\times\Pi_2^{a_2}\times\cdots\times\Pi_n^{a_n}, \tag{5.10}$$

where $|\sigma|=\sum ia_i$ and $|\pi|=\sum a_i$. Let $\mathbf{\Pi}=(\Pi_1,\Pi_2,\dots)$. For each $n\in\mathbb{P}$, let $f_n\in I(\Pi_n,K)$, the incidence algebra of Π_n. Suppose that the sequence $f=(f_1,f_2,\dots)$ satisfies the following property: There is a function (also denoted f) $f:\mathbb{P}\to K$ such that if $\sigma\leq\pi$ in Π_n and $[\sigma,\pi]$ satisfies (5.10), then

$$f_n(\sigma,\pi)=f(1)^{a_1}f(2)^{a_2}\cdots f(n)^{a_n}. \tag{5.11}$$

We then call f a *multiplicative function* on $\mathbf{\Pi}$.

For instance, if ζ_n is the zeta function of Π_n, then $\zeta=(\zeta_1,\zeta_2,\dots)$ is multiplicative with $\zeta(n)=1$ for all $n\in\mathbb{P}$. If μ_n is the Möbius function of Π_n, then by Proposition 3.8.2 and equation (3.37) we see that $\mu=(\mu_1,\mu_2,\dots)$ is multiplicative with $\mu(n)=(-1)^{n-1}(n-1)!$.

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Let $f = (f_1, f_2, \ldots)$ and $g = (g_1, g_2, \ldots)$, where $f_n, g_n \in I(\Pi_n, K)$. We can define the *convolution* $fg = ((fg)_1, (fg)_2, \ldots)$ by

$$(fg)_n = f_n g_n \quad \text{(convolution in } I(\Pi_n, K)). \tag{5.12}$$

5.1.10 Lemma. *If f and g are multiplicative on $\mathbf{\Pi}$, then so is fg.*

Proof. Let P and Q be locally finite posets, and let $u \in I(P,K), v \in I(Q,K)$. Define $u \times v \in I(P \times Q, K)$ by

$$u \times v((x,x'),(y,y')) = u(x,y)v(x',y').$$

Then a straightforward argument as in the proof of Proposition 3.8.2 shows that $(u \times v)(u' \times v') = uu' \times vv'$. Thus from (5.10) we have

$$\begin{aligned}(fg)_n(\sigma,\pi) &= f_1 g_1(\hat{0},\hat{1})^{a_1} \cdots f_n g_n(\hat{0},\hat{1})^{a_n} \\ &= fg(1)^{a_1} \cdots fg(n)^{a_n}.\end{aligned}$$

□

It follows from Lemma 5.1.10 that the set $M(\mathbf{\Pi}) = M(\mathbf{\Pi}, K)$ of multiplicative functions on $\mathbf{\Pi}$ forms a semigroup under convolution. In fact, $M(\mathbf{\Pi})$ is even a monoid (= semigroup with identity), since the identity function $\delta = (\delta_1, \delta_2, \ldots)$ is multiplicative with $\delta(n) = \delta_{1n}$. (CAVEAT: $M(\mathbf{\Pi})$ is *not* closed under addition!)

5.1.11 Theorem. *Define a map $\phi : M(\mathbf{\Pi}) \to xK[[x]]$ (the monoid of power series with zero constant term under composition) by*

$$\phi(f) = E_f(x) = \sum_{n \geq 1} f(n) \frac{x^n}{n!}.$$

Then ϕ is an anti-isomorphism of monoids, that is, ϕ is a bijection and

$$\phi(fg) = E_g(E_f(x)).$$

Proof. Clearly ϕ is a bijection. Since fg is multiplicative by Lemma 5.1.10, it suffices to show that

$$\sum_{n \geq 1} fg(n) \frac{x^n}{n!} = E_g(E_f(x)).$$

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By definition of $fg(n)$, we have in $I(\Pi_n, K)$

$$
\begin{aligned}
fg(n) &= \sum_{\pi=\{B_1,\ldots,B_k\}\in\Pi_n} f_n(\hat{0},\pi)g_n(\pi,\hat{1}) \\
&= \sum_{\pi} f(\#B_1)\cdots f(\#B_k)g(k). \qquad (5.13)
\end{aligned}
$$

Since (5.13) agrees with (5.3), the proof follows from Theorem 5.1.4. $\square$

The next result follows from Theorem 5.1.11 in the same way that Proposition 3.18.5 follows from Theorem 3.18.4 (using Proposition 5.4.1), so the proof is omitted. (A direct proof avoiding Theorem 5.1.11 can also be given.) If $f = (f_1, f_2, \ldots)$ where $f_n \in I(\Pi_n, K)$ and each f_n^{-1} exists in $I(\Pi_n, K)$, then we write $f^{-1} = (f_1^{-1}, f_2^{-1}, \ldots)$.

5.1.12 Proposition. *Suppose f is multiplicative and f^{-1} exists. Then f^{-1} is multiplicative.*

5.1.13 Example. Let $\zeta, \delta, \mu \in M(\mathbf{\Pi})$ have the same meanings as above, so $\zeta\mu = \mu\zeta = \delta$. Now

$$
E_\zeta(x) = \sum_{n\geq 1} \frac{x^n}{n!} = e^x - 1
$$

$$
E_\delta(x) = x,
$$

so by Theorem 5.1.11

$$
\begin{aligned}
[\exp E_\mu(x)] - 1 &= x \\
\Rightarrow E_\mu(x) &= \log(1+x) \\
&= \sum_{n\geq 1} (-1)^{n-1}(n-1)!\frac{x^n}{n!} \\
\Rightarrow \mu(n) &= (-1)^{n-1}(n-1)!.
\end{aligned}
$$

Thus we have another derivation of the Möbius function of Π_n (equation (3.37)).

5.1.14 Example. Let $h(n)$ be the number of ways to partition the set $[n]$, and then partition each block into blocks of odd cardinality. We are asking for the number of chains $\hat{0} \leq \pi \leq \sigma \leq \hat{1}$ in Π_n such that all block sizes of π are odd. Define $f \in M(\mathbf{\Pi})$ by

$$
f(n) = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}
$$

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Then clearly $h = f\zeta^2$, so by Theorem 5.1.11,

$$\begin{aligned} E_h(x) &= E_\zeta(E_\zeta(E_f(x))) \\ &= \exp\left[\left(\exp\sum_{n\geq 0}\frac{x^{2n+1}}{(2n+1)!}\right) - 1\right] - 1 \\ &= \exp(e^{\sinh x} - 1) - 1. \end{aligned}$$

We have discussed in this section the combinatorial significance of multiplying and composing exponential generating functions. Three further operations are important to understand combinatorially: addition, multiplication by x (really a special case of arbitrary multiplication, but of special significance), and differentiation.

5.1.15 Proposition. *Let S be a finite set. Given functions $f, g : \mathbb{N} \to K$, define new functions h_1, h_2, h_3, and h_4 as follows:*

$$h_1(\#S) = f(\#S) + g(\#S) \tag{5.14}$$

$$h_2(\#S) = (\#S)f(\#T), \quad \textit{where } \#T = \#S - 1 \tag{5.15}$$

$$h_3(\#S) = f(\#T), \quad \textit{where } \#T = \#S + 1 \tag{5.16}$$

$$h_4(\#S) = (\#S)f(\#S). \tag{5.17}$$

Then

$$E_{h_1}(x) = E_f(x) + E_g(x) \tag{5.18}$$

$$E_{h_2}(x) = xE_f(x) \tag{5.19}$$

$$E_{h_3}(x) = E'_f(x) \tag{5.20}$$

$$E_{h_4}(x) = xE'_f(x). \tag{5.21}$$

Proof. Easy. □

Equation (5.14) corresponds to a choice of two structures to place on S, one enumerated by f and one by g. In equation (5.15), we "root" a vertex v of S (i.e., we choose a distinguished vertex v, often called the *root*) and then place a structure on the remaining vertices $T = S - \{x\}$. Equation (5.16) corresponds to adjoining an extra element to S and then placing a structure enumerated by f. Finally in equation (5.17) we are simply placing a structure on S and rooting a vertex.

As we will see in subsequent sections, many structures have a recursive nature by which we can obtain from the results of this section functional equations or differential equations for the corresponding exponential generating

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function. Let us illustrate these ideas here by interpreting combinatorially the formula $E'_h(x) = E'_f(x)E_h(x)$ of equation (5.8). The left-hand side corresponds to the following construction: take a (finite) set S, adjoin a new element t, and then place on $S \cup \{t\}$ a structure enumerated by h (or "h-structure"). The right-hand side says: choose a subset T of S, adjoin an element t to T, place on $T \cup \{t\}$ an f-structure, and place on $S - T$ an h-structure. Clearly if h and f are related by (5.4) (so that h-structures are unique disjoint unions of f-structures) then the combinatorial interpretations of $E'_h(x)$ and $E'_f(x)E_h(x)$ are equivalent.

5.2 Applications of the Exponential Formula

The most straightforward applications of Corollary 5.1.6 concern structures which have an obvious decomposition into "connected components."

5.2.1 Example. The number of graphs (without loops or multiple edges) on an n-element vertex set S is clearly $2^{\binom{n}{2}}$. (Each of the $\binom{n}{2}$ pairs of vertices may or may not be joined by an edge.) Let $c(\#S) = c(n)$ be the number of *connected* graphs on the vertex set S. Since a graph on S is obtained by choosing a partition π of S and then placing a connected graph on each block of π, we see that equation (5.5) holds for $h(n) = 2^{\binom{n}{2}}$ and $f(n) = c(n)$. Hence by Corollary 5.1.6,

$$\begin{aligned} E_h(x) &= \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!} \\ &= \exp E_c(x) \\ &= \exp \sum_{n\geq 1} c(n) \frac{x^n}{n!}. \end{aligned}$$

Equivalently,

$$\sum_{n\geq 1} c(n) \frac{x^n}{n!} = \log \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}. \tag{5.22}$$

Note that the generating functions $E_h(x)$ and $E_c(x)$ have zero radius of convergence; nevertheless, they still have combinatorial meaning.

Of course there is nothing special about graphs in the above example. If, for instance, $h(n)$ is the number of posets (or digraphs, topologies, triangle-free graphs, ...) on an n-set and $c(n)$ is the number of connected posets (digraphs, topologies, triangle-free graphs, ...) on an n-set, then the fundamental relation $E_h(x) = \exp E_c(x)$ continues to hold. In some cases (such as graphs and digraphs) we have an explicit formula for $h(n)$, but this is an incidental "bonus."

5.2.2 Example. Suppose we are interested in not just the number of connected graphs on an n-element vertex set, but rather the number of such graphs with exactly k components. Let $c_k(n)$ denote this number, and define

$$F(x,t) = \sum_{n\geq 0}\sum_{k\geq 0} c_k(n)t^k\frac{x^n}{n!}. \tag{5.23}$$

There are two ways to obtain this generating function from Theorem 5.1.4 and Corollary 5.1.6. We can either set $f(n) = c(n)$ and $g(k) = t^k$ in (3), or set $f(n) = c(n)t$ in (5.5). In either case we have

$$h(n) = \sum_{k\geq 0} c_k(n)t^k.$$

Thus

$$\begin{aligned} F(x,t) &= \exp t\sum_{n\geq 1} c(n)\frac{x^n}{n!} \\ &= \left(\sum_{n\geq 0} 2^{\binom{n}{2}}\frac{x^n}{n!}\right)^t. \end{aligned}$$

Again the same reasoning works equally as well for posets, digraphs, topologies, ... In general, if $E_h(x)$ is the exponential generating function for the total number of structures on an n-set (where of course each structure is a unique disjoint union of connected components), then $E_h(x)^t$ also keeps track of the number of components, as in (5.23). Equivalently, if $h(n)$ is the number of structures on an n-set and $c_k(n)$ the number with k components, then

$$\begin{aligned} \sum_{k\geq 0} t^k E_{c_k}(x) &= E_h(x)^t \\ &= \exp tE_{c_1}(x) \\ &= \sum_{k\geq 0} t^k \frac{E_{c_1}(x)^k}{k!}, \end{aligned} \tag{5.24}$$

so

$$E_{c_k}(x) = \frac{1}{k!}E_{c_1}(x)^k = \frac{1}{k!}(\log E_h(x))^k,$$

where we set $c_k(0) = \delta_{0k}$ and $h(0) = 1$. In particular, if $h(n) = n!$ (the number of permutations of an n-set) then $E_h(x) = (1-x)^{-1}$ while $c_k(n) = c(n,k)$, the number of permutations of an n-set with k cycles. In other words, $c(n,k)$ is a signless Stirling number of the first kind (see Chapter 1.3); and we get

$$\sum_{n\geq 0} c(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left[\log(1-x)^{-1}\right]^k. \tag{5.25}$$

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Let us give one further "direct" application, concocted for the elegance of the final answer.

5.2.3 Example. Suppose we have a room of n children. The children gather into circles by holding hands, and one child stands in the center of each circle. A circle may consist of as few as one child (clasping his or her hands), but each circle must contain a child inside it. In how many ways can this be done? Let this number be $h(n)$. An allowed arrangement of children is obtained by choosing a partition of the children, choosing a child c from each block B to be in the center of the circle, and arranging the other children in the block B in a circle about c. If $\#B = i \geq 2$, then there are $i \cdot (i-2)!$ ways to do this, and no ways if $i = 1$. Hence (setting $h(0) = 1$),

$$\begin{aligned} E_h(x) &= \exp \sum_{i \geq 2} i \cdot (i-2)! \frac{x^i}{i!} \\ &= \exp x \sum_{i \geq 1} \frac{x^i}{i} \\ &= \exp x \log(1-x)^{-1} \\ &= (1-x)^{-x}. \end{aligned}$$

Similarly, if $c_k(n)$ denotes the number of arrangements of n children with exactly k circles, then

$$\sum_{n \geq 0} \sum_{k \geq 0} c_k(n) t^k \frac{x^n}{n!} = (1-x)^{-xt}.$$

We next consider some problems concerned with successively partitioning the blocks of a partition.

5.2.4 Example. Let $B(n) = B_1(n)$ denote the nth Bell number, that is, $B(n) = \#\Pi_n$ (Chapter 1.9). Setting each $f(i) = 1$ in (5.4), we obtain

$$E_B(x) = \sum_{n \geq 0} B(n) \frac{x^n}{n!} = \exp(e^x - 1).$$

(See equation (1.94f).) Now let $B_2(n)$ be the number of ways to partition an n-set S, and then partition each block. Equivalently, $B_2(n)$ is the number of chains $\hat{0} \leq \pi_1 \leq \pi_2 \leq \hat{1}$ in Π_n. Putting each $f(i) = B(i)$ in (5.4), or equivalently using Theorem 5.1.11 to compute $\phi(\zeta^3)$, we obtain

$$\sum_{n \geq 0} B_2(n) \frac{x^n}{n!} = \exp[\exp(e^x - 1) - 1].$$

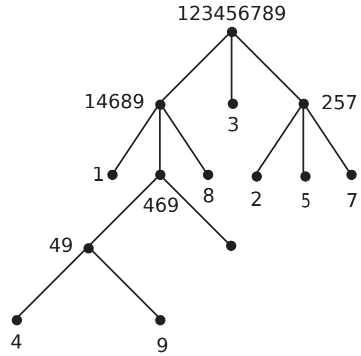

Figure 5.3 A total partition of [9] represented as a tree

Continuing in this manner, if $B_k(n)$ denotes the number of chains $\hat{0} \leq \pi_1 \leq \pi_2 \leq \cdots \leq \pi_k \leq \hat{1}$ in Π_n, then

$$\sum_{n \geq 0} B_k(n) \frac{x^n}{n!} = 1 + e^{\langle k+1 \rangle}(x),$$

where $e(x) = e^{\langle 1 \rangle}(x) = e^x - 1$ and $e^{\langle k+1 \rangle}(x) = e(e^{\langle k \rangle}(x))$.

5.2.5 Example. The preceding example was quite straightforward. Consider now the following variation. Begin with an n-set S, and for $n \geq 2$ partition S into at least two blocks. Then partition each non-singleton block into at least two blocks. Continue partitioning each non-singleton block into at least two blocks, until only singletons remain. Call such a procedure a *total partition* of S. A total partition can be represented in a natural way by an (unordered) tree, as illustrated in Figure 5.3 for $S = [9]$. Notice that only the endpoints (leaves) need to be labelled; the other labels are superfluous. Let $t(n)$ denote the number of total partitions of S (with $t(0) = 0$). Thus $t(1) = 1$, $t(2) = 1$, $t(3) = 4$, $t(4) = 26$.

Consider what happens when we choose a partition π of S and then a total partition of each block of π. If $|\pi| = 1$, then we have done the equivalent of choosing a total partition of S. On the other hand, partitioning S into at least two blocks and then choosing a total partition of each block is equivalent to choosing a total partition of S itself. Thus altogether we obtain each total partition of S *twice*, provided $\#S \geq 2$. If $\#S = 1$, then we obtain the unique total partition of S only once. If $\#S = 0$ (i.e., $S = \emptyset$) then our procedure can be done in one way (i.e., do nothing), but by our convention there are no total partitions of S. Hence from Corollary 5.1.6 we obtain

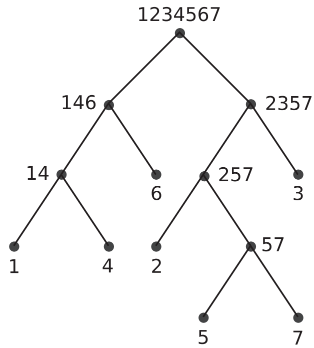

Figure 5.4 A binary total partition represented as a tree

$$\exp E_t(x) = 2E_t(x) - x + 1. \tag{5.26}$$

In other words, writing $F^{\langle -1 \rangle}(x)$ for the compositional inverse of $F(x) = ax + bx^2 + \cdots$ where $a \neq 0$, that is,

$$F(F^{\langle -1 \rangle}(x)) = F^{\langle -1 \rangle}(F(x)) = x,$$

we have

$$E_t(x) = (1 + 2x - e^x)^{\langle -1 \rangle}. \tag{5.27}$$

It does not seem possible to obtain a simpler result. In particular, in Section 5.4 we will discuss methods for computing the coefficients of compositional inverses, but these methods don't seem to yield anything interesting when $F(x) = 1 + 2x - e^x$. For some enumeration problems closely related to total partitions, see Exercises 26 and 40.

5.2.6 Example. Consider the variation of the preceding example where each non-singleton block must be partitioned into *exactly two* blocks. Call such a procedure a *binary total partition* of S, and denote the number of them by $b(\#S)$. As with total partitions, set $b(0) = 0$. The tree representing a binary total partition is a complete (unordered) binary tree, as illustrated in Figure 5.4. (Thus $b(n)$ is just the number of (unordered) complete binary trees with n labelled endpoints.) It now follows from Theorem 5.1.4 (with $g(k) = \delta_{2k}$) or just by (5.3) (with $k = 2$ and $g(2) = 1$), in a similar way to how we obtained (5.26), that

$$\frac{1}{2} E_b(x)^2 = E_b(x) - x. \tag{5.28}$$

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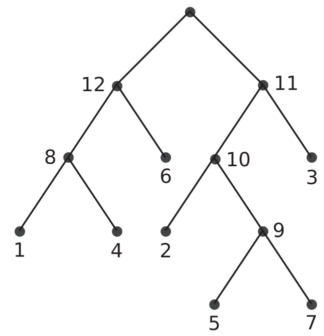

Figure 5.5 A decreasing labelled tree corresponding to a binary total partition

Solving this quadratic equation yields

$$\begin{aligned} E_b(x) &= 1 - \sqrt{1-2x} \\ &= 1 - \sum_{n\geq 0} \binom{1/2}{n} (-2)^n x^n \\ &= \sum_{n\geq 1} 1 \cdot 3 \cdot 5 \cdots (2n-3) \frac{x^n}{n!}, \end{aligned}$$

whence

$$b(n) = 1 \cdot 3 \cdot 5 \cdots (2n-3).$$

As usual, when such a simple answer is obtained a direct combinatorial proof is desired. Now $1 \cdot 3 \cdot 5 \cdots (2n-3)$ is easily seen to be the number of partitions of $[2n-2]$ of type (2^{n-1}), that is, with $n-1$ 2-element blocks. Given a binary total partition β of $[n]$, we obtain a partition π of $[2n-2]$ of type (2^{n-1}) as follows. In the tree representing β (such as Figure 5.4), delete all the labels except the endpoints (leaves). Now iterate the following procedure until all vertices are labelled except the root. If labels $1, 2, \ldots, m$ have been used, then label by $m+1$ the vertex v satisfying: (a) v is unlabelled and both successors of v are labelled, and (b) among all unlabelled vertices with both successors labelled, the vertex having the successor with the *least* label is v. Figure 5.5 illustrates this procedure carried out for the tree in Figure 5.4. Finally let the blocks of π consist of the vertex labels of the two successors of a non-endpoint vertex. Thus from Figure 5.5 we obtain

$$\pi = \{\{1,4\}, \{2,9\}, \{3,10\}, \{5,7\}, \{6,8\}, \{11,12\}\}.$$

We leave the reader to check (not entirely trivial) that this procedure yields the desired bijection.

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Certain problems involving symmetric matrices are well-suited for use of the exponential formula. (Analogous results for arbitrary matrices are discussed in Section 5.5.) The basic idea is that a symmetric matrix $A = (a_{uv})$ whose rows and columns are indexed by a set V may be identified with a graph $G = G_A$ on the vertex set V, with an edge uv connecting u and v labelled by a_{uv}. (If $a_{uv} = 0$, then we simply omit the edge uv, rather than labelling it by 0. More generally, if $a_{uv} \in \mathbb{P}$ then it is often convenient to draw a_{uv} (unlabelled) edges between u and v.) Sometimes the connected components of G_A have a simple structure, so that the exponential formula can be used to enumerate all the graphs (or matrices).

5.2.7 Example. As in Proposition 4.6.21, let $S_n(2)$ denote the number of $n \times n$ symmetric $\mathbb{N}$-matrices A with every row (and hence every column) sum equal to two. The graph G_A has every vertex of degree two (counting loops once only). Hence the connected components of G_A must be (a) a single vertex with two loops, (b) a double edge between two vertices, (c) a cycle of length ≥ 3, or (d) a path of length ≥ 1 with a loop at each end. There are $\frac{1}{2}(n-1)!$ (undirected) cycles on $n \geq 3$ vertices, and $\frac{1}{2}n!$ (undirected) paths on $n \geq 2$ vertices with a loop at each end. Hence by Corollary 5.1.6,

$$\begin{aligned}
\sum_{n\geq 0} S_n(2)\frac{x^n}{n!} &= \exp\left(x + \frac{x^2}{2!} + \frac{1}{2}\sum_{n\geq 3}(n-1)!\frac{x^n}{n!} + \frac{1}{2}\sum_{n\geq 2} n!\frac{x^n}{n!}\right) \\
&= \exp\left(\frac{x^2}{4} + \frac{1}{2}\sum_{n\geq 1}\frac{x^n}{n} + \frac{1}{2}\sum_{n\geq 1} x^n\right) \\
&= \exp\left(\frac{x^2}{4} + \frac{1}{2}\log(1-x)^{-1} + \frac{x}{2(1-x)}\right) \\
&= (1-x)^{-1/2}\exp\left(\frac{x^2}{4} + \frac{x}{2(1-x)}\right).
\end{aligned}$$

Using the technique of Exercise 24(c), we obtain the recurrence (writing $S_m = S_m(2)$)

$$S_{n+1} = (2n+1)S_n - (n)_2\, S_{n-1} - (n)_2\, S_{n-2} + \frac{1}{2}(n)_3\, S_{n-3}, \quad n \geq 0.$$

5.2.8 Example. Suppose that in the previous example A must be a 0–1 matrix (i.e., the entry two is not allowed). Now the components of G_A of type (a) or

(b) above are not allowed. If we let $S_n^*(2)$ denote the number of such matrices, it follows that

$$\begin{aligned}\sum_{n\geq 0} S_n^*(2)\frac{x^n}{n!} &= e^{-x-\frac{x^2}{2}}\sum_{n\geq 0} S_n(2)\frac{x^n}{n!}\\ &= (1-x)^{-1/2}\exp\left(-x-\frac{x^2}{4}+\frac{x}{2(1-x)}\right).\end{aligned}$$

As a further variation, suppose we again allow two as an entry, but that $\operatorname{tr} A = 0$ (i.e., all main diagonal entries are zero). Now the components of G_A cannot have loops, so are of type (b) or (c). Hence, letting $T_n(2)$ be the number of such matrices, we have

$$\begin{aligned}\sum_{n\geq 0} T_n(2)\frac{x^n}{n!} &= \exp\left(\frac{x^2}{2!}+\frac{1}{2}\sum_{n\geq 3}(n-1)!\frac{x^n}{n!}\right)\\ &= (1-x)^{-1/2}\exp\left(-\frac{x}{2}+\frac{x^2}{4}\right).\end{aligned}$$

Similarly, if $T_n^*(2)$ denotes the number of traceless symmetric $n\times n$ 0-1 matrices with line sum 2, then

$$\begin{aligned}\sum_{n\geq 0} T_n^*(2)\frac{x^n}{n!} &= \exp\left(\frac{1}{2}\sum_{n\geq 3}(n-1)!\frac{x^n}{n!}\right)\\ &= (1-x)^{-1/2}\exp\left(-\frac{x}{2}-\frac{x^2}{4}\right).\end{aligned} \tag{5.29}$$

The recurrence relations for $S_n^*(2)$, $T_n(2)$, and $T_n^*(2)$ turn out to be (using the technique of Exercise 24(c))

$$\begin{aligned}S_{n+1}^*(2) &= 2nS_n^*(2) - n(n-2)\,S_{n-1}^*(2) - \frac{1}{2}(n)_3\,S_{n-3}^*(2)\\ T_{n+1}(2) &= nT_n(2) + nT_{n-1}(2) - \tbinom{n}{2}T_{n-2}(2)\\ T_{n+1}^*(2) &= nT_n^*(2) + \tbinom{n}{2}T_{n-2}^*(2),\end{aligned}$$

all valid for $n \geq 0$.

The next example is an interesting variation of the preceding two, where it is not *a priori* evident that the exponential formula is relevant.

5.2.9 Example. Let $X_n = (x_{ij})$ be an $n \times n$ symmetric matrix whose entries x_{ij} are independent indeterminates (over $\mathbb{R}$, say), except that $x_{ij} = x_{ji}$. Let $L(n)$

be the number of terms (i.e., distinct monomials) in the expansion of $\det X_n$, where we use only the variables x_{ij} for $i \le j$. For instance,

$$\det X_3 = x_{11}x_{22}x_{33} + 2x_{12}x_{23}x_{13} - x_{13}^2 x_{22} - x_{11}x_{23}^2 - x_{12}^2 x_{33}.$$

Hence $L(3) = 5$, since the above sum has five terms. One might ask whether we should count a monomial which does arise in the expansion of $\det X_n$ but whose coefficient because of cancellation turns out to be zero. But we will soon see that no cancellation is possible; all occurrences of a given monomial have the same sign. Suppose now that $\pi \in \mathfrak{S}_n$. Define

$$M_\pi = x_{1,\pi(1)}x_{2,\pi(2)} \cdots x_{n,\pi(n)},$$

where we set $x_{ji} = x_{ij}$ if $j > i$. Thus M_π is the monomial corresponding to π in the expansion of $\det X_n$. Define a graph G_π whose vertex set is $[n]$, and with an (undirected) edge between i and $\pi(i)$ for $1 \le i \le n$. Thus the components of G_π are cycles of length ≥ 1. (A cycle of length 1 is a loop, and of length 2 is a double edge.) The crucial observation, whose easy proof we omit, is that $M_\pi = M_\sigma$ if and only if $G_\pi = G_\sigma$. Since a permutation π is even (respectively, odd) if and only if G_π has an even number (respectively, odd number) of cycles of even length, it follows that if $M_\pi = M_\sigma$ then M_π and M_σ occur in the expansion of $\det X_n$ with the same sign. In other words, there is no cancellation in the expansion of $\det X_n$. Also note that a graph G on $[n]$, every component of which is a cycle, is equal to G_π for some $\pi \in \mathfrak{S}_n$. (In fact, the number of such π is exactly $2^{d(\pi)}$, where π has $d(\pi)$ cycles of length ≥ 3.) It follows that $L(n)$ is simply the number of graphs on $[n]$ for which every component is a cycle (including loops and double edges). Hence

$$\begin{aligned}
\sum_{n\ge 0} L(n)\frac{x^n}{n!} &= \exp\left(x + \frac{x^2}{2!} + \frac{1}{2}\sum_{n\ge 3}(n-1)!\frac{x^n}{n!}\right) \\
&= (1-x)^{-1/2}\exp\left(\frac{x}{2} + \frac{x^2}{4}\right).
\end{aligned}$$

Note also that if P_π is the permutation matrix corresponding to $\pi \in \mathfrak{S}_n$, then $G_\pi = G_\sigma$ if and only if $P_\pi + P_\pi^{-1} = P_\sigma + P_\sigma^{-1}$. Hence $L(n)$ is the number of distinct matrices of the form $P_\pi + P_\pi^{-1}$, where $\pi \in \mathfrak{S}_n$. Equivalently, if we define $\pi, \sigma \in \mathfrak{S}_n$ to be *equivalent* if every cycle of π is a cycle or inverse of a cycle of σ, then $L(n)$ is the number of equivalence classes.

We conclude this section with some examples where it is more natural to use the permutation version of the exponential formula (Corollary 5.1.9).

5.2.10 Example. Let $\pi \in \mathfrak{S}_n$ be a permutation. Suppose that π has $c_i = c_i(\pi)$ cycles of length i, so $\sum ic_i = n$. Form a monomial

$$Z(\pi) = Z(\pi, t) = t_1^{c_1} t_2^{c_2} \cdots t_n^{c_n}$$

in the variables $t = (t_1, t_2, \dots)$. We call $Z(\pi)$ the *cycle index* (or *cycle indicator* or *cycle monomial*) of π. Define the *cycle index* or *cycle index polynomial* (or *cycle indicator*, etc.) $Z(\mathfrak{S}_n)$ (also denoted $Z_{\mathfrak{S}_n}(t), P_{\mathfrak{S}_n}(t), \mathrm{Cyc}(\mathfrak{S}_n, t)$, etc.) of $\mathfrak{S}_n$ by

$$Z(\mathfrak{S}_n) = Z(\mathfrak{S}_n, t) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} Z(\pi).$$

(In Chapter 7.7.24 we will consider the cycle index of other permutation groups.) It is sometimes more convenient to work with the *augmented cycle index*

$$\tilde{Z}(\mathfrak{S}_n) = n!\, Z(\mathfrak{S}_n) = \sum_{\pi \in \mathfrak{S}_n} Z(\pi).$$

For instance

$$\begin{aligned}
\tilde{Z}(\mathfrak{S}_1) &= t_1 \\
\tilde{Z}(\mathfrak{S}_2) &= t_1^2 + t_2 \\
\tilde{Z}(\mathfrak{S}_3) &= t_1^3 + 3t_1t_2 + 2t_3 \\
\tilde{Z}(\mathfrak{S}_4) &= t_1^4 + 6t_1^2t_2 + 8t_1t_3 + 3t_2^2 + 6t_4.
\end{aligned}$$

Clearly if we define $f : \mathbb{P} \to K$ by $f(n) = t_n$, then

$$\tilde{Z}(\mathfrak{S}_n) = \sum_{\pi \in \mathfrak{S}_n} f(\#C_1) f(\#C_2) \cdots f(\#C_k),$$

where $C_1, C_2, \cdots, C_k$ are the cycles of π. Hence by Corollary 5.1.9,

$$\sum_{n \geq 0} \tilde{Z}(\mathfrak{S}_n) \frac{x^n}{n!} = \exp \sum_{i \geq 1} t_i \frac{x^i}{i}. \tag{5.30}$$

There are many interesting specializations of (5.30) related to enumerative properties of $\mathfrak{S}_n$. For instance, fix $r \in \mathbb{P}$ and let $e_r(n)$ be the number of $\pi \in \mathfrak{S}_n$ satisfying $\pi^r = \mathrm{id}$. A permutation π satisfies $\pi^r = \mathrm{id}$ if and only if $c_d(\pi) = 0$ whenever $d \nmid r$. Hence

$$e_r(n) = \tilde{Z}(\mathfrak{S}_n) \Bigg|_{t_d = \begin{cases} 1, & d \mid r \\ 0, & d \nmid r. \end{cases}}$$

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It follows that

$$\sum_{n\geq 0} e_r(n)\frac{x^n}{n!} = \exp\left(\sum_{d\geq 1} t_d \frac{x^d}{d}\right)\Bigg|_{t_d=\left\{\begin{array}{ll} 1, & d|r \\ 0, & d\nmid r\end{array}\right.}$$

$$= \exp\left(\sum_{d|r}\frac{x^d}{d}\right). \tag{5.31}$$

In particular, the number $e_2(n)$ of involutions in $\mathfrak{S}_n$ satisfies

$$\sum_{n\geq 0} e_2(n)\frac{x^n}{n!} = \exp\left(x+\frac{x^2}{2}\right). \tag{5.32}$$

This is the same generating function encountered way back in equation (1.11). Now we are able to understand its combinatorial significance more clearly.

There is a surprising connection between (a) Corollary 5.1.9, (b) the relationship between linear and circular words obtained in Proposition 4.7.13, and (c) the bijection $\pi \mapsto \hat{\pi}$ discussed in Chapter 1.3 between permutations written as products of cycles and as words. Basically, such a connection arises from a formula of the type

$$\sum_{n\geq 0} n!\,f(n)\frac{x^n}{n!} = \exp\sum_{n\geq 1}(n-1)!\,g(n)\frac{x^n}{n!}$$

because $n!$ is the number of linear words (permutations) on $[n]$, while $(n-1)!$ is the number of circular words (cycles). The next example may be regarded as the archetype for this line of thought.

5.2.11 Example. Let

$$F(x) = \prod_k (1-t_k x)^{-1}, \tag{5.33}$$

where k ranges over some index set, say $k \in \mathbb{P}$. Thus

$$\log F(x) = \sum_k \log(1-t_k x)^{-1}$$

$$= \sum_k \sum_{n\geq 1} t_k^n \frac{x^n}{n}$$

$$= \sum_{n\geq 1} p_n(t)\frac{x^n}{n}, \tag{5.34}$$

where $p_n(t) = \sum_k t_k^n$. On the other hand, it is clear from (5.33) that

$$F(x) = \sum_{n\geq 0} h_n(t)x^n, \tag{5.35}$$

where $h_n(t)$ is the sum of all monomials of degree n in $t = (t_1, t_2, \cdots)$, that is,

$$h_n(t) = \sum_{\substack{a_1+a_2+\cdots=n\\ a_i\in\mathbb{N}}} t_1^{a_1} t_2^{a_2} \cdots .$$

(In Chapter 7 we will analyze the symmetric functions $p_n(t)$ and $h_n(t)$, as well as many others, in much greater depth.) From (5.34) and (5.35) we conclude

$$\sum_{n\geq 0} n!\, h_n(t)\frac{x^n}{n!} = \exp \sum_{n\geq 1} p_n(t)\frac{x^n}{n}. \tag{5.36}$$

We wish to give a direct combinatorial proof. By Corollary 5.1.9, the right-hand side is the exponential generating function for the following structure: Choose a permutation $\pi \in \mathfrak{S}_n$, and weight each cycle C of π by a monomial $t_k^{\#C}$ for some k. Define the total weight of π to be the product of the weights of each cycle. For instance the list of structures of weight u^2v (where $u = t_1$ and $v = t_2$, say) is given by

$$\begin{array}{ccc} \begin{matrix}(1)&(2)&(3)\\ u&u&v\end{matrix} & \qquad & \begin{matrix}(12)&(3)\\ uu&v\end{matrix} \\[1ex] \begin{matrix}(1)&(2)&(3)\\ u&v&u\end{matrix} & & \begin{matrix}(13)&(2)\\ uu&v\end{matrix} \\[1ex] \begin{matrix}(1)&(2)&(3)\\ v&u&u\end{matrix} & & \begin{matrix}(1)&(23)\\ v&uu\end{matrix} \end{array} \tag{5.37}$$

Moreover, the left-hand side of (5.36) is clearly the exponential generating function for pairs (π, t^a), where $\pi \in \mathfrak{S}_n$ and t^a is a monomial of degree n in t. Thus the structures of weight u^2v are given by

$$\begin{array}{ll} 123,\ u^2v & \qquad 231,\ u^2v \\ 132,\ u^2v & \qquad 312,\ u^2v \\ 213,\ u^2v & \qquad 321,\ u^2v. \end{array} \tag{5.38}$$

In both (5.37) and (5.38) there are six items.

In general, in order to prove (5.36) bijectively, we need to do the following. Given a monomial t^a of degree n, find a bijection $\phi : \mathcal{C}_a \to \mathfrak{S}_n$, where $\mathcal{C}_a$ is the set of all permutations π in $\mathfrak{S}_n$, with each cycle C weighted by $t_j^{\#C}$ for some $j = j(C)$, such that the total weight $\Pi_C t_{j(C)}^{\#C}$ is equal to t^a. To describe ϕ,

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first impose some linear ordering on the t_j's, say $t_1 < t_2 < \cdots$. For fixed j, take all the cycles C of π with weight $t_j^{\#C}$ and write their standard representation (in the sense of Proposition 1.3.1), that is, the largest element of each cycle is written first in the cycle, and the cycles are written left-to-right in increasing order of their largest elements. Remove the parentheses from this standard representation, obtaining a word w_j. Finally set $\phi(\pi) = (w, t^a)$, where $w = w_1 w_2 \cdots$ (juxtaposition of words). For instance, suppose π is the weighted permutation

$$\pi = \begin{matrix} (19) & (82) & (3) & (547) & (6) \\ vv & uu & v & uuu & u \end{matrix}$$

where $u < v$. The cycles weighted by u's and v's, respectively, have standard form

$$\begin{matrix} (6)(754)(82) & \quad (\text{weight } u^6) \\ (3)(91) & \quad (\text{weight } v^3). \end{matrix}$$

Hence

$$w_1 = 675482, \quad w_2 = 391,$$

$$\phi(\pi) = (675482391, u^6 v^3).$$

It is easy to check that ϕ is a bijection. Given (w, t^a), the monomial t^a determines the words w_j with their weights $t_j^{|w_j|}$. Each word w_j then corresponds to a collection of cycles C (with weight $t_j^{|C|}$) using the inverse of the bijection $\pi \mapsto \hat{\pi}$ of Chapter 1.3.

A similar argument leads to a direct combinatorial proof of equation (4.41); see Exercise 21.

5.3 Enumeration of trees

Trees have a recursive structure which makes them highly amenable to the methods of this chapter. We will develop in this section some basic properties of trees as a prelude to the Lagrange inversion formula of the next section. Trees are also fascinating objects of study for their own sake, so we will cover some topics not strictly germane to the composition of generating functions.

For the basic definitions and terminology concerning trees, see Appendix A of Volume 1. We also define a *planted forest* (also called a *rooted forest* or *forest of rooted trees*) to be a graph for which every connected component is a (rooted) tree. We begin with an investigation of the total number $p_k(n)$ of planted forests with k components on the vertex set $[n]$. Note that $p_1(n)$ is just the number $r(n)$ of rooted trees on $[n]$. If $S \subseteq [n]$ and $\#S = k$, then define $p_S(n)$

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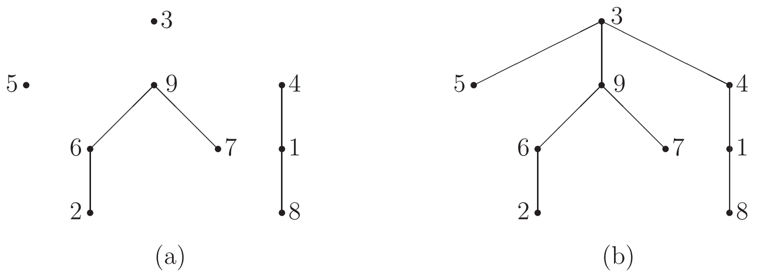

Figure 5.6 A plane tree built from the subtrees of its root

to be the number of planted forests on $[n]$ with k components, whose set of roots is S. Thus $p_k(n) = \binom{n}{k} p_S(n)$, since clearly $p_S(n) = p_T(n)$ if $\#S = \#T$.

5.3.1 Proposition. *Let*

$$y = R(x) = \sum_{n\geq 1} r(n)\frac{x^n}{n!},$$

where $r(n)$ *as above is the number of rooted trees on the vertex set* $[n]$ *(with* $r(0) = 0$*). Then* $y = xe^y$*, or equivalently (since* $x = ye^{-y}$*),*

$$y = (xe^{-x})^{\langle -1\rangle}. \tag{5.39}$$

Moreover, for $k \in \mathbb{P}$ *we have*

$$\frac{1}{k!}y^k = \sum_{n\geq 1} p_k(n)\frac{x^n}{n!}. \tag{5.40}$$

Proof. By Corollary 5.1.6, e^y is the exponential generating function for planted forests on the vertex set $[n]$. By equation (5.19), xe^y is the exponential generating function for the following structure on $[n]$. Choose a root vertex i, and place a planted forest F on the remaining vertices $[n] - \{i\}$. But this structure is equivalent to a tree with root i, whose subtrees of the root are the components of F. (See Figure 5.6.) Thus xe^y is just the exponential generating function for trees, so $y = xe^y$. Equation (5.40) then follows from Proposition 5.1.3. □

In the functional equation $y = xe^y$ of Proposition 5.3.1, substitute xe^y for the occurrence of y on the right-hand side to obtain

$$y = xe^{xe^y}$$

Again making the same substitution yields

$$y = xe^{xe^{xe^{y}}}.$$

Iterating this procedure yields the "formula"

$$y = xe^{xe^{xe^{x\cdot^{\cdot^{\cdot}}}}}. \tag{5.41}$$

The precise meaning of (5.41) is the following. Define $y_0 = x$ and for $k \geq 0$, $y_{k+1} = xe^{y_k}$. Then $\lim_{k\to\infty} y_k = y$, where the limit exists in the formal power series sense of Chapter 1.1. Moreover, by Corollary 5.1.6 and the second case of Proposition 5.1.15, we see that

$$y_k = \sum_{n\geq 1} r_k(n) \frac{x^n}{n!},$$

where $r_k(n)$ in the number of rooted trees on $[n]$ of length $\leq k$. For instance,

$$y_1 = xe^x = \sum_{n\geq 1} n \frac{x^n}{n!},$$

so $r_1(n) = n$ (as is obvious from the definition of $r_1(n)$).

The following quantities are closely related to the number $r(n)$ of rooted trees on the vertex set $[n]$:

$$\begin{aligned} t(n) &= \text{number of free trees on } [n] \\ f(n) &= \text{number of free forests (i.e., disjoint unions of free trees) on } [n] \\ p(n) &= \text{number of planted forests on } [n]. \end{aligned}$$

We set $t(0) = 0, f(0) = 1, p(0) = 1$. Also write $T(x) = E_t(x), F(x) = E_f(x)$, and $P(x) = E_p(x)$. It is easy to verify the following relations:

$$r(n) = np(n-1) = nt(n), \;\; p(n) = t(n+1)$$

$$F(x) = e^{T(x)}, \;\; P(x) = e^{R(x)} \tag{5.42}$$

$$P(x) = T'(x), \;\; R(x) = xP(x).$$

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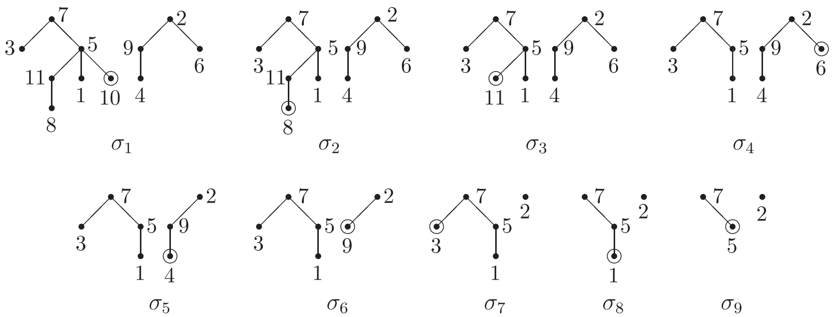

Figure 5.7 Constructing the Prüfer sequence of a labelled forest

5.3.2 Proposition. *We have* $p_S(n) = kn^{n-k-1}$ *for any* $S \in \binom{[n]}{k}$. *Thus*

$$p_k(n) = k\binom{n}{k}n^{n-k-1} = \binom{n-1}{k-1}n^{n-k}$$
$$r(n) = n^{n-1}$$
$$t(n) = n^{n-2}$$
$$p(n) = (n+1)^{n-1}.$$

First proof. The case $n = k$ is trivial, so assume $n \geq k + 1$. The number of sequences $s = (s_1, \ldots, s_{n-k})$ with $s_i \in [n]$ for $1 \leq i \leq n - k - 1$ and $s_{n-k} \in S$ is equal to kn^{n-k-1}. Hence we seek a bijection $\gamma : \mathcal{T}_{n,S} \to [n]^{n-k-1} \times S$, where $\mathcal{T}_{n,S}$ is the set of planted forests on $[n]$ with root set S. Given a forest $\sigma \in \mathcal{T}_{n,S}$, define a sequence $\sigma_1, \sigma_2, \ldots, \sigma_{n-k+1}$ of subforests (all with root set S) of σ as follows: Set $\sigma_1 = \sigma$. If $i < n - k + 1$ and σ_i has been defined, then define σ_{i+1} to be the forest obtained from σ_i by removing its *largest nonroot endpoint* v_i (and the edge incident to v_i). Then define s_i to be the unique vertex of σ_i adjacent to v_i, and let $\gamma(\sigma) = (s_1, s_2, \ldots, s_{n-k})$. The sequence $\gamma(\sigma)$ is called the *Prüfer sequence* or *Prüfer code* of the planted forest σ. Figure 5.7 illustrates this construction with a forest $\sigma = \sigma_1 \in \mathcal{T}_{11,\{2,7\}}$ and the subforests σ_i, with vertex v_i circled. Hence for this example $\gamma(\sigma) = (5, 11, 5, 2, 9, 2, 7, 5, 7)$.

We claim that the map $\gamma : \mathcal{T}_{n,S} \to [n]^{n-k-1} \times S$ is a bijection. The crucial fact is that the largest element of $[n] - S$ missing from the sequence $(s_1, \ldots, s_{n-k})$ must be v_1 [why?]. Since v_1 and s_1 are adjacent, we are reduced to computing σ_2. But $\gamma(\sigma_2) = (s_2, s_3, \ldots, s_{n-k})$ (keeping in mind that the vertex set of σ_2 is $[n] - \{v_1\}$, and not $[n-1]$). Hence by induction on n (the case $n = k+1$ being easy) we can recover σ uniquely from any $(s_1, \ldots, s_{n-k})$, so the proof is complete. □

5.3.3 Example. Let $S = \{2, 7\}$ and $(s_1, \ldots, s_9) = (5, 11, 5, 2, 9, 2, 7, 5, 7)$, so $n - k = 9$ and $n = 11$. The largest element of $[11]$ missing from $(s_1, \ldots, s_9)$

is 10. Hence 10 is an endpoint of σ adjacent to $s_1 = 5$. The largest element of $[11] - \{10\}$ missing from $(s_2, \ldots, s_9) = (11, 5, 2, 9, 2, 7, 5, 7)$ is 8. Hence 8 is an endpoint of σ_2 adjacent to $s_2 = 11$. The largest element of $[11] - \{8, 10\}$ missing from $(s_3, \ldots, s_9) = (5, 2, 9, 2, 7, 5, 7)$ is 11. Hence 11 is an endpoint of σ_3 adjacent to $s_3 = 5$. Continuing in this manner, we obtain the sequence of endpoints $10, 8, 11, 6, 4, 9, 3, 1, 5$. By beginning with the roots 2 and 7, and successively adding the endpoints in reverse order to the vertices $(s_9, \ldots, s_1) = (7, 5, 7, 2, 9, 2, 5, 11, 5)$, we obtain the forest $\sigma = \sigma_1$ of Figure 5.7.

Second proof of Proposition 5.3.2. We will show by a suitable bijection that

$$np_k(n) = k \binom{n}{k} n^{n-k}. \tag{5.43}$$

The underlying idea of the bijection is that a permutation can be represented both as a *word* and as a disjoint union of *cycles*. The bijection can be simplified somewhat for the case of rooted trees ($k = 1$), so we will present this special case first. Given a rooted tree τ on the vertex set $[n]$, circle a vertex $s \in [n]$. Let $w = w_1 w_2 \cdots w_k$ be the sequence (or word) of vertices in the unique path P in τ from the root r to s. Regard w as a permutation of its elements written in increasing order. For instance, if $w = 57283$, then w represents the permutation given by $w(2) = 5$, $w(3) = 7$, $w(5) = 2$, $w(7) = 8$, $w(8) = 3$, which in cycle notation is $(2, 5)(3, 7, 8)$. Let D_w be the directed graph with vertex set $A = \{w_1, \ldots, w_k\}$, and with an edge from j to $w(j)$ for all $j \in A$. Thus D_w is a disjoint union of (directed) cycles. When we remove from τ the edges of the path from r to s, we obtain a disjoint union of trees. Attach these trees to D_w by identifying vertices with the same label, and direct all the edges of these trees toward D_w. We obtain a digraph $D(\tau, s)$ for which all vertices have outdegree one. Moreover, the rooted tree τ, together with the distinguished vertex s, can be uniquely recovered from $D(\tau, s)$ by reversing the above steps. Since there are n choices for the vertex s, it follows that $nr(n)$ is equal to the number of digraphs on the vertex set $[n]$ for which every vertex has outdegree one. But such a digraph is just the digraph D_f of a function $f : [n] \to [n]$ (i.e., for each $j \in [n]$ draw an edge from j to $f(j)$). Since there are n^n such functions, we get $nr(n) = n^n$, so $r(n) = n^{n-1}$ as desired.

If we try the same idea for arbitrary planted forests σ, we end up at the end needing to count functions f from some subset B of $[n]$ to $[n]$ such that the digraph D_f with vertex set $[n]$ and edges $j \to f(j)$ is *nonacyclic* (i.e., has at least one directed cycle). Since there is no obvious way to count such functions, some modification of the above bijection is needed. Note that the nonacyclicity condition is irrelevant when $k = 1$, since when $B = [n]$ the digraph D_f is always nonacyclic.

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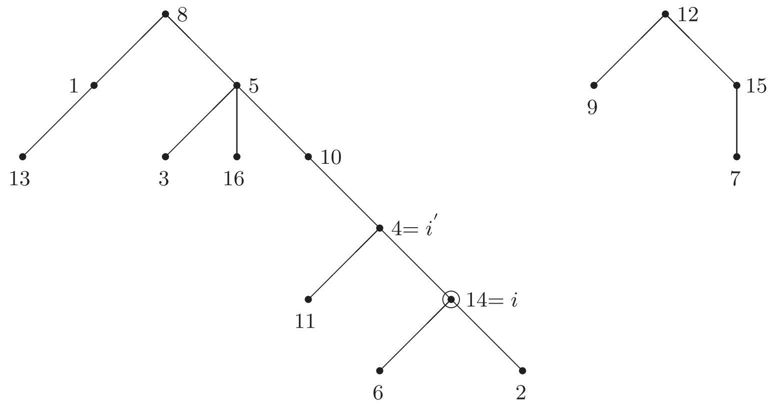

Figure 5.8 An illustration of the second proof of Proposition 5.3.2

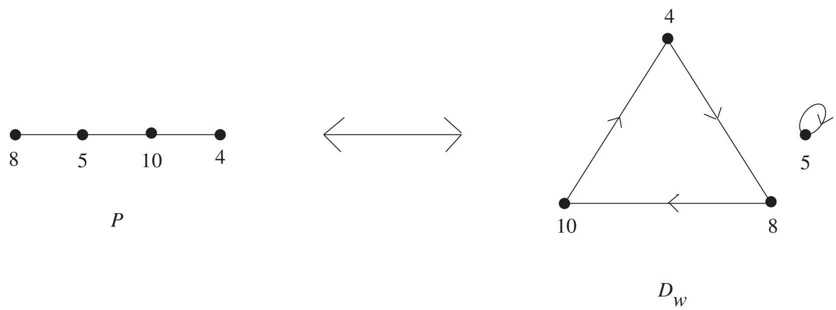

Figure 5.9 Two graphical representations of a permutation

We now proceed to the correct bijection in the general case. Let σ be a planted forest on $[n]$ with k components. Circle a vertex i of σ. Figure 5.8 illustrates the case $n = 16$, $k = 2$, $i = 14$. The vertex i belongs to a component τ of σ. Remove from τ the complete subtree τ_i with root i (keeping i circled). If i is not the root of τ then let i' be the unique predecessor of i in τ. (If i is a root, then ignore all steps below involving i'.) Let $w = w_1 w_2 \cdots w_k$ be the sequence (or word) of vertices in the unique path P in $\tau - \tau_i$ from the root r to i'. Let $A = \{w_1, \ldots, w_k\}$ be the set of vertices of P. In the example of Figure 5.8, we have $w = 8, 5, 10, 4$. Regard w as a permutation of its elements written in increasing order. For our example, the permutation is given by $w(4) = 8, w(5) = 5, w(8) = 10, w(10) = 4$, which in cycle notation is $(4, 8, 10)(5)$. Let D_w be the directed graph with vertex set A, and with an edge from j to $w(j)$ for all $j \in A$. (See Figure 5.9.)

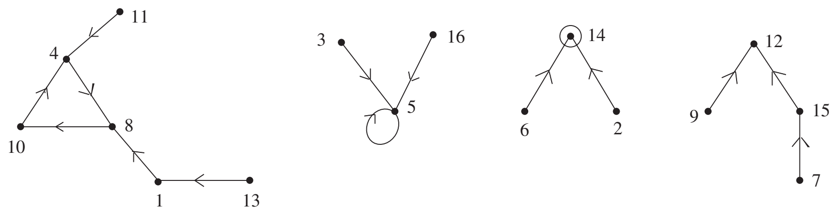

Figure 5.10 The digraph $D(\sigma, i)$

When we remove from $\tau - \tau_i$ the edges of the path P, we obtain a collection of (rooted) trees whose roots are the vertices in P. Attach these trees to D_w by identifying vertices with the same label. Direct all the edges of these trees toward D_w. For each component of σ other than τ, and for τ_i, direct their edges toward the root. We obtain a digraph $D(\sigma, i)$ on $[n]$ for which k vertices have outdegree zero and the remaining $n-k$ vertices have outdegree one. Moreover, one of the vertices of outdegree zero is circled. (See Figure 5.10.) If i is a root of τ, then $D(\sigma, i)$ is just σ with all edges directed toward roots.

Let B be the set of vertices of $D(\sigma, i)$ of outdegree one. We may identify $D(\sigma, i)$ with the function $f : B \to [n]$ defined by $f(a) = b$ if $a \to b$ is an edge of $D(\sigma, i)$. Moreover, the circled vertex i belongs to $[n] - B$.

It is not difficult to reverse all the steps and obtain the pair (σ, i) from (f, i). There are $np_k(n)$ pairs (σ, i). We can choose B to be any $(n-k)$-subset of $[n]$ in $\binom{n}{k}$ ways, then choose $i \in [n]-B$ in k ways, and finally choose $f : B \to [n]$ in n^{n-k} ways. Hence (5.43) follows. $\square$

The surprising formula

$$R(xe^{-x}) = \sum_{n\geq 1} n^{n-1} \frac{(xe^{-x})^n}{n!} = x \tag{5.44}$$

inherent in equation (5.39) and the formula $r(n) = n^{n-1}$ of Proposition 5.3.2, can be proved directly as follows:

$$\begin{aligned}
\sum_{n\geq 1} n^{n-1} \frac{(xe^{-x})^n}{n!} &= \sum_{n\geq 1} \frac{n^{n-1}x^n}{n!} \sum_{k\geq 0} \frac{(-nx)^k}{k!} \\
&= \sum_{m\geq 1} \frac{x^m}{m!} \sum_{j=1}^{m} \binom{m}{j} (-1)^{m+j} j^{m-1} \\
&= \sum_{m\geq 1} \frac{x^m}{m!} \left(\Delta^m 0^{m-1} - (-1)^m 0^{m-1} \right),
\end{aligned} \tag{5.45}$$

by applying (1.98) to the function $f(j) = j^{m-1}$. Here we must interpret $0^0 = 1$. Then by Proposition 1.9.2(a), the sum in (5.45) collapses to the single term x.

The two proofs of Proposition 5.3.2 lead to an elegant refinement of the formula $r(n) = n^{n-1}$. Given a vertex v of a planted forest σ, define the *degree* $\deg v$ of v to be the number of successors of v. Thus v is an endpoint of σ if and only if $\deg v = 0$. If the vertex set of σ is $[n]$, then define the *ordered degree sequence* $\Delta(\sigma) = (\delta_1, \ldots, \delta_n)$, where $\delta_i = \deg i$. It is easy to see that a sequence $(\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$ is the ordered degree sequence of some planted forest σ on $[n]$ with k components if and only if

$$\sum_{i=1}^{n} \delta_i = n - k. \tag{5.46}$$

5.3.4 Theorem. *Let $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$ with $\sum \delta_i = n - k$. The number $N(\delta)$ of planted forests σ on the vertex set $[n]$ (necessarily with k components) with ordered degree sequence $\Delta(\sigma) = \delta$ is given by*

$$N(\delta) = \binom{n-1}{k-1}\binom{n-k}{\delta_1, \delta_2, \ldots, \delta_n}.$$

Equivalently,

$$\sum_{\sigma} x_1^{\deg 1} \cdots x_n^{\deg n} = \binom{n-1}{k-1}(x_1 + \cdots + x_n)^{n-k}, \tag{5.47}$$

where σ ranges over all planted forests on $[n]$ with k components.

First proof. Consider the first proof of Proposition 5.3.2. The number of times $j \in [n]$ appears in the sequence $\gamma(\sigma)$ is clearly equal to $\deg j$, since j is the predecessor of exactly $\deg j$ vertices v_i. Hence for fixed root set S,

$$\sum_{\sigma \in \mathcal{T}_{n,S}} x_1^{\deg 1} \cdots x_n^{\deg n} = (x_1 + \cdots + x_n)^{n-k-1}\left(\sum_{i \in S} x_i\right).$$

Now sum over all $S \in \binom{[n]}{k}$ to obtain (5.47). □

Second proof. Now consider the second proof of Proposition 5.3.2. The key observation here is that for each $j \in [n]$, the degree of vertex j in the planted forest σ is equal to the indegree of j in the digraph $D(\sigma, i)$, or equivalently, $\deg j = \#f^{-1}(j)$. Hence

$$n \sum_{\sigma} x_1^{\deg 1} \cdots x_n^{\deg n} = k \sum_{\substack{B \subseteq [n] \\ \#B = n-k}} \sum_{f: B \to [n]} x_1^{\#f^{-1}(1)} \cdots x_n^{\#f^{-1}(n)}, \tag{5.48}$$

where σ ranges over all k-component planted forests on $[n]$. The inner sum in the right-hand side of (5.48) is independent of B and is equal to $(x_1 + \cdots + x_n)^{n-k}$. Hence

$$n \sum_{\sigma} x_1^{\deg 1} \cdots x_n^{\deg n} = k \binom{n}{k} (x_1 + \cdots + x_n)^{n-k},$$

which is equivalent to (5.47). □

There is an alternative way of stating Theorem 5.3.4 which is sometimes more convenient. Given a planted forest σ, define the *type* of σ to be the sequence

$$\text{type } \sigma = (r_0, r_1, \dots),$$

where r_i vertices of σ have degree i. We also write type $\sigma = (r_0, r_1, \dots, r_m)$ if $r_j = 0$ for $j > m$. It follows easily from (5.46) that a sequence $\boldsymbol{r} = (r_0, r_1, \dots)$ of nonnegative integers is the type of some planted forest with n vertices and k components if and only if

$$\sum r_i = n, \quad \sum (i-1) r_i = -k. \tag{5.49}$$

5.3.5 Corollary. *Let* $\mathbf{r} = (r_0, r_1, \dots)$ *be a sequence of nonnegative integers satisfying (5.49). Then the number* $M(\boldsymbol{r})$ *of planted forests* σ *on the vertex set* $[n]$ *(necessarily with k components) of type* $\mathbf{r}$ *is given by*

$$\begin{aligned} M(\boldsymbol{r}) &= \binom{n-1}{k-1} \frac{(n-k)!}{0!^{r_0} 1!^{r_1} \cdots} \binom{n}{r_0, r_1, \dots} \\ &= \frac{k}{n} \binom{n}{k} \frac{(n-k)!}{0!^{r_0} 1!^{r_1} \cdots} \binom{n}{r_0, r_1, \dots}. \end{aligned}$$

We have been considering up to now the case of *labelled trees*, that is, trees whose vertices are distinguishable. We next will deal with *unlabelled plane forests* σ, so the vertices of σ are regarded as indistinguishable, but the subtrees at any vertex (as well as the components themselves of σ) are linearly ordered. This automatically makes the vertices of σ distinguishable (in other words, an unlabelled plane forest has only the trivial automorphism), so it really makes no difference whether or not the vertices of σ are labelled. (An unlabelled plane forest with n vertices has $n!$ labellings.) Thus all plane forests will henceforth be assumed to be unlabelled. We continue to define the *degree* of a vertex v to be the number of successors (children) of v, and the *type* of σ is $\boldsymbol{r} = (r_0, r_1, \dots)$ if r_i vertices have degree i. Equation (5.49) continues to be the condition on nonnegative integers $r_0, r_1, \dots$ for there to exist a plane forest with n vertices,

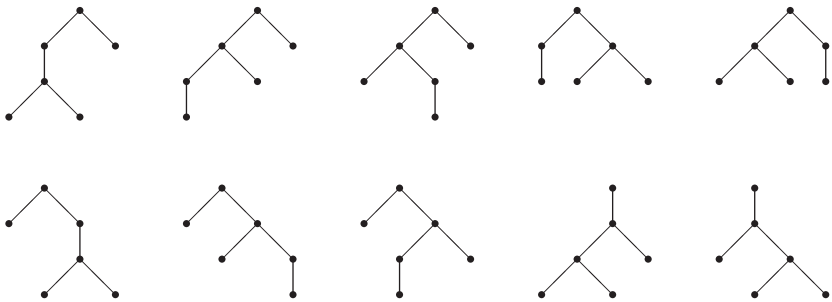

Figure 5.11 The ten plane trees of type $(3, 1, 2)$

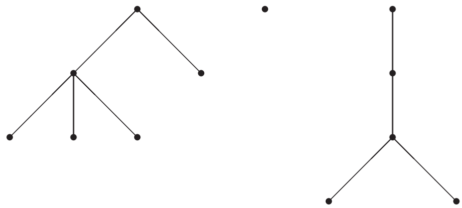

Figure 5.12 A plane forest of type $(7, 2, 2, 1)$

k components, and type $\boldsymbol{r} = (r_0, r_1, \ldots)$. Thus, for example, Figure 5.11 shows the ten plane trees of type $(3, 1, 2)$, while Figure 5.12 illustrates a plane forest with 12 vertices, 3 components, and type $(7, 2, 2, 1)$.

Our goal here is to enumerate unlabelled plane forests of a given type. This result will be used in the next section to prove the Lagrange inversion formula. It is convenient to work in the context of words in free monoids, as discussed in Section 4.7. Our alphabet $\mathcal{A}$ will consist of letters $x_0, x_1, x_2, \ldots$ (For plane forests with maximum degree m, it will suffice to take $\mathcal{A} = \{x_0, \ldots, x_m\}$.) The empty word is denoted by 1. Define the *weight* $\phi(x_i)$ of the letter x_i by $\phi(x_i) = i - 1$, and extend ϕ to $\mathcal{A}^*$ by

$$\phi(w_1 w_2 \cdots w_j) = \phi(w_1) + \phi(w_2) + \cdots + \phi(w_j),$$

where each $w_i \in \mathcal{A}$. (Set $\phi(1) = 0$.) Define a subset $\mathcal{B} \subset \mathcal{A}^*$ by

$$\mathcal{B} = \{w \in \mathcal{A}^* : \phi(w) = -1; \text{and if } w = uv \text{ where } v \neq 1, \text{then } \phi(u) \geq 0\}. \tag{5.50}$$

The elements of $\mathcal{B}$ are called *Łukasiewicz words*; see Example 6.6.7 for further information.

5.3.6 Lemma. *The monoid $\mathcal{B}^*$ generated by $\mathcal{B}$ is very pure (and hence free) with basis $\mathcal{B}$. (See Chapter 4.7 for relevant definitions.)*

Proof. Let $w = w_1 \cdots w_m \in \mathcal{B}^*$, where $w_i \in \mathcal{A}$. Let j be the least integer for which $\phi(w_1 \cdots w_j) < 0$, so in fact $\phi(w_1 \cdots w_j) = -1$ and $u = w_1 \cdots w_j \in \mathcal{B}$. Clearly if $w = vv'$ with $v \in \mathcal{B}$ then $u = v$. Thus by induction on the length of w, we obtain a unique factorization of w into elements of $\mathcal{B}$, so $\mathcal{B}^*$ is free with basis $\mathcal{B}$.

To show that $\mathcal{B}^*$ is very pure, it suffices to show [why?] that we cannot have $u, v, w \in \mathcal{A}^+ := \mathcal{A}^* - \{1\}$ with $uv \in \mathcal{B}$ and $vw \in \mathcal{B}$. But if $uv \in \mathcal{B}$ then $\phi(u) \geq 0$ and $\phi(u) + \phi(v) = -1$, so $\phi(v) < 0$. This contradicts $vw \in \mathcal{B}$, so $\mathcal{B}^*$ is very pure. $\square$

Recall from Chapter 4.7 that if $w = w_1 w_2 \cdots w_m \in \mathcal{A}^*$ with $w_i \in \mathcal{A}$, then a cyclic shift $w_i w_{i+1} \cdots w_m w_1 \cdots w_{i-1}$ of w is called a *conjugate* (or $\mathcal{A}$-conjugate if there is a possibility of confusion) of w. (The reason for this terminology is that in a group G, the elements $w_1 w_2 \cdots w_m$ and $w_i w_{i+1} \cdots w_{i-1}$ are conjugate in the usual group-theoretic sense.)

5.3.7 Lemma. *A word $w \in \mathcal{A}^*$ is a conjugate of a word in $\mathcal{B}^*$ if and only if $\phi(w) < 0$.*

First proof. Since $\phi(w)$ is unaffected by conjugation, clearly $\phi(w) < 0$ for every conjugate of a word in $\mathcal{B}^*$. We show the converse by induction on the length (in $\mathcal{A}^*$) $\ell(w)$ of w. The assertion is clear for $\ell(w) = 0$ (so $w = 1$), so assume it for words of length $< m$ and let $w = w_1 \cdots w_m$ where $w_i \in \mathcal{A}$ and $\phi(w) < 0$. Since $\phi(w_1) + \cdots + \phi(w_m) < 0$ and since $\phi(w_i) < 0$ only when $\phi(w_i) = -1$, it is easily seen that some conjugate w' of w has the form $w' = x_{s+1} x_0^s v$ for some $s \geq 0$. Since $\phi(v) = \phi(w') < 0$, it follows by induction that some conjugate v' of v lies in $\mathcal{B}^*$. Specifically, say that $v = yz$ where $zy \in \mathcal{B}^*$ and $y \neq 1$ (so that if v itself is in $\mathcal{B}^*$, then we take $y = v$ and $z = 1$). But then it is easily seen that $z x_{s+1} x_0^s y \in \mathcal{B}^*$. Since $z x_{s+1} x_0^s y$ is a conjugate of w, the proof follows by induction. $\square$

Second proof (sketch). The previous proof was straightforward but not particularly enlightening. We sketch another proof based on geometrical considerations which is more intuitive. Given any word $u = u_1 \cdots u_m \in \mathcal{A}^*$, with $u_i \in \mathcal{A}$, associate with u a lattice path $LP(u)$ with m steps in $\mathbb{R}^2$ as follows. Begin at $(0, 0)$, and let the i-th step, for $1 \leq i \leq m$, be $(1, \phi(u_i))$. Now suppose $w \in \mathcal{A}^*$ and $\phi(w) < 0$, and consider the path $LP(w^2)$. Figure 5.13 illustrates $LP(w^2)$ for $w = x_0 x_1 x_0^2 x_2 x_0^2 x_2 x_0 x_3$. Suppose $\phi(w) = -k$. Let B be the leftmost lowest point on $LP(w^2)$, and let A be the leftmost point which is exactly k levels higher than B. (See Figure 5.13.) The horizontal distance between A and B is exactly m. If we translate the part of $LP(w^2)$ between A and B so that A is at

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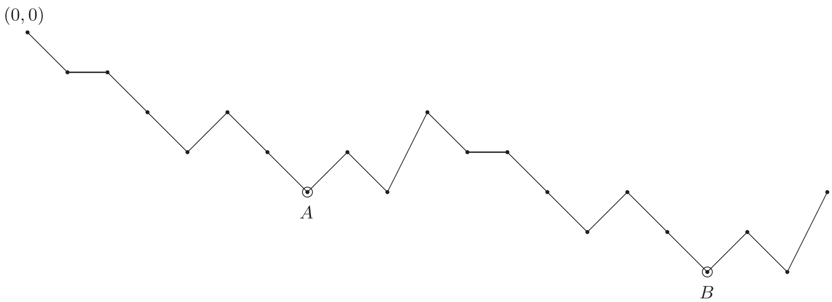

Figure 5.13 A lattice path $LP(w^2)$

the origin, then the resulting path is equal to $LP(v)$, where v is a conjugate of w belonging to $\mathcal{B}^*$. □

5.3.8 Example. Let $w = x_0x_1x_0^2x_2x_0^2x_2x_0x_3$ as in the second proof above. From Figure 5.13 we see that if A is translated to $(0,0)$ then the path between A and B is $LP(v)$ where $v = x_2x_0x_3x_0x_1x_0^2x_2x_0^2$. The unique factorization of v into elements of $\mathcal{B}$ is

$$v = (x_2x_0x_3x_0x_1x_0^2)(x_2x_0^2).$$

Since $\phi(w) = -2$ and $\mathcal{B}^*$ is very pure, there are precisely two conjugates of w which belong to $\mathcal{B}^*$, namely, v and

$$u = (x_2x_0^2)(x_2x_0x_3x_0x_1x_0^2).$$

In general, if $\phi(w) = -k$ then precisely k conjugates of w belong to $\mathcal{B}^*$. However, these conjugates might not be all distinct elements of $\mathcal{A}^*$. For instance, if $w = x_0^k$ then all k conjugates of w are equal to w.

We now wish to associate with an unlabelled plane forest σ with n vertices a word $w(\sigma)$ in $\mathcal{A}^*$ of length n (and weight $\phi(w(\sigma)) = -k$, where σ has k components). To do this, we first need to define a certain canonical linear ordering on the vertices of σ, called *depth first order* or *preorder*, and denoted $\mathrm{ord}(\sigma)$. It is defined recursively as follows:

(a) If σ has $k \geq 2$ components $\tau_1, \ldots, \tau_k$ (listed in the order defining σ as a plane forest), then set

$$\mathrm{ord}(\sigma) = \mathrm{ord}(\tau_1), \ \ldots, \ \mathrm{ord}(\tau_k) \text{ (concatenation of words).}$$

(b) If σ has one component, then let $\tau_1, \ldots, \tau_j$ be the subtrees of the root v (listed in the order defining σ as a plane tree). Set

$$\mathrm{ord}(\sigma) = v, \ \mathrm{ord}(\tau_1), \ \ldots, \ \mathrm{ord}(\tau_j) \text{ (concatenation of words).}$$

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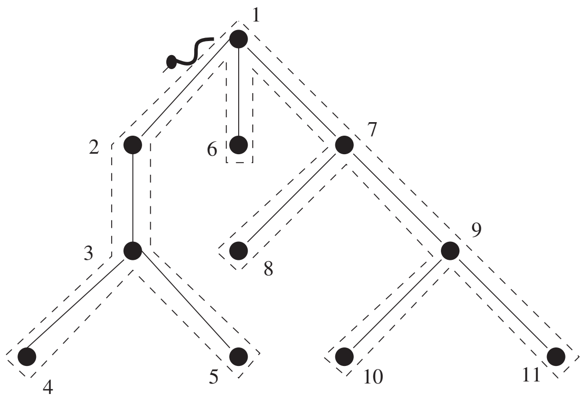

Figure 5.14 A plane tree traversed in preorder

The preorder on a plane tree has an alternative informal description as follows. Imagine that the edges of the tree are wooden sticks, and that a worm begins just left of the root and crawls along the outside of the sticks until (s)he (or it) returns to the starting point. Then the order in which vertices are seen for the first time is preorder. Figure 5.14 shows the path of the worm on a plane tree τ, with the vertices labelled 1 to 11 in preorder.

Given a plane forest σ, let $\operatorname{ord}(\sigma) = (v_1, \ldots, v_n)$, and set $\delta_i = \deg v_i$ (the number of successors of v_i). Now define a word $w(\sigma) \in \mathcal{A}^*$ by

$$w(\sigma) = x_{\delta_1} x_{\delta_2} \cdots x_{\delta_n}.$$

For the forest σ of Figure 5.12 we have

$$w(\sigma) = x_2 x_3 x_0^5 x_1^2 x_2 x_0^2,$$

while for the tree τ of Figure 5.14,

$$w(\tau) = x_3 x_1 x_2 x_0^3 x_2 x_0 x_2 x_0^2.$$

The following fundamental lemma has a fairly straightforward proof by induction and will be omitted here.

5.3.9 Lemma. *The map $\sigma \mapsto w(\sigma)$ is a bijection from the set of plane forests σ to $\mathcal{B}^*$.*

We now have all the ingredients necessary for our main result on plane forests.

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5.3.10 Theorem. *Let $\boldsymbol{r} = (r_0, r_1, \ldots, r_m) \in \mathbb{N}^{m+1}$, with $\sum r_i = n$ and $\sum(1-i)r_i = k > 0$. Then the number $P(\boldsymbol{r})$ of plane forests (necessarily with n vertices and k components) of type $\boldsymbol{r}$ (i.e., r_i vertices have i successors) is given by*

$$P(\boldsymbol{r}) = \frac{k}{n}\binom{n}{r_0, r_1, \ldots, r_m}.$$

First proof. The proof is an immediate consequence of Lemma 4.7.14, but for convenience we repeat the argument here. By Lemma 5.3.9, $P(\boldsymbol{r})$ is equal to the number of words $w \in \mathcal{B}^*$ with r_i x_i's for all i. (Regard $r_i = 0$ for $i > m$.) Denote by $\mathcal{B}^*_{\boldsymbol{r}}$ the set of all $P(\boldsymbol{r})$ such words, and similarly let $\mathcal{A}^*_{\boldsymbol{r}}$ be the set of all words in $\mathcal{A}^*$ with r_i x_i's for all i. Define a map $\psi : \mathcal{B}^*_{\boldsymbol{r}} \times [n] \to \mathcal{A}^*_{\boldsymbol{r}} \times [k]$ as follows. Let $w = w_1 w_2 \cdots w_n = u_1 u_2 \cdots u_k \in \mathcal{B}^*_{\boldsymbol{r}}$, where $w_i \in \mathcal{A}$ and $u_i \in \mathcal{B}$. Choose $i \in [n]$ and suppose w_i is a letter of u_j. Then set

$$\psi(w, i) = (w_i w_{i+1} \cdots w_{i-1}, j).$$

By Lemma 5.3.6 ψ is injective, while by Lemma 5.3.7 (and the fact that $\phi(w) = -k$ if $w \in \mathcal{B}^*_r$) ψ is surjective. Hence

$$nP(\boldsymbol{r}) = k(\#\mathcal{A}^*_{\boldsymbol{r}}).$$

But clearly by the formula (1.22) for $\#\mathfrak{S}(M)$ we have

$$\#\mathcal{A}^*_{\boldsymbol{r}} = \binom{n}{r_0, r_1, \ldots, r_m}, \tag{5.51}$$

and the proof follows. □

Second proof. Let $w \in \mathcal{A}^*_{\boldsymbol{r}}$ (as defined in the first proof), and let $C(w)$ be the set of all *distinct* conjugates of w. If $\#C(w) = m$ then m divides n, and every element of $C(w)$ occurs exactly n/m times among the n conjugates of w. It follows from Lemma 5.3.6 that exactly k conjugates of w belong to $\mathcal{B}^*$. Hence there are exactly $(k/n)m$ *distinct* conjugates of w belonging to $\mathcal{B}^*$, so the total number of distinct conjugates of elements of $\mathcal{A}^*_{\boldsymbol{r}}$ belonging to $\mathcal{B}^*$ is $(k/n)(\#\mathcal{A}^*_{\boldsymbol{r}})$. The proof follows from Lemma 5.3.9 and (5.51). □

The situation of the previous proof is simplest when $k = 1$. Here the n conjugates $w_i w_{i+1} \cdots w_{i-1}$ of $w = w_1 w_2 \cdots w_n \in \mathcal{A}^*_{\boldsymbol{r}}$ are all distinct, and exactly one of them lies in $\mathcal{B}^*$. Thus $P(\boldsymbol{r}) = (1/n)(\#\mathcal{A}^*_{\boldsymbol{r}})$. The fact that the conjugates of w are all distinct may be seen directly from the formula $\sum(i-1)r_i = -k$, since if $w = v^p$ then $p|r_i$ for all i, so $p|k$.

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5.3.11 Example. How many plane trees have three endpoints, one vertex of degree one, and two of degree two? This is the case $\mathbf{r} = (3, 1, 2)$. Since $\sum(i - 1)r_i = -1 \cdot 3 + 0 \cdot 1 + 1 \cdot 2 = -1$, such trees exist; and

$$P(\boldsymbol{r}) = \frac{1}{6}\binom{6}{3, 1, 2} = 10,$$

in agreement with Figure 5.11.

5.3.12 Example. How many plane binary trees τ have $n + 1$ endpoints? ("Binary" means here that every non-endpoint vertex has two successors. Without the adjective "plane", "binary" has a different meaning as explained in Appendix A of Volume 1.) One sees easily that τ has exactly n vertices of degree two. Hence $\boldsymbol{r} = (n + 1, 0, n)$, and

$$P(\boldsymbol{r}) = \frac{1}{2n + 1}\binom{2n + 1}{n} = \frac{1}{n + 1}\binom{2n}{n},$$

a Catalan number. These numbers made several appearances in Volume 1 and will be discussed in more detail in the next chapter (see in particular Exercise 6.19). Note that in the context of the second proof of Theorem 5.3.10, we obtain the expression $\frac{1}{2n+1}\binom{2n+1}{n}$ because there are $\binom{2n+1}{n}$ sequences of n 1's and $n + 1$ -1's, and each of them have $2n+1$ distinct conjugates of which exactly one has all its partial sums (except for the last sum) nonnegative. Alternatively, there are $\binom{2n}{n}$ sequences of n 1's and $n + 1$ -1's that end with a -1. Each of them has $n + 1$ distinct conjugates ending with a -1, of which exactly one has all partial sums nonnegative except for the last partial sum. This gives directly the expression $\frac{1}{n+1}\binom{2n}{n}$ for the number of plane binary trees with $n+1$ endpoints.

5.4 The Lagrange inversion formula

The set $xK[[x]]$ of all formal power series $a_1x + a_2x^2 + \cdots$ with zero constant term over a field K forms a monoid under the operation of functional composition. The identity element of this monoid is the power series x. Recall from Example 5.2.5 that if $f(x) = a_1x + a_2x^2 + \cdots \in K[[x]]$, then we call a power series $g(x)$ a *compositional inverse* of f if $f(g(x)) = g(f(x)) = x$, in which case we write $g(x) = f^{\langle -1\rangle}(x)$. The following simple proposition explains when $f(x)$ has a compositional inverse.

5.4.1 Proposition. *A power series $f(x) = a_1x + a_2x^2 + \cdots \in K[[x]]$ has a compositional inverse $f^{\langle -1\rangle}(x)$ if and only if $a_1 \neq 0$, in which case $f^{\langle -1\rangle}(x)$ is*

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unique. Moreover, if $g(x) = b_1x + b_2x^2 + \cdots$ *satisfies either* $f(g(x)) = x$ *or* $g(f(x)) = x$*, then* $g(x) = f^{\langle -1\rangle}(x)$*.*

Proof. Assume that $g(x) = b_1x + b_2x^2 + \cdots$ satisfies $f(g(x)) = x$. We then have

$$a_1(b_1x+b_2x^2+b_3x^3+\cdots)+a_2(b_1x+b_2x^2+\cdots)^2+a_3(b_1x+\cdots)^3+\cdots = x.$$

Equating coefficients on both sides yields the infinite system of equations

$$\begin{aligned} a_1b_1 &= 1 \\ a_1b_2 + a_2b_1^2 &= 0 \\ a_1b_3 + 2a_2b_1b_2 + a_3b_1^3 &= 0 \\ &\vdots \end{aligned}$$

We can solve the first equation (uniquely) for b_1 if and only if $a_1 \neq 0$. We can then solve the second equation uniquely for b_2, the third for b_3, etc. Hence $g(x)$ exists if and only if $a_1 \neq 0$, in which case it is unique. The remaining assertions are special cases of the fact that in a group every left or right inverse is a two-sided inverse. For the present situation, suppose for instance that $f(g(x)) = x$ and $h(f(x)) = x$. Substitute $g(x)$ for x in the second equation to get $h(x) = g(x)$, etc. $\square$

In some cases the equation $y = f(x)$ can be solved directly for x, yielding $x = f^{\langle -1\rangle}(y)$. For instance, one can verify in this way that

$$(e^x - 1)^{\langle -1\rangle} = \log(1+x)$$

$$\left(\frac{a+bx}{c+dx}\right)^{\langle -1\rangle} = \frac{-a+cx}{b-dx}, \text{ if } ad \neq bc.$$

In most cases, however, a simple explicit formula for $f^{\langle -1\rangle}(x)$ will not exist. We can still ask if there is a nice formula or combinatorial interpretation of the *coefficients* of $f^{\langle -1\rangle}(x)$. For instance, from (5.44) we have

$$(xe^{-x})^{\langle -1\rangle} = \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}. \tag{5.52}$$

Recall that we are always assuming that $\operatorname{char} K = 0$. With this assumption, the Lagrange inversion formula will express the coefficients of $f^{\langle -1\rangle}(x)$ in terms of coefficients of certain other power series. This will allow us to derive results such as (5.52) in a routine, systematic way. Somewhat more generally, we obtain an expression for the coefficients of $(f^{\langle -1\rangle}(x))^k$ for any $k \in \mathbb{Z}$. In effect this determines $g(f^{\langle -1\rangle}(x))$ for any $g(x)$, since if $g(x) = \sum b_k x^k$ then $g(f^{\langle -1\rangle}(x)) = \sum b_k (f^{\langle -1\rangle}(x))^k$.

We will give three proofs of the Lagrange inversion formula. The first proof is a direct algebraic argument. The second proof regards power series as *ordinary* generating functions and is based on our enumeration of plane forests (Theorem 5.3.10). Our final proof regards power series as *exponential* generating functions and is based on our enumeration of planted forests (Theorem 5.3.4). Thus we will give two combinatorial proofs of Lagrange inversion, one using (unlabelled) plane forests and the other (labelled) planted forests.

5.4.2 Theorem (the Lagrange inversion formula)**.** *Let* $F(x) = a_1x + a_2x^2 + \cdots \in xK[[x]]$*, where* $a_1 \neq 0$ *(and* $\operatorname{char} K = 0$*), and let* $k, n \in \mathbb{Z}$*. Then*

$$n[x^n]F^{\langle -1\rangle}(x)^k = k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n = k[x^{-k}]F(x)^{-n}. \tag{5.53}$$

(The second equality is trivial.) Equivalently, suppose $G(x) \in K[[x]]$ *with* $G(0) \neq 0$*, and let* $f(x)$ *be defined by*

$$f(x) = xG(f(x)). \tag{5.54}$$

Then

$$n[x^n]f(x)^k = k[x^{n-k}]G(x)^n. \tag{5.55}$$

NOTE 1. If $k < 0$ then $F^{\langle -1\rangle}(x)^k$ and $f(x)^k$ are Laurent series of the form $\sum_{i\geq k} p_i x^i$. Note also that if $n < k$ then both sides of (5.53) and (5.55) are 0.

NOTE 2. Equations (5.53) and (5.55) are equivalent since the statement that $f(x) = F^{\langle -1\rangle}(x)$ is easily seen to mean the same as $f(x) = xG(f(x))$ where $G(x) = x/F(x)$.

First proof of Theorem 5.4.2. The first proof is based on the following innocuous observation: If $y = \sum_{n\in\mathbb{Z}} c_n x^n$ is a Laurent series, then

$$[x^{-1}]y' = 0, \tag{5.56}$$

that is, the derivative of a Laurent series has no x^{-1} term.

Now set

$$F^{\langle -1\rangle}(x)^k = \sum_{i\geq k} p_i x^i,$$

so

$$x^k = \sum_{i\geq k} p_i F(x)^i.$$

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Differentiate both sides to obtain

$$kx^{k-1} = \sum_{i\geq k} ip_i F(x)^{i-1}F'(x)$$

$$\Rightarrow \frac{kx^{k-1}}{F(x)^n} = \sum_{i\geq k} ip_i F(x)^{i-n-1}F'(x). \tag{5.57}$$

Here we are expanding both sides of (5.57) as elements of $K((x))$, that is, as Laurent series with finitely many negative exponents. For instance,

$$\begin{aligned}\frac{kx^{k-1}}{F(x)^n} &= \frac{kx^{k-1}}{(a_1x + a_2x^2 + \cdots)^n} \\ &= kx^{k-n-1}(a_1 + a_2x + \cdots)^{-n}.\end{aligned}$$

We wish to take the coefficient of x^{-1} on both sides of (5.57). Since

$$F(x)^{i-n-1}F'(x) = \frac{1}{i-n}\frac{d}{dx}F(x)^{i-n}, i \neq n,$$

it follows from (5.56) that the coefficient of x^{-1} on the right-hand side of (5.57) is

$$\begin{aligned}[x^{-1}]np_nF(x)^{-1}F'(x) &= [x^{-1}]np_n\left(\frac{a_1 + 2a_2x + \cdots}{a_1x + a_2x^2 + \cdots}\right) \\ &= [x^{-1}]np_n\left(\frac{1}{x} + \cdots\right) \\ &= np_n.\end{aligned}$$

Hence

$$[x^{-1}]\frac{kx^{k-1}}{F(x)^n} = np_n = n[x^n]F^{\langle -1\rangle}(x)^k,$$

which is equivalent to (5.53). □

Second proof (only for $k \geq 1$). Let $t_0, t_1, \ldots$ be (commuting) indeterminates, and set

$$G(x) = t_0 + t_1x + \cdots .$$

If σ is a plane forest, set

$$t^\sigma = \prod_{i\geq 0} t_i^{r_i(\sigma)}, \tag{5.58}$$

where $r_i(\sigma)$ is the number of vertices of σ of degree i. Now set

$$s_n = \sum_\tau t^\tau,$$

summed over all plane trees with n vertices. For instance,

$$s_1 = t_0, \;\; s_2 = t_0 t_1, \;\; s_3 = t_0 t_1^2 + t_0^2 t_2.$$

Let

$$f(x) = \sum_{n \geq 1} s_n x^n. \tag{5.59}$$

If τ is a plane tree whose root has j subtrees, then τ is obtained by choosing j (nonempty) plane trees, arranging them in linear order, and adjoining a root of degree j attached to the roots of the j plane trees. Thus

$$t_j x f(x)^j = \sum_{n \geq 1} \left(\sum_{\tau} t^{\tau} \right) x^n \tag{5.60}$$

where τ runs over all plane trees with n vertices whose root is of degree j. Summing over all $j \geq 1$ yields

$$xG(f(x)) = f(x). \tag{5.61}$$

Now let $k \in \mathbb{P}$. By the definition (5.59) of $f(x)$, we have

$$f(x)^k = \sum_{n \geq 1} \left(\sum_{\sigma} t^{\sigma} \right) x^n, \tag{5.62}$$

where σ runs over all plane forests with n vertices and k components. On the other hand, from Theorem 5.3.10 we have

$$[x^n] f(x)^k = \sum_{\sigma} t^{\sigma} = \frac{k}{n} \sum_{r_0, r_1, \ldots} \binom{n}{r_0, r_1, \ldots} t_0^{r_0} t_1^{r_1} \cdots,$$

summed over all $\mathbb{N}$-sequences $r_0, r_1, \ldots$ satisfying $\sum r_i = n$ and $\sum (i-1) r_i = -k$, or equivalently $\sum r_i = n$ and $\sum i r_i = n - k$. But

$$\begin{aligned} G(x)^n &= (t_0 + t_1 x + \cdots)^n \\ &= \sum_{r_0 + r_1 + \cdots = n} \binom{n}{r_0, r_1, \ldots} t_0^{r_0} t_1^{r_1} \cdots x^{\Sigma i r_i}. \end{aligned}$$

Thus

$$[x^n] f(x)^k = \frac{k}{n} [x^{n-k}] G(x)^n,$$

which is equivalent to (5.55). Since $G(x)$ has "general coefficients" (i.e., independent indeterminates), the proof follows. $\square$

Note that this proof yields an explicit combinatorial formula (5.62) for the coefficients of $F^{\langle -1 \rangle}(x)^k = f(x)^k$ in terms of the coefficients of $x/F(x) = G(x)$.

Third proof of Theorem 5.4.2 (again only for $k \geq 1$). This proof is analogous to the previous proof, but instead of plane forests we use planted forests on $[n]$. Since the vertices are labelled (by elements of $[n]$), it is necessary to use exponential rather than ordinary generating functions. Thus we set

$$G(x) = \sum_{n \geq 0} t_n \frac{x^n}{n!}.$$

If σ is a planted forest on $[n]$, then let $r_i(\sigma)$ be the number of vertices of degree i, and as in (5.58) set $t^\sigma = \Pi t_i^{r_i(\sigma)}$. Now set

$$s_n = \sum_\tau t^\tau,$$

summed over all rooted trees on $[n]$, and let

$$\begin{aligned} f(x) &= \sum_{n \geq 1} s_n \frac{x^n}{n!} \\ &= t_0 x + 2t_0 t_1 \frac{x^2}{2!} + (6t_0 t_1^2 + 3t_0^2 t_2)\frac{x^3}{3!} + \cdots . \end{aligned}$$

If τ is a rooted tree on $[n]$ whose root has degree k, then τ is obtained by choosing a root $r \in [n]$ and then placing k rooted trees on the remaining vertices $[n] - \{r\}$. By Proposition 5.1.3, we have

$$f(x)^k = \sum_{n \geq 1} \left(\sum_\zeta t^\zeta \right) \frac{x^n}{n!},$$

where ζ runs over all *ordered* k-tuples of rooted trees with total vertex set $[n]$. Thus (since rooted trees are nonempty, so there are $k!$ ways to order k of them on $[n]$),

$$\frac{1}{k!} f(x)^k = \sum_{n \geq 1} \left(\sum_\sigma t^\sigma \right) \frac{x^n}{n!}, \tag{5.63}$$

where σ runs over all planted forests on $[n]$ with k components. Hence by Proposition 5.1.15 (equations (5.15) and (5.19)), we have that

$$\frac{t_k}{k!} x f(x)^k = \sum_{n \geq 1} \left(\sum_\zeta t^\zeta \right) \frac{x^n}{n!},$$

where now ζ runs over all rooted trees on $[n]$ whose root has degree k. Summing over all $k \geq 1$ yields, as in (5.61), $f(x) = xG(f(x))$.

Now let $k \in \mathbb{P}$. We have from (5.63) and Corollary 5.3.5 that

$$\left[\frac{x^n}{n!}\right]\frac{1}{k!}f(x)^k = \frac{k}{n}\binom{n}{k}\sum_{r_0,r_1,\ldots}\frac{(n-k)!t_0^{r_0}t_1^{r_1}\cdots}{0!^{r_0}1!^{r_1}\cdots}\binom{n}{r_0,r_1,\ldots},$$

summed over all $\mathbb{N}$-sequences $r_0, r_1, \ldots$ satisfying $\sum r_i = n$ and $\sum ir_i = n-k$. Equivalently,

$$[x^n]f(x)^k = \frac{k}{n}\sum_{\substack{r_0,r_1,\ldots\\ \Sigma r_i=n\\ \Sigma ir_i=n-k}}\binom{n}{r_0,r_1,\ldots}\frac{t_0^{r_0}t_1^{r_1}\cdots}{0!^{r_0}1!^{r_1}\cdots}.$$

But

$$\begin{aligned} G(x)^n &= \left(t_0 + t_1\frac{x}{1!} + t_2\frac{x^2}{2!} + \cdots\right)^n \\ &= \sum_{r_0+r_1+\cdots=n}\binom{n}{r_0,r_1,\ldots}\frac{t_0^{r_0}t_1^{r_1}\cdots}{0!^{r_0}1!^{r_1}\cdots}x^{\Sigma ir_i}. \end{aligned}$$

Thus

$$[x^n]f(x)^k = \frac{k}{n}[x^{n-k}]G(x)^n,$$

as desired. □

5.4.3 Corollary. *Preserve the notation of Theorem 5.4.2. Then for any power series $H(x) \in K[[x]]$ (or Laurent series $H(x) \in K((x))$) we have*

$$n[x^n]H(F^{\langle -1\rangle}(x)) = [x^{n-1}]H'(x)\left(\frac{x}{F(x)}\right)^n. \tag{5.64}$$

Equivalently,

$$n[x^n]H(f(x)) = [x^{n-1}]H'(x)G(x)^n, \tag{5.65}$$

where $f(x) = xG(f(x))$.

Proof. By linearity (for *infinite* linear combinations) it suffices to prove (5.64) or (5.65) for $H(x) = x^k$. But this is equivalent to (5.53) or (5.55). □

Let us consider some simple examples of the use of the Lagrange inversion formula. Additional applications appear in the exercises.

5.4.4 Example. We certainly should be able to deduce the formula

$$(xe^{-x})^{\langle -1\rangle} = \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}$$

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(equation (5.52)) directly from Theorem 5.4.2. Indeed letting $F(x) = xe^{-x}$ and $k = 1$ in (5.53) gives

$$\begin{aligned}[x^n](xe^{-x})^{\langle -1\rangle} &= \frac{1}{n}[x^{n-1}]e^{nx} \\ &= \frac{1}{n}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}.\end{aligned}$$

More generally, for any $k \in \mathbb{Z}$ we get

$$\begin{aligned}[x^n]\left((xe^{-x})^{\langle -1\rangle}\right)^k &= \frac{k}{n}[x^{n-k}]e^{nx} \\ &= \frac{k}{n}\cdot\frac{n^{n-k}}{(n-k)!}.\end{aligned} \tag{5.66}$$

Thus the number of k-component planted forests on $[n]$ is equal to

$$\frac{n!}{k!}\cdot\frac{k}{n}\cdot\frac{n^{n-k}}{(n-k)!} = \binom{n-1}{k-1}n^{n-k},$$

agreeing with Proposition 5.3.2. Note also that setting $k = -1$ in (5.66) yields

$$[x^n]\frac{1}{(xe^{-x})^{\langle -1\rangle}} = -\frac{n^n}{(n+1)!},\ \ n \geq -1\ \ (\text{with } 0^0 = 1).$$

Hence

$$\left(\sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}\right)^{-1} = -\sum_{n\geq -1} n^n\frac{x^n}{(n+1)!}.$$

A little rearranging yields the interesting identity

$$\left(1 - \sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right)^{-1} = \sum_{n\geq 0}(n+1)^{n-1}\frac{x^n}{n!}. \tag{5.67}$$

Compare Exercise 42.

5.4.5 Example. Let A be a subset of $\{2, 3, \ldots\}$. Let $t_A(n)$ denote the number of ways of beginning with an n-set S, then partitioning S into k blocks where $k \in A$, then partitioning each nonsingleton block into k blocks where $k \in A$, etc., until only singleton blocks remain. (In particular, we can never have a block whose cardinality is strictly between 1 and $\min A$.) Set $t_A(0) = 0$, and set $y = E_{t_A}(x)$. Then, as a generalization of both (5.26) and (5.28), we have

$$\sum_{n\in A}\frac{y^n}{n!} = y - x.$$

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Hence

$$y = \left(x - \sum_{k\in A} \frac{x^k}{k!}\right)^{\langle -1\rangle},$$

so by Theorem 5.4.2,

$$t_A(n) = \left[\frac{x^n}{n!}\right] y = \left[\frac{x^{n-1}}{(n-1)!}\right]\left(1 - \sum_{k\in A}\frac{x^{k-1}}{k!}\right)^{-n}.$$

When A consists of a single element k, then we have

$$\begin{aligned}\left(1 - \frac{x^{k-1}}{k!}\right)^{-n} &= \sum_{j\geq 0}\binom{n+j-1}{j}\frac{x^{j(k-1)}}{k!^j}\\ &= \sum_{j\geq 0}\binom{n+j-1}{j}\frac{(j(k-1))!}{k!^j}\frac{x^{j(k-1)}}{(j(k-1))!}.\end{aligned}$$

Thus (writing t_k for $t_{\{k\}}$) $t_k(n) = 0$ unless $n = j(k-1)+1$ for some $j \in \mathbb{N}$, and

$$\begin{aligned}t_k(j(k-1)+1) &= \binom{jk}{j}\frac{(j(k-1))!}{k!^j}\\ &= \frac{(jk)!}{j!\,k!^j}.\end{aligned}$$

A combinatorial proof can be given along the lines of Example 5.2.6.

5.5 Exponential structures

There are many possible generalizations of the compositional and exponential formulas (Theorem 5.1.4 and Corollary 5.1.6). We will consider here a generalization involving partially ordered sets much in the spirit of binomial posets (Chapter 3.18).

5.5.1 Definition. An *exponential structure* is a sequence $\boldsymbol{Q} = (Q_1, Q_2, \ldots)$ of posets satisfying the following three axioms:

(E1) For each $n \in \mathbb{P}$, Q_n is finite and has a unique maximal element $\hat{1}_n$ (denoted simply by $\hat{1}$), and every maximal chain of Q_n has n elements (or length $n-1$).

(E2) If $\pi \in Q_n$, then the interval $[\pi, \hat{1}]$ is isomorphic to Π_k (the lattice of partitions of $[k]$) for some k. We then write $|\pi| = k$. Thus if $|\pi| = k$, then every saturated chain from π to $\hat{1}$ has k elements.

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(E3) Suppose $\pi \in Q_n$ and ρ is a minimal element of Q_n satisfying $\rho \leq \pi$. Thus by (E1) and (E2), $[\rho, \hat{1}] \cong \Pi_n$. It follows from Example 3.10.4 that $[\rho, \pi] \cong \Pi_1^{a_1} \times \Pi_2^{a_2} \times \cdots \times \Pi_n^{a_n}$ for unique $a_1, a_2, \ldots, a_n \in \mathbb{N}$ satisfying $\sum i a_i = n$ (and $\sum a_i = |\pi|$). We require that the subposet $\Lambda_\pi = \{\sigma \in Q_n : \sigma \leq \pi\}$ of Q_n be isomorphic to $Q_1^{a_1} \times Q_2^{a_2} \times \cdots \times Q_n^{a_n}$. In particular, if ρ' is another minimal element of Q_n satisfying $\rho' \leq \pi$, then $[\rho, \pi] \cong [\rho', \pi]$. We call $(a_1, a_2, \ldots, a_n)$ the *type* of π.

Intuitively, one should think of Q_n as forming a set of "decompositions" of some structure S_n of "size" n into "pieces" which are smaller S_i's. Then (E2) states that given a decomposition of S_n, one can take any partition of the pieces of the decomposition and join together the pieces in each block in a unique way to obtain a coarser decomposition. Moreover, (E3) states that each piece can be decomposed independently to form a finer decomposition.

If $\boldsymbol{Q} = (Q_1, Q_2, \ldots)$ is an exponential structure, then let $M(n)$ denote the number of minimal elements of Q_n. As will be seen below, all the basic combinatorial properties of $\boldsymbol{Q}$ can be deduced from the numbers $M(n)$. We call the sequence $\mathbf{M} = (M(1), M(2), \ldots)$ the *denominator sequence* of $\boldsymbol{Q}$. $M(n)$ turns out to play a role for exponential structures analogous to that of the factorial function of a binomial poset (see Definition 3.18.2(c)).

We now proceed to some examples of exponential structures.

5.5.2 Example. (a) The prototypical example of an exponential structure is given by $Q_n = \Pi_n$. In this case we have $M(n) = 1$.

(b) Let $V_n = V_n(q)$ be an n-dimensional vector space over the finite field $\mathbb{F}_q$. Let Q_n consist of all collections $\{W_1, W_2, \ldots, W_k\}$ of subspaces of V_n such that $\dim W_i > 0$ for all i, and such that $V_n = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ (direct sum). An element of Q_n is called a *direct sum decomposition* of V_n. We order Q_n in an obvious way by refinement, namely, $\{W_1, W_2, \ldots, W_k\} \leq \{W_1', W_2', \ldots, W_j'\}$ if each W_r is contained in some W_s'. It is easily seen that $(Q_1, Q_2, \ldots)$ is an exponential structure with

$$M(n) = q^{\binom{n}{2}} (\boldsymbol{n})!/n!,$$

where $(\boldsymbol{n})! = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$ as in Chapter 1.3.

(c) Let $\boldsymbol{Q} = (Q_1, Q_2, \ldots)$ be an exponential structure with denominator sequence $\boldsymbol{M} = (M(1), M(2), \ldots)$. Fix $r \in \mathbb{P}$, and define $Q_n^{(r)}$ to be the subposet of Q_{rn} consisting of all $\pi \in Q_{rn}$ of type $(a_1, a_2, \ldots, a_{rn})$ such that $a_i = 0$ unless r divides i. Then $\boldsymbol{Q}^{(r)} = (Q_1^{(r)}, Q_2^{(r)}, \ldots)$ is an exponential structure with denominator sequence $\boldsymbol{M}^{(r)} = (M_r(1), M_r(2), \ldots)$ given by

$$M_r(n) = \frac{M(rn)(rn)!}{M(r)^n\, n!\, r!^n}. \tag{5.68}$$

(Equation (5.68) can be seen by a direct argument and is also a special case of Lemma 5.5.3.) For instance, if $\boldsymbol{Q} = \boldsymbol{\Pi} = (\Pi_1, \Pi_2, \dots)$, then $\Pi_n^{(r)}$ consists of all partitions of $[rn]$ whose block sizes are divisible by r.

(d) Let $r \in \mathbb{P}$, and let S be an n-set. An *r-partition of S* is a set

$$\pi = \{(B_{11}, B_{12}, \dots, B_{1r}), (B_{21}, B_{22}, \dots, B_{2r}), \dots, (B_{k1}, B_{k2}, \dots, B_{kr})\}$$

satisfying:

(i) For each $j \in [r]$, the set $\pi_j = \{B_{1j}, B_{2j}, \dots, B_{kj}\}$ forms a partition of S (into k blocks), and
(ii) For fixed i, $\#B_{i1} = \#B_{i2} = \cdots = \#B_{ir}$.

The set $Q_n = Q_n(S)$ of all r-partitions of S has an obvious partial ordering by refinement which makes $(Q_1, Q_2, \dots)$ into an exponential structure with $M(n) = n!^{r-1}$. (A minimal element ρ of $Q_n(S)$ may be identified with an $(r-1)$-tuple $(w_1, \dots, w_{r-1})$ of permutations $w_i \in \mathfrak{S}(S)$ (the group of all permutations of the set S) *via*

$$\rho = \{(x, w_1(x), \dots, w_{r-1}(x)), (y, w_1(y), \dots, w_{r-1}(y)), \dots\},$$

where $S = \{x, y, \dots\}$, and where we abbreviate a one-element set $\{z\}$ as z.) The type $(a_1, a_2, \dots, a_n)$ of $\pi \in Q_n$ is equal to the type of any of the partitions π_j, that is, π_j has a_i blocks of size i. (By (ii), all the π_j's have the same type.)

The basic combinatorial properties of exponential structures will be obtained from the following lemma.

5.5.3 Lemma. *Let $\boldsymbol{Q} = (Q_1, Q_2, \dots)$ be an exponential structure with denominator sequence $(M(1), M(2), \dots)$. Then the number of $\pi \in Q_n$ of type $(a_1, a_2, \dots, a_n)$ is equal to*

$$\frac{n!\, M(n)}{1!^{a_1} \cdots n!^{a_n} a_1! \cdots a_n!\, M(1)^{a_1} \cdots M(n)^{a_n}}.$$

Proof. Let $N = N(a_1, \dots, a_n)$ be the number of pairs (ρ, π) where ρ is a minimal element of Q_n such that $\rho \le \pi$ and type $\pi = (a_1, \dots, a_n)$. On the one hand we can pick ρ in $M(n)$ ways, and then pick $\pi \ge \rho$. The number of choices for π is the number of elements of Π_n of type $(a_1, \dots, a_n)$,

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which is easily seen to equal (e.g., by a simple variation of Proposition 1.3.2) $n!/\,(1!^{a_1}\cdots n!^{a_n}a_1!\cdots a_n!)$. Hence

$$N = \frac{n!\,M(n)}{1!^{a_1}\cdots n!^{a_n}a_1!\cdots a_n!}. \tag{5.69}$$

On the other hand, if K is the desired number of $\pi \in Q_n$ of type $(a_1,\ldots,a_n)$, then we can pick π in K ways and then choose $\rho \le \pi$. Since Q_n has $M(n)$ minimal elements, the poset $\Lambda_\pi \cong Q_1^{a_1} \times \cdots \times Q_n^{a_n}$ has $M(1)^{a_1}\cdots M(n)^{a_n}$ minimal elements. Hence there are $M(1)^{a_1}\cdots M(n)^{a_n}$ choices for ρ, so

$$N = K\cdot M(1)^{a_1}\cdots M(n)^{a_n}. \tag{5.70}$$

The proof follows from (5.69) and (5.70). □

We come to the main result of this section.

5.5.4 Theorem (the compositional formula for exponential structures)**.** *Let $(Q_1,Q_2,\ldots)$ be an exponential structure with denominator sequence $(M(1),M(2),\ldots)$. Given functions $f:\mathbb{P}\to K$ and $g:\mathbb{N}\to K$ with $g(0)=1$, define a new function $h:\mathbb{N}\to K$ by*

$$\begin{aligned} h(n) &= \sum_{\pi\in Q_n} f(1)^{a_1}f(2)^{a_2}\cdots f(n)^{a_n}g(|\pi|),\ n\ge 1,\\ h(0) &= 1, \end{aligned}$$

where type $\pi = (a_1,a_2,\ldots,a_n)$(so $|\pi| = a_1+a_2+\cdots+a_n$). Define formal power series $F,G,H\in K[[x]]$ by

$$\begin{aligned} F(x) &= \sum_{n\ge 1} f(n)\frac{x^n}{n!\,M(n)}\\ G(x) &= E_g(x) = \sum_{n\ge 0} g(n)\frac{x^n}{n!}\\ H(x) &= \sum_{n\ge 0} h(n)\frac{x^n}{n!\,M(n)}. \end{aligned}$$

Then $H(x) = G(F(x))$.

Proof. By Theorem 5.1.4, we have

$$\left[\frac{x^n}{n!M(n)}\right]G(F(x)) = M(n)\sum_{\pi\in\Pi_n}\left(\frac{f(1)}{M(1)}\right)^{a_1}\cdots\left(\frac{f(n)}{M(n)}\right)^{a_n} g(|\pi|), \tag{5.71}$$

where type $\pi = (a_1, \ldots, a_n)$. Write $t(Q_n; a_1, a_2, \ldots)$ for the number of $\pi \in Q_n$ of type $(a_1, a_2, \ldots)$. By Lemma 5.5.3 we have

$$\frac{t(Q_n; a_1, a_2, \ldots)}{t(\Pi_n; a_1, a_2, \ldots)} = \frac{M(n)}{M(1)^{a_1} \cdots M(n)^{a_n}}.$$

Hence (5.71) may be rewritten

$$\left[\frac{x^n}{n!M(n)}\right] G(F(x)) = \sum_{\pi \in Q_n} f(1)^{a_1} \cdots f(n)^{a_n} g(|\pi|),$$

as desired. □

Putting $g(n) = 1$ for all $n \geq 0$ yields:

5.5.5 Corollary (the exponential formula for exponential structures)**.** *Let $(Q_1, Q_2, \ldots)$ be an exponential structure with denominator sequence $(M(1), M(2), \ldots)$. Given a function $f : \mathbb{P} \to K$, define a new function $h : \mathbb{N} \to K$ by*

$$h(n) = \sum_{\pi \in Q_n} f(1)^{a_1} \cdots f(n)^{a_n}, \; n \geq 1,$$
$$h(0) = 1,$$

where type $\pi = (a_1, \ldots, a_n)$. Define $F(x)$ and $H(x)$ as in Theorem 5.5.4. Then

$$H(x) = \exp F(x).$$

Let us turn to some examples of the use of Corollary 5.5.5.

5.5.6 Example. Let $(Q_1, Q_2, \ldots)$ be an exponential structure with denominator sequence $(M(1), M(2), \ldots)$, and write $q(n) = \#Q_n$. Letting $f(i) = 1$ for all i in Corollary 5.5.5 yields $h(n) = q(n)$, so

$$\sum_{n \geq 0} q(n) \frac{x^n}{n!M(n)} = \exp \sum_{n \geq 1} \frac{x^n}{n!M(n)}.$$

For instance, if $n!M(n) = q^{\binom{n}{2}}\boldsymbol{(n)!}$, then by Example 5.5.2(b) we have that $q(n)$ is the number of ways to express $V_n(q)$ as a direct sum (without regard to order) of nontrivial subspaces.

More generally, let $S_{\boldsymbol{Q}}(n,k)$ denote the number of $\pi \in Q_n$ satisfying $|\pi| = k$ (so for $\boldsymbol{Q} = \boldsymbol{\Pi}$, $S_{\mathbf{Q}}(n,k)$ becomes the Stirling number $S(n,k)$ of the second kind). Define a polynomial

$$W_n(t) = \sum_{\pi \in Q_n} t^{|\pi|} = \sum_{k=1}^{n} S_{\boldsymbol{Q}}(n,k) t^k, \tag{5.72}$$

with $W_0(t) = 1$. Putting $f(i) = 1$ and $g(k) = t^k$ in Theorem 5.5.4 (or $f(i) = t$ in Corollary 5.5.5) leads to

$$\sum_{n\geq 0} W_n(t)\frac{x^n}{n!M(n)} = \exp\left(t\sum_{n\geq 1}\frac{x^n}{n!M(n)}\right), \tag{5.73}$$

which is analogous to Example 5.2.2.

5.5.7 Example. We now consider a generalization of the previous example. Let $r \in \mathbb{P}$, and define a polynomial

$$\begin{aligned} P_n(r,t) &= \sum_{\pi_1\leq\cdots\leq\pi_r} t^{|\pi_r|}, \quad n \geq 1, \\ P_0(r,t) &= 1, \end{aligned}$$

where the sum ranges over all r-element multichains in Q_n. In particular, $P_n(1,t) = W_n(t)$ and $P_n(r,1) = Z(Q_n, r+1)$, where $Z(Q_n,\cdot)$ is the zeta polynomial of Q_n (see Chapter 3.12). Now let $\bar{Q}_n$ denote Q_n with a $\hat{0}$ adjoined, and let ζ denote the zeta function of $\bar{Q}_n$ (as defined in Chapter 3.6). Then clearly for $n \geq 1$,

$$P_n(r,t) = \sum_{\pi\in Q_n}\left[\zeta^r(\hat{0},\pi) - \zeta^{r-1}(\hat{0},\pi)\right]t^{|\pi|}. \tag{5.74}$$

The right-hand side makes sense for any $r \in \mathbb{Z}$ and thus yields an interpretation of $P_n(r,t)$ for $r \leq 0$. In particular, since $\zeta^0(\hat{0},\pi) = 0$ for all $\pi \in Q_n$, putting $r = 0$ in (5.74) yields

$$\begin{aligned} P_n(0,t) &= -\sum_{\pi\in Q_n}\mu_n(\hat{0},\pi)t^{|\pi|} \\ &= t^{n+1} - t\chi(\bar{Q}_n, t), \end{aligned} \tag{5.75}$$

where μ_n denotes the Möbius function and χ the characteristic polynomial (as defined in Chapter 3.10) of $\bar{Q}_n$. Note that

$$\mu_n := \mu_n(\hat{0},\hat{1}) = -[t]P_n(0,t) = -\frac{d}{dt}P_n(0,t)\mid_{t=0}. \tag{5.76}$$

Now put $f(i) = P_i(r-1,1)$ and $g(k) = t^k$ in Theorem 5.5.4 to deduce

$$\begin{aligned} \sum_{n\geq 0} P_n(r,t)\frac{x^n}{n!M(n)} &= \exp\left(t\sum_{n\geq 1}P_n(r-1,1)\frac{x^n}{n!M(n)}\right) \\ &= \left(\sum_{n\geq 0}P_n(r,1)\frac{x^n}{n!M(n)}\right)^t. \end{aligned} \tag{5.77}$$

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Note that from (5.75) we have

$$\begin{aligned} P_n(0,1) &= -\sum_{\pi\in Q_n} \mu_n(\hat{0},\pi) \\ &= \mu_n(\hat{0},\hat{0}) = 1, \end{aligned}$$

by the recurrence (3.15) for Möbius functions. (This also follows from putting $r = 1$ in (5.77) and comparing with (5.73).) Hence setting $r = 0$ in (5.77) yields

$$\sum_{n\geq 0} P_n(0,t)\frac{x^n}{n!M(n)} = \left(\sum_{n\geq 0}\frac{x^n}{n!M(n)}\right)^t.$$

Applying $\frac{d}{dt}$ to both sides and setting $t = 0$ yields from (5.77) that

$$-\sum_{n\geq 1}\mu_n\frac{x^n}{n!M(n)} = \log\left(\sum_{n\geq 0}\frac{x^n}{n!M(n)}\right). \tag{5.78}$$

For instance, suppose $Q_n = \Pi_n^{(2)}$, the poset of partitions of $[2n]$ with even block sizes (Example 5.5.2(c)). By (5.68) we have $M_2(n) = (2n)!/2^n n!$. Hence

$$-\sum_{n\geq 1}\mu_n\frac{2^n x^n}{(2n)!} = \log\left(\sum_{n\geq 0}\frac{2^n x^n}{(2n)!}\right).$$

Put $2x = y^2$ to obtain

$$-\sum_{n\geq 1}\mu_n\frac{y^{2n}}{(2n)!} = \log(\cosh y),$$

or equivalently (by applying $\frac{d}{dy}$),

$$\begin{aligned} -\sum_{n\geq 1}\mu_n\frac{y^{2n-1}}{(2n-1)!} &= \tanh y \\ &= \sum_{n\geq 1}(-1)^{n-1}E_{2n-1}\frac{y^{2n-1}}{(2n-1)!}, \end{aligned}$$

where E_{2n-1} denotes an Euler (or tangent) number (see Chapter 1.6.1). Thus for $Q_n = \Pi_n^{(2)}$, we have

$$\mu_n = (-1)^n E_{2n-1}.$$

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A primary reason for our discussion of exponential structures is to provide a general framework for extending our results on symmetric matrices with equal row and column sums (Examples 5.2.7–5.2.8) to arbitrary square matrices. (For rectangular matrices, see Exercise 65.) Thus let $\mathcal{M}(n, r)$ denote the set of all $n \times n$ $\mathbb{N}$-matrices $A = (A_{ij})$ for which every row and column sums to r. For instance, $\mathcal{M}(n, 0)$ consists of the $n \times n$ zero matrix, while $\mathcal{M}(n, 1)$ consists of the $n!$ $n \times n$ permutation matrices. We assume that the rows and columns of A are indexed by $[n]$. By a *k-component* of $A \in \mathcal{M}(n, r)$, we mean a pair (S, T) of nonempty subsets of $[n]$ satisfying the following two properties:

(i) $\#S = \#T = k$,
(ii) Let $A(S, T)$ be the $k \times k$ submatrix of A whose rows are indexed by S and whose columns are indexed by T, that is, $A(S, T) = (A_{ij})$, where $(i, j) \in S \times T$. Then every row and column of $A(S, T)$ sums to r, that is, $A(S, T) \in \mathcal{M}(k, r)$.

We call (S, T) a *component* of A if it is a k-component for some k. A component (S, T) is *irreducible* if any component (S', T') with $S' \subseteq S$ and $T' \subseteq T$ satisfies $(S', T') = (S, T)$. The matrix $A(S, T)$ is then also called *irreducible*. For instance, $(\{i\}, \{j\})$ is a 1-component (in which case it is irreducible) if and only if $A_{ij} = r$. It is easily seen that the set of irreducible components of A forms a 2-partition $\pi = \pi_A$ of $[n]$, as defined in Example 5.5.2(d). Conversely, we obtain (uniquely) a matrix $A \in \mathcal{M}(n, r)$ by choosing a 2-partition π of $[n]$ and then "attaching" an irreducible matrix to each block $(S, T) \in \pi$. There follows from Corollary 5.5.5 in the case where Q_n consists of the 2-partitions of $[n]$ the following result.

5.5.8 Proposition. *Let $h_r(a_1, \ldots, a_n)$ denote the number of matrices $A \in \mathcal{M}(n, r)$ such that A has a_i irreducible i-components (or equivalently, type $\pi_A = (a_1, \ldots, a_n)$). Let $f_r(n)$ be the number of irreducible $n \times n$ matrices $A \in \mathcal{M}(n, r)$. Then*

$$\sum_{n\geq 0} \sum_{a_1,\ldots,a_n} h_r(a_1, \ldots, a_n) t_1^{a_1} \cdots t_n^{a_n} \frac{x^n}{n!^2} = \exp \sum_{n\geq 1} f_r(n) t_n \frac{x^n}{n!^2}.$$

5.5.9 Corollary. *(a) Let $H(n, r) = \#\mathcal{M}(n, r)$. Then*

$$\sum_{n\geq 0} H(n, r) \frac{x^n}{n!^2} = \exp \sum_{n\geq 1} f_r(n) \frac{x^n}{n!^2}.$$

(b) Let $H^*(n,r)$ denote the number of matrices in $\mathcal{M}(n,r)$ with no entry equal to r. Then

$$\sum_{n\geq 0} H^*(n,r)\frac{x^n}{n!^2} = \exp \sum_{n\geq 2} f_r(n)\frac{x^n}{n!^2}$$

$$= e^{-x}\sum_{n\geq 0} H(n,r)\frac{x^n}{n!^2}. \tag{5.79}$$

Proof. (a) Put each $t_i = 1$ in Proposition 5.5.8.

(b) Since $(\{i\},\{j\})$ is an (irreducible) 1-component if and only if $A_{ij} = r$, the proof follows by setting $t_1 = 0, t_2 = t_3 = \cdots = 1$ in Proposition 5.5.8 (and noting that $f_r(1) = 1$). □

There is a simple graph-theoretic interpretation of the 2-partition π_A associated with the matrix $A \in \mathcal{M}(n,r)$. Let $X = \{x_1,\ldots,x_n\}$ and $Y = \{y_1,\ldots,y_n\}$, and define a bipartite graph $\Gamma(A)$ with vertex bipartition (X,Y) by placing A_{ij} edges (or a single edge weighted by A_{ij}) between x_i and y_j. Thus $\Gamma(A)$ is regular of degree r. Then the connected components of $\Gamma(A)$ correspond to the irreducible components of A. More precisely, if Γ' is a connected component of Γ, then define

$$\begin{aligned} S &= \{j : x_j \text{ is a vertex of } \Gamma'\} \\ T &= \{j : y_j \text{ is a vertex of } \Gamma'\}. \end{aligned}$$

Then (S,T) is an irreducible component of A (or block of π_A), and conversely all irreducible components of A are obtained in this way. Thus an irreducible $A \in \mathcal{M}(n,r)$ corresponds to a *connected* regular bipartite graph of degree r with $2n$ vertices. As an example, suppose

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \end{bmatrix}$$

The bipartite graph $\Gamma(A)$ is shown in Figure 5.15. The 2-partition π_A is given by

$$\pi = \{(234,146),(16,25),(5,3)\},$$

of type $(1,1,1)$.

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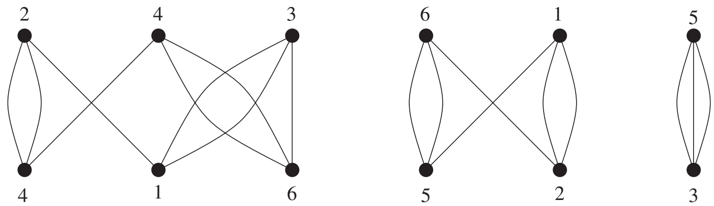

Figure 5.15 A bipartite graph $\Gamma(A)$

It is not difficult to compute $f_2(n)$. Indeed, an irreducible matrix $A \in \mathcal{M}(n, 2)$ is of the form $P + PQ$, where P is a permutation matrix and Q a cyclic permutation matrix. In graph-theoretic terms, $\Gamma(A)$ is a connected bipartite graph of degree two (and therefore a cycle of even length ≥ 2) with vertex bipartition (X, Y) where $\#X = \#Y = n$. There are easily seen to be $\frac{1}{2}(n-1)!\, n!$ such cycles for $n \geq 2$, and of course just one for $n = 1$. (Equivalently, there are $n!$ choices for P and $(n-1)!$ choices for Q. If $n > 1$ then P and PQ could have been chosen in reverse order.) There follows from Proposition 5.5.8:

5.5.10 Proposition. *We have*

$$\sum_{n \geq 0} \sum_{a_1, \ldots, a_n} h_2(a_1, \ldots, a_n) t_1^{a_1} \cdots t_n^{a_n} \frac{x^n}{n!^2} = \exp\left(t_1 x + \frac{1}{2} \sum_{n \geq 2} t_n \frac{x^n}{n}\right). \tag{5.80}$$

5.5.11 Corollary. *We have*

$$\sum_{n \geq 0} H(n, 2) \frac{x^n}{n!^2} = (1 - x)^{-\frac{1}{2}} e^{\frac{1}{2}x}$$

$$\sum_{n \geq 0} H^*(n, 2) \frac{x^n}{n!^2} = (1 - x)^{-\frac{1}{2}} e^{-\frac{1}{2}x}. \tag{5.81}$$

Proof. Put $t_i = 1$ in (5.80) to obtain

$$\begin{aligned} \exp\left(x + \frac{1}{2} \sum_{n \geq 2} \frac{x^n}{n}\right) &= \exp\left(\frac{1}{2}x + \frac{1}{2} \sum_{n \geq 1} \frac{x^n}{n}\right) \\ &= \exp\left(\frac{1}{2}x + \frac{1}{2} \log(1 - x)^{-1}\right) \\ &= (1 - x)^{-\frac{1}{2}} e^{\frac{1}{2}x}. \end{aligned}$$

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Similarly put $t_1 = 0$ and $t_2 = t_3 = \cdots = 1$ (or use (5.79) directly) to obtain (5.81). $\square$

5.6 Oriented trees and the Matrix-Tree Theorem

A famous problem that goes back to Euler asks for what graphs G is there a closed walk that uses every edge exactly once. (There is also a version for non-closed walks.) Such a walk is called an *Eulerian tour* (also known as an *Eulerian cycle*). A graph which has an Eulerian tour is called an *Eulerian graph*. Euler's famous theorem (the first real theorem of graph theory) states that G is Eulerian if and only if it is connected (except for isolated vertices) and every vertex has even degree. Here we will be concerned with the analogous theorem for directed graphs D. We want to know not just whether an Eulerian tour exists, but how many there are. We reduce this problem to that of counting certain subtrees of D called "oriented trees." We will prove an elegant determinantal formula for this number, and from it derive a determinantal formula, known as the *Matrix-Tree Theorem*, for the number of spanning trees of any (undirected) graph. An application of the enumeration of Eulerian tours is given to the enumeration of de Bruijn sequences. For the case of undirected graphs no analogous formula is known for the number of Eulerian tours, explaining why we consider only the directed case.

We will use the terminology and notation associated with directed graphs introduced at the beginning of Chapter 4.7. Let $D = (V, E, \varphi)$ be a digraph with vertex set $V = \{v_1, \ldots, v_p\}$ and edge set $E = \{e_1, \ldots, e_q\}$. We say that D is *connected* if it is connected as an undirected graph. A *tour* in D is a sequence $e_1, e_2, \ldots, e_r$ of *distinct* edges such that the final vertex of e_i is the initial vertex of e_{i+1} for all $1 \le i \le r-1$, and the final vertex of e_r is the initial vertex of e_1. A tour is *Eulerian* if every edge of D occurs at least once (and hence exactly once). A digraph that has no isolated vertices and contains an Eulerian tour is called an *Eulerian digraph*. Clearly an Eulerian digraph is connected. (Even more strongly, there is a directed path between any pair of vertices.) The *outdegree* of a vertex v, denoted outdeg(v), is the number of edges of G with initial vertex v. Similarly the *indegree* of v, denoted indeg(v), is the number of edges of D with final vertex v. A loop (edge of the form (v, v)) contributes one to both the indegree and outdegree. A digraph is *balanced* if indeg(v) = outdeg(v) for all vertices v.

5.6.1 Theorem. *A digraph D without isolated vertices is Eulerian if and only if it is connected and balanced.*

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Proof. Assume D is Eulerian, and let $e_1, \ldots, e_q$ be an Eulerian tour. As we move along the tour, whenever we enter a vertex v we must exit it, except that at the very end we enter the final vertex v of e_q without exiting it. However, at the beginning we exited v without having entered it. Hence every vertex is entered as often as it is exited and so must have the same outdegree as indegree. Therefore D is balanced, and as noted above D is clearly connected.

Now assume that D is balanced and connected. We may assume that D has at least one edge. We first claim that for any edge e of D, D has a tour (not necessarily Eulerian) for which $e = e_1$. If e_1 is a loop we are done. Otherwise we have entered the vertex $\text{fin}(e_1)$ for the first time, so since D is balanced there is some exit edge e_2. Either $\text{fin}(e_2) = \text{init}(e_1)$ and we are done, or else we have entered the vertex $\text{fin}(e_2)$ once more than we have exited it. Since D is balanced there is a new edge e_3 with $\text{fin}(e_2) = \text{init}(e_3)$. Continuing in this way, either we complete a tour or else we have entered the current vertex once more than we have exited it, in which case we can exit along a new edge. Since D has finitely many edges, eventually we must complete a tour. Thus D does have a tour for which $e = e_1$.

Now let $e_1, \ldots, e_r$ be a tour C of maximum length. We must show that $r = q$, the number of edges of D. Assume to the contrary that $r < q$. Since in moving along C every vertex is entered as often as it is exited (with $\text{init}(e_1)$ exited at the beginning and entered at the end), when we remove the edges of C from D we obtain a digraph H which is still balanced, though it need not be connected. However, since D is connected, at least one connected component H_1 of H contains at least one edge and has a vertex v in common with C. Since H_1 is balanced, there is an edge e of H_1 with initial vertex v. See Figure 5.16, where the edges of a tour C are drawn as solid lines, and the remaining edges as dotted lines. The argument of the previous paragraph shows that H_1 has a tour C' of positive length beginning with the edge e. But then when moving along C, when we reach v we can take the "detour" C' before continuing with C. This gives a tour of length longer than r, a contradiction. Hence $r = q$, and the theorem is proved. □

Our primary goal is to count the number of Eulerian tours of a connected balanced digraph. A key concept in doing so is that of an oriented tree. An *oriented tree* with root v is a (finite) digraph T with v as one of its vertices, such that there is a unique directed path from any vertex u to v. In other words, for every vertex u there is a unique sequence of edges $e_1, \ldots, e_r$ such that (a) $\text{init}(e_1) = u$, (b) $\text{fin}(e_r) = v$, and (c) $\text{fin}(e_i) = \text{init}(e_{i+1})$ for $1 \le i \le r-1$. It is easy to see that this means that the underlying undirected graph (i.e., "erase" all the arrows from the edges of T) is a tree, and that all arrows in T "point toward"

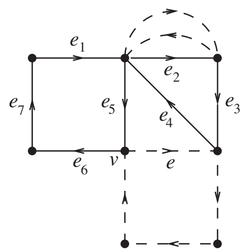

Figure 5.16 A nonmaximal tour in a balanced digraph

v. There is a surprising connection between Eulerian tours and oriented trees, given by the next result.

5.6.2 Theorem. *Let D be a connected balanced digraph with vertex set V. Fix an edge e of D, and let $v = \mathrm{init}(e)$. Let $\tau(D, v)$ denote the number of oriented (spanning) subtrees of D with root v, and let $\epsilon(D, e)$ denote the number of Eulerian tours of D starting with the edge e. Then*

$$\epsilon(D, e) = \tau(D, v) \prod_{u \in V} (\mathrm{outdeg}(u) - 1)!. \tag{5.82}$$

Proof. Let $e = e_1, e_2, \ldots, e_q$ be an Eulerian tour E in D. For each vertex $u \neq v$, let $e(u)$ be the "last exit" from u in the tour, that is, let $e(u) = e_j$ where $\mathrm{init}(e_j) = u$ and $\mathrm{init}(e_k) \neq u$ for any $k > j$.

Claim #1. The vertices of D, together with the edges $e(u)$ for all vertices $u \neq v$, form an oriented subtree of D with root v.

Proof of Claim #1. This is a straightforward verification. Let T be the spanning subgraph of D with edges $e(u)$, $u \neq v$. Thus if $\#V = p$, then T has p vertices and $p - 1$ edges. We now make the following three observations.

(a) T does not have two edges f and f' satisfying $\mathrm{init}(f) = \mathrm{init}(f')$. This is clear since both f and f' can't be last exits from the same vertex.
(b) T does not have an edge f with $\mathrm{init}(f) = v$. This is clear since by definition the edges of T consist only of last exits from vertices other than v, so no edge of T can exit from v.
(c) T does not have a (directed) cycle C. For suppose C were such a cycle. Let f be that edge of C which occurs after all the other edges of C in the Eulerian tour E. Let f' be the edge of C satisfying $\mathrm{fin}(f) = \mathrm{init}(f')$ $(= u$, say). We can't have $u = v$ by (b). Thus when we enter u *via* f, we must exit u. We can't exit u *via* f' since f occurs after f' in E. (Note that we cannot

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have $f = f'$ since then f would be a loop and therefore not a last exit.) Hence f' is not the last exit from u, contradicting the definition of T.

It is easy to see that conditions (a)–(c) imply that T is an oriented tree with root v, proving the claim.

Claim #2. We claim that the following converse to Claim #1 is true. Given a connected balanced digraph D and a vertex v, let T be an oriented (spanning) subtree of D with root v. Then we can construct an Eulerian tour Δ as follows. Choose an edge e_1 with $\mathrm{init}(e_1) = v$. Then continue to choose any edge possible to continue the tour, except we never choose an edge f of T unless we have to, that is, unless it's the only remaining edge exiting the vertex at which we stand. Then we never get stuck until all edges are used, so we have constructed an Eulerian tour Δ. Moreover, the set of last exits of Δ from vertices $u \neq v$ of D coincides with the set of edges of the oriented tree T.

Proof of Claim #2. Since D is balanced, the only way to get stuck is to end up at v with no further exits available, but with an edge still unused. Suppose this is the case. At least one unused edge must be a last exit edge, that is, an edge of T. Let u be a vertex of T closest to v in T such that the unique edge f of T with $\mathrm{init}(f) = u$ is not in the tour. Let $y = \mathrm{fin}(f)$. Suppose $y \neq v$. Since we enter y as often as we leave it, we don't use the last exit from y. Thus $y = v$. But then we can leave v, a contradiction. This proves Claim #2.

We have shown that every Eulerian tour Δ beginning with the edge e has associated with it a "last exit" oriented subtree $T = T(\Delta)$ with root $v = \mathrm{init}(e)$. Conversely, we have also shown that given an oriented subtree T with root v, we can obtain all Eulerian tours Δ beginning with e and satisfying $T = T(\Delta)$ by choosing for each vertex $u \neq v$ the order in which the edges from u, except the edge of T, appear in Δ; as well as choosing the order in which all the edges from v except for e appear in Δ. Thus for each vertex u we have $(\mathrm{outdeg}(u)-1)!$ choices, so for each T we have $\prod_u (\mathrm{outdeg}(u) - 1)!$ choices. Since there are $\tau(D, v)$ choices for T, the proof follows. □

5.6.3 Corollary. *Let D be a connected balanced digraph, and let v be a vertex of D. Then the number $\tau(D, v)$ of oriented subtrees with root v is independent of v.*

Proof. Let e be an edge with initial vertex v. By equation (5.82), we need to show that the number $\epsilon(G, e)$ of Eulerian tours beginning with e is independent of e. But $e_1 e_2 \cdots e_q$ is an Eulerian tour if and only if $e_i e_{i+1} \cdots e_q e_1 e_2 \cdots e_{i-1}$ is also an Eulerian tour, and the proof follows. □

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In order for Theorem 5.6.2 to be of use, we need a formula for $\tau(G, v)$. To this end, define the *Laplacian matrix* $\boldsymbol{L} = \boldsymbol{L}(D)$ of a directed graph D with vertex set $V = \{v_1, \ldots, v_p\}$ to be the $p \times p$ matrix

$$\boldsymbol{L}_{ij} = \begin{cases} -m_{ij}, & \text{if } i \neq j \text{ and there are } m_{ij} \text{ edges with} \\ & \quad \text{initial vertex } v_i \text{ and final vertex } v_j \\ \text{outdeg}(v_i) - m_{ii}, & \text{if } i = j. \end{cases}$$

Note that the diagonal entry outdeg(v_i) $-$ m_{ii} is just the number of *nonloop* edges of D with initial vertex v_i. Hence the Laplacian matrix $\boldsymbol{L}(D)$ is independent of the loops of D. Note also that if every vertex of D has the same outdegree d, then the adjacency matrix $\boldsymbol{A}$ (defined in Chapter 4.7) and Laplacian matrix $\boldsymbol{L}$ of D are related by $\boldsymbol{L} = d\boldsymbol{I} - \boldsymbol{A}$, where $\boldsymbol{I}$ denotes the $p \times p$ identity matrix. In particular, if $\boldsymbol{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_p$, then $\boldsymbol{L}$ has eigenvalues $d - \lambda_1, \ldots, d - \lambda_p$.

5.6.4 Theorem. *Let D be a loopless digraph with vertex set $V = \{v_1, \ldots, v_p\}$, and let $1 \leq k \leq p$. Let $\boldsymbol{L}$ be the Laplacian matrix of D, and define $\boldsymbol{L}_0$ to be $\boldsymbol{L}$ with the k-th row and column deleted. Then*

$$\det \boldsymbol{L}_0 = \tau(D, v_k). \tag{5.83}$$

Proof. Induction on q, the number of edges of D. First note that the theorem is true if D is not connected, since clearly $\tau(D, v_k) = 0$, while if D_1 is the component of D containing v_k and D_2 is the rest of D, then $\det \boldsymbol{L}_{\boldsymbol{0}}(D) = \det \boldsymbol{L}_{\boldsymbol{0}}(D_1) \cdot \det \boldsymbol{L}(D_2) = 0$. Thus we may assume that D is connected. In this case the least number of edges that D can have is $p-1$. Suppose then that D has $p - 1$ edges, so that as an undirected graph D is a tree. If D is not an oriented tree with root v_k, then some vertex $v_i \neq v_k$ of D has outdegree 0. Then $\boldsymbol{L}_0$ has a zero row, so $\det \boldsymbol{L}_0 = 0 = \tau(D, v_k)$. If on the other hand D is an oriented tree with root v_k, then there is an ordering of the set $V - \{v_k\}$ so that $\boldsymbol{L}_0$ is upper triangular with 1's on the main diagonal. Hence $\det \boldsymbol{L}_0 = 1 = \tau(D, v_k)$.

Now suppose that D has $q > p - 1$ edges, and assume the theorem for digraphs with at most $q - 1$ edges. We may assume that no edge f of D has initial vertex v_k, since such an edge belongs to no oriented tree with root v_k and also makes no contribution to $\boldsymbol{L}_0$. It then follows, since D has at least p edges, that there exists a vertex $u \neq v_k$ of D of outdegree at least two. Let e be an edge with init(e) $= u$. Let D_1 be D with the edge e removed. Let D_2 be D with all edges e' removed such that init(e) $=$ init(e') and $e' \neq e$. (Note that D_2 is strictly smaller than D since outdeg(u) ≥ 2.) By induction, we have $\det \boldsymbol{L}_0(D_1) = \tau(D_1, v_k)$ and $\det \boldsymbol{L}_0(D_2) = \tau(D_2, v_k)$. Clearly $\tau(D, v_k) = \tau(D_1, v_k) + \tau(D_2, v_k)$, since in an oriented tree T with root v_k, there is exactly one edge whose initial vertex coincides with that of e. On the other hand, it follows immediately from the multilinearity of the determinant that

$$\det \boldsymbol{L_0}(D) = \det \boldsymbol{L_0}(D_1) + \det \boldsymbol{L_0}(D_2).$$

From this the proof follows by induction. □

The operation of removing a row and column from $\boldsymbol{L}(D)$ may seem somewhat contrived. In the case when D is balanced (so $\tau(D, v)$ is independent of v), we would prefer a description of $\tau(D, v)$ directly in terms of $\boldsymbol{L}(D)$. Such a description will follow from the next lemma.

5.6.5 Lemma. *Let $\boldsymbol{M}$ be a $p \times p$ matrix (with entries in a field) such that the sum of the entries in every row and column is* 0. *Let $\boldsymbol{M_0}$ be the matrix obtained from $\boldsymbol{M}$ by removing the i-th row and j-th column. Then the coefficient of x in the characteristic polynomial* $\det(\boldsymbol{M}-x\boldsymbol{I})$ *of $\boldsymbol{M}$ is equal to* $(-1)^{i+j+1}p\cdot\det(\boldsymbol{M_0})$. *(Moreover, the constant term of* $\det(\boldsymbol{M} - x\boldsymbol{I})$ *is* 0.*)*

Proof. The constant term of $\det(\boldsymbol{M} - x\boldsymbol{I})$ is $\det(\boldsymbol{M})$, which is 0 since the columns of $\boldsymbol{M}$ sum to 0.

For definiteness we prove the rest of the lemma only for removing the last row and column, though the proof works just as well for any row and column. Add all the rows of $\boldsymbol{M}-x\boldsymbol{I}$ except the last row to the last row. This doesn't affect the determinant, and will change the entries of the last row all to $-x$ (since the columns of $\boldsymbol{M}$ sum to 0). Factor out $-x$ from the last row, yielding a matrix $\boldsymbol{N}(x)$ satisfying $\det(\boldsymbol{M} - x\boldsymbol{I}) = -x\det(\boldsymbol{N}(x))$. Hence the coefficient of x in $\det(\boldsymbol{M} - x\boldsymbol{I})$ is given by $-\det(\boldsymbol{N}(0))$. Now add all the columns of $\boldsymbol{N}(0)$ except the last column to the last column. This does not effect $\det(\boldsymbol{N}(0))$. Because the rows of $\boldsymbol{M}$ sum to 0, the last column of $\boldsymbol{N}(0)$ becomes the column vector $[0, 0, \ldots, 0, p]^t$. Expanding the determinant by the last column shows that $\det(\boldsymbol{N}(0)) = p \cdot \det(\boldsymbol{M_0})$, and the proof follows. □

Suppose that the eigenvalues of the matrix $\boldsymbol{M}$ of Lemma 5.6.5 are equal to $\mu_1, \ldots, \mu_p$ with $\mu_p = 0$. Since $\det(\boldsymbol{M} - x\boldsymbol{I}) = -x\prod_{j=1}^{p-1}(\mu_j - x)$, we see that

$$(-1)^{i+j+1}p \cdot \det(M_0) = -\mu_1 \cdots \mu_{p-1}. \tag{5.84}$$

This equation allows Theorem 5.6.4, in the case of balanced digraphs, to be restated as follows.

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5.6.6 Corollary. *Let D be a balanced digraph with p vertices and with Laplacian matrix $\boldsymbol{L}$. Suppose that the eigenvalues of $\boldsymbol{L}$ are $\mu_1, \ldots, \mu_p$ with $\mu_p = 0$. Then for any vertex v of D,*

$$\tau(D, v) = \frac{1}{p}\mu_1 \cdots \mu_{p-1}.$$

Combining Theorems 5.6.2 and 5.6.4 yields a formula for the number of Eulerian tours in a balanced digraph.

5.6.7 Corollary. *Let D be a connected balanced digraph with p vertices. Let e be an edge of D. Then the number $\epsilon(D, e)$ of Eulerian tours of D with first edge e is given by*

$$\epsilon(D, e) = (\det \boldsymbol{L_0}(D)) \prod_{u \in V} (\mathrm{outdeg}(u) - 1)!.$$

Equivalently (using Corollary 5.6.6), if $\boldsymbol{L}(D)$ has eigenvalues $\mu_1, \ldots, \mu_p$ with $\mu_p = 0$, then

$$\epsilon(D, e) = \frac{1}{p}\mu_1 \cdots \mu_{p-1} \prod_{u \in V} (\mathrm{outdeg}(u) - 1)!.$$

Let us consider an important special case of Corollary 5.6.7. The *Laplacian matrix* $\boldsymbol{L} = \boldsymbol{L}(G)$ of the *undirected* graph G with vertex set $V = \{v_1, \ldots, v_p\}$ is the $p \times p$ matrix

$$\boldsymbol{L}_{ij} = \begin{cases} -m_{ij}, & \text{if } i \neq j \text{ and there are } m_{ij} \text{ edges between} \\ & \quad \text{vertices } v_i \text{ and } v_j \\ \deg(v_i) - m_{ii}, & \text{if } i = j, \end{cases}$$

where $\deg(v_i)$ denotes the degree (number of incident edges) of v_i. Let $\widehat{G}$ be the digraph obtained from G by replacing each edge $e = uv$ of G with a pair of directed edges $u \to v$ and $v \to u$. Clearly $\widehat{G}$ is balanced and connected. Choose a vertex v of G. There is an obvious one-to-one correspondence between spanning trees T of G and oriented spanning trees $\widehat{T}$ of $\widehat{G}$ with root v, namely, direct each edge of T toward v. Moreover, $\boldsymbol{L}(G) = \boldsymbol{L}(\widehat{G})$. Let $c(G)$ denote the number of spanning trees (or *complexity*) of G. Then as an immediate consequence of Theorem 5.6.4 we obtain the following determinantal formula for $c(G)$. This formula is known as the *Matrix-Tree Theorem*.

5.6.8 Theorem (the Matrix-Tree theorem)**.** *Let G be a finite connected p-vertex graph without loops, with Laplacian matrix $\boldsymbol{L} = \boldsymbol{L}(G)$. Let $1 \leq i \leq p$, and let $\boldsymbol{L_0}$ denote $\boldsymbol{L}$ with the i-th row and column removed. Then*

$$c(G) = \det(\boldsymbol{L_0}).$$

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Equivalently, if $\boldsymbol{L}$ *has eigenvalues* $\mu_1, \ldots, \mu_p$ *with* $\mu_p = 0$*, then*

$$c(G) = \frac{1}{p}\mu_1 \cdots \mu_{p-1}.$$

Let us look at some examples of the use of the results we have just proved.

5.6.9 Example. Let $G = K_p$, the complete graph on p vertices. We have $\boldsymbol{L}(K_p) = p\boldsymbol{I} - \boldsymbol{J}$, where $\boldsymbol{J}$ is the $p \times p$ matrix of all 1's, and $\boldsymbol{I}$ is the $p \times p$ identity matrix. Since $\boldsymbol{J}$ has rank one, $p - 1$ of its eigenvalues are equal to 0. Since $\mathrm{tr}(\boldsymbol{J}) = p$, the other eigenvalue is equal to p. (Alternatively, the column vector of all 1's is an eigenvector with eigenvalue p.) Hence the eigenvalues of $p\boldsymbol{I} - \boldsymbol{J}$ are p ($p - 1$ times) and 0 (once). By the Matrix-Tree Theorem we get

$$c(K_p) = \frac{1}{p}p^{p-1} = p^{p-2},$$

agreeing with the formula for $t(n)$ in Proposition 5.3.2.

5.6.10 Example. Let Γ be the group $(\mathbb{Z}/2\mathbb{Z})^n$ of n-tuples of 0's and 1's under componentwise addition modulo 2. Define a "scalar product" $\alpha \cdot \beta$ on Γ by

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = \sum a_i b_i \in \mathbb{Z}/2\mathbb{Z}.$$

Note that since $(-1)^m$ depends only on the value of the integer m modulo 2, such expressions as $(-1)^{\alpha\cdot\beta+\gamma\cdot\delta}$ are well-defined for $\alpha, \beta, \gamma, \delta \in \Gamma$ whether we interpret the addition in the exponent as taking place in $\mathbb{Z}/2\mathbb{Z}$ or in $\mathbb{Z}$. In particular, there continues to hold the law of exponents $(-1)^{\alpha+\beta} = (-1)^\alpha(-1)^\beta$. Let C_n be the graph whose vertices are the elements of Γ, with two vertices α and β connected by an edge whenever $\alpha + \beta$ has exactly one component equal to 1. Thus C_n may be regarded as the graph formed by the vertices and edges of an n-dimensional cube. Equivalently, C_n is the Hasse diagram of the boolean algebra B_n, regarded as a graph. Let V be the vector space of all functions $f : \Gamma \to \mathbb{Q}$. Define a linear transformation $\Phi : V \to V$ by

$$(\Phi f)(\alpha) = nf(\alpha) - \sum_\beta f(\beta),$$

where β ranges over all elements of Γ adjacent to α in C_n. Note that the matrix of Φ with respect to some ordering of the basis Γ of V is just the Laplacian matrix $\boldsymbol{L}(C_n)$ (with respect to the same ordering of the vertices of C_n). Now for each $\gamma \in \Gamma$ define a function $\chi_\gamma \in V$ by

$$\chi_\gamma(\alpha) = (-1)^{\alpha\cdot\gamma}.$$

Then

$$(\Phi\chi_\gamma)(\alpha) = n(-1)^{\alpha\cdot\gamma} - \sum_\beta (-1)^{\beta\cdot\gamma},$$

with β as above. If γ has exactly k 1's, then for exactly $n-k$ values of β do we have $\beta\cdot\gamma = \alpha\cdot\gamma$, while for the remaining k values of β we have $\beta\cdot\gamma = \alpha\cdot\gamma + 1$. Hence

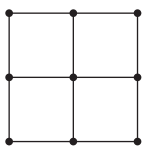
$$\begin{aligned}(\Phi\chi_\gamma)(\alpha) &= (n - ((n-k) - k))\,(-1)^{\alpha\cdot\gamma} \\ &= 2k\,\chi_\gamma(\alpha).\end{aligned}$$

It follows that χ_γ is an eigenvector of Φ with eigenvalue $2k$. It is easy to see that the χ_γ's are linearly independent, so we have found all 2^n eigenvalues of $\boldsymbol{L}$, namely, $2k$ is an eigenvalue of multiplicity $\binom{n}{k}$, $0 \le k \le n$. Hence from the Matrix-Tree Theorem there follows the remarkable result

$$\begin{aligned}c(C_n) &= \frac{1}{2^n}\prod_{k=1}^{n}(2k)^{\binom{n}{k}} \\ &= 2^{2^n-n-1}\prod_{k=1}^{n} k^{\binom{n}{k}}. \qquad (5.85)\end{aligned}$$

A bijective proof of this formula is not known.

5.6.11 Example (the efficient mail carrier)**.** A mail carrier has an itinerary of city blocks to which he (or she) must deliver mail. He wants to accomplish this by walking along each block twice, once in each direction, thus passing along houses on each side of the street. The blocks form the edges of a graph G, whose vertices are the intersections of streets. The mail carrier wants simply to walk along an Eulerian tour in the digraph $\widehat{G}$ defined after Corollary 5.6.7. Making the plausible assumption that the graph is connected, not only does an Eulerian tour always exist, but we can tell the mail carrier how many there are. Thus he will know how many different routes he can take to avoid boredom. For instance, suppose G is the 3×3 grid illustrated below.



This graph has 192 spanning trees. Hence the number of mail carrier routes beginning with a fixed edge (in a given direction) is $192\cdot 1!^4 2!^4 3! = 18432$.

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The total number of routes is thus 18432 times twice the number of edges, namely, $18432 \times 24 = 442368$. Assuming the mail carrier delivered mail 250 days a year, it would be 1769 years before he would have to repeat a route!

5.6.12 Example (binary de Bruijn sequences)**.** A *binary sequence* is just a sequence of 0's and 1's. A *(binary) de Bruijn sequence* of degree n is a binary sequence $A = a_1 a_2 \cdots a_{2^n}$ such that every binary sequence $b_1 \cdots b_n$ of length n occurs exactly once as a "circular factor" of A, that is, as a sequence $a_i a_{i+1} \cdots a_{i+n-1}$, where the subscripts are taken modulo 2^n if necessary. Note that there are exactly 2^n binary sequences of length n, so the only possible length of a de Bruijn sequence of degree n is 2^n. Clearly any conjugate (cyclic shift) $a_i a_{i+1} \cdots a_{2^n} a_1 a_2 \cdots a_{i-1}$ of a de Bruijn sequence $a_1 a_2 \cdots a_{2^n}$ is also a de Bruijn sequence, and we call two such sequences *equivalent*. This relation of equivalence is obviously an equivalence relation, and every equivalence class contains exactly one sequence beginning with n 0's. Up to equivalence, there is one de Bruijn sequence of degree two, namely, 0011. It's easy to check that there are two inequivalent de Bruijn sequences of degree three, namely, 00010111 and 00011101. However, it's not clear at this point whether de Bruijn sequences exist for all n. By a clever application of Theorems 5.6.2 and 5.6.4, we will not only show that such sequences exist for all positive integers n, but we will also count the number of them. It turns out that there are *lots* of them. For instance, the number of inequivalent de Bruijn sequences of degree eight is equal to

$$1329227995784915872903807060280344576.$$

Our method of enumerating de Bruijn sequences will be to set up a correspondence between them and Eulerian tours in a certain directed graph D_n, the *de Bruijn graph* of degree n. The graph D_n has 2^{n-1} vertices, which we will take to consist of the 2^{n-1} binary sequences of length $n-1$. A pair $(a_1 a_2 \cdots a_{n-1}, b_1 b_2 \cdots b_{n-1})$ of vertices forms an edge of D_n if and only if $a_2 a_3 \cdots a_{n-1} = b_1 b_2 \cdots b_{n-2}$, that is, e is an edge if the last $n-2$ terms of $\operatorname{init}(e)$ agree with the first $n-2$ terms of $\operatorname{fin}(e)$. Thus every vertex has indegree two and outdegree two, so D_n is balanced. The number of edges of D_n is 2^n. Moreover, it's easy to see that D_n is connected (see Lemma 5.6.13). The graphs D_3 and D_4 are shown in Figure 5.17.

Suppose that $E = e_1 e_2 \cdots e_{2^n}$ is an Eulerian tour in D_n. If $\operatorname{fin}(e_i)$ is the binary sequence $a_{i1} a_{i2} \cdots a_{i,n-1}$, then replace e_i in E by the last bit $a_{i,n-1}$. It is easy to see that the resulting sequence $\beta(E) = a_{1,n-1} a_{2,n-1} \cdots a_{2^n,n-1}$ is a de Bruijn sequence, and conversely every de Bruijn sequence arises in this way. In particular, since D_n is balanced and connected there exists at least one de

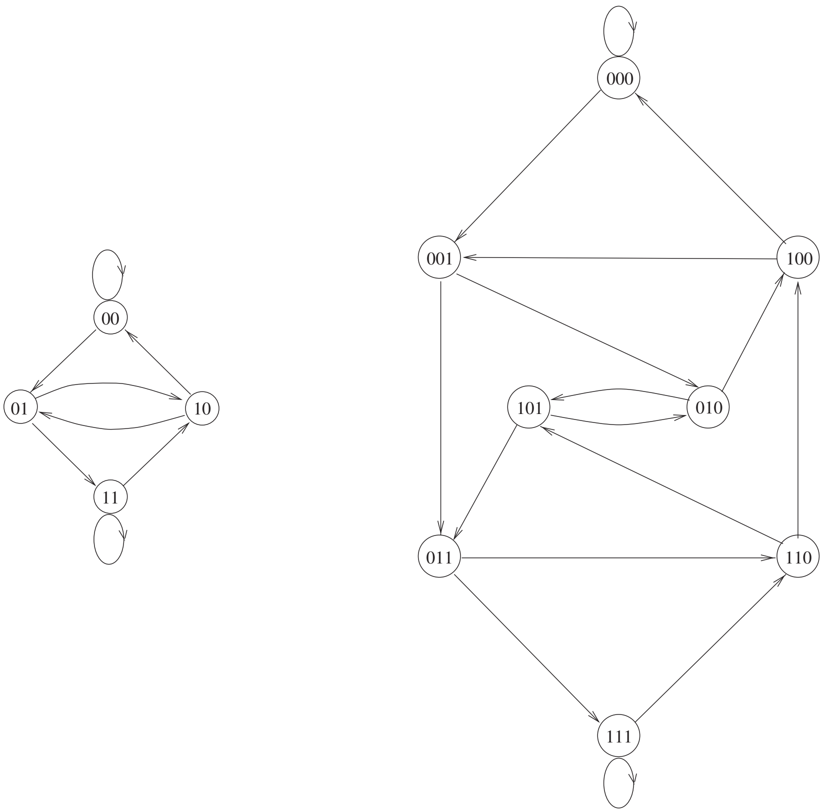

Figure 5.17 The de Bruijn graphs D_3 and D_4

Bruijn sequence. In order to count the total number of such sequences, we need to compute $\det \boldsymbol{L_0}(D_n)$. One way to do this is by a clever but messy sequence of elementary row and column operations which transforms the determinant into triangular form. We will give instead an elegant computation of the eigenvalues of $\boldsymbol{L}(D_n)$ (and hence of $\det \boldsymbol{L_0}$) based on the following simple lemma.

5.6.13 Lemma. *Let u and v be any two vertices of D_n. Then there is a unique (directed) walk from u to v of length $n-1$.*

Proof. Suppose $u = a_1a_2\cdots a_{n-1}$ and $v = b_1b_2\cdots b_{n-1}$. Then the unique path of length $n-1$ from u to v has vertices

$$a_1a_2\cdots a_{n-1},\ a_2a_3\cdots a_{n-1}b_1,\ a_3a_4\cdots a_{n-1}b_1b_2,\ \ldots,$$
$$a_{n-1}b_1\cdots b_{n-2}, b_1b_2\cdots b_{n-1}.$$

□

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5.6.14 Lemma. *The eigenvalues of* $\boldsymbol{L}(D_n)$ *are* 0 *(with multiplicity one) and* 2 *(with multiplicity* $2^{n-1} - 1$).

Proof. Let $\boldsymbol{A}(D_n)$ denote the directed adjacency matrix of D_n, that is, the rows and columns are indexed by the vertices, with

$$\boldsymbol{A}_{uv} = \begin{cases} 1, & \text{if } (u, v) \text{ is an edge} \\ 0, & \text{otherwise.} \end{cases}$$

Now Lemma 5.6.13 is equivalent to the assertion that $\boldsymbol{A}^{n-1} = \boldsymbol{J}$, the $2^{n-1} \times 2^{n-1}$ matrix of all 1's. If the eigenvalues of $\boldsymbol{A}$ are $\lambda_1, \dots \lambda_{2^{n-1}}$, then the eigenvalues of $\boldsymbol{J} = \boldsymbol{A}^{n-1}$ are $\lambda_1^{n-1}, \dots, \lambda_{2^{n-1}}^{n-1}$. By Example 5.6.9, the eigenvalues of $\boldsymbol{J}$ are 2^{n-1} (once) and 0 ($2^{n-1} - 1$ times). Hence the eigenvalues of $\boldsymbol{A}$ are 2ζ (once, where ζ is an $(n-1)$-st root of unity to be determined), and 0 ($2^{n-1} - 1$ times). Since the trace of $\boldsymbol{A}$ is 2, it follows that $\zeta = 1$, and we have found all the eigenvalues of A.

Now $\boldsymbol{L}(D_n) = 2\boldsymbol{I} - \boldsymbol{A}(D_n)$. Hence the eigenvalues of $\boldsymbol{L}$ are $2 - \lambda_1, \dots, 2 - \lambda_{2^{n-1}}$, and the proof follows from the above determination of $\lambda_1, \dots, \lambda_{2^{n-1}}$. □

5.6.15 Corollary. *The number* $B_0(n)$ *of de Bruijn sequences of degree n beginning with n* 0*'s is equal to* $2^{2^{n-1}-n}$*. The total number* $B(n)$ *of de Bruijn sequences of degree n is equal to* $2^{2^{n-1}}$.

Proof. By the above discussion, $B_0(n)$ is the number of Eulerian tours in D_n whose first edge is the loop at vertex $00\cdots0$. Moreover, the outdegree of every vertex of D_n is two. Hence by Corollary 5.6.7 and Lemma 5.6.14 we have

$$B_0(n) = \frac{1}{2^{n-1}} 2^{2^{n-1}-1} = 2^{2^{n-1}-n}.$$

Finally, $B(n)$ is obtained from $B_0(n)$ by multiplying by the number 2^n of edges, and the proof follows. □

Notes[2]

The compositional formula (Theorem 5.1.4) and the exponential formula (Corollary 5.1.6) had many precursors before blossoming into their present form. A purely formal formula for the coefficients of the composition of two exponential generating functions goes back to Faà di Bruno [86][87] in 1855 and 1857,

[2] A reference such as [3.13] refers to reference 13 of the Notes section to Volume 1, Chapter 3. A reference such as [13] refers to the References of the present volume.

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and is known as *Faà di Bruno's formula*. For additional references on this formula, see [2.3, p. 137]. An early precursor of the exponential formula is due to Jacobi [159]. The idea of interpreting the coefficients of $e^{F(x)}$ combinatorially was considered in certain special cases by Touchard [329] and by Riddell and Uhlenbeck [270]. Touchard was concerned with properties of permutations and obtained our equation (5.30), from which he derived many consequences. Equation (5.30) was earlier obtained by Pólya [250, Sect. 13] but he was not interested in general combinatorial applications. It is also apparent from the work of Frobenius (see [104, bottom of p. 152 of GA]) and Hurwitz [156, §4] that they were aware of (5.30), even if they did not state it explicitly. Riddell and Uhlenbeck, on the other hand, were concerned with graphical enumeration and obtained our Example 5.2.1 and related results.

It was not until the early 1970s that a general combinatorial interpretation of $e^{F(x)}$ was developed independently by Foata–Schützenberger [91], Bender–Goldman [3.3], and Doubilet–Rota–Stanley [3.12]. The approach most like the one taken here is that of Foata–Schützenberger. Doubilet–Rota–Stanley use an incidence algebra approach and prove a result (Theorem 5.1) equivalent to our Theorem 5.1.11. The most sophisticated combinatorial theory of power series composition is the theory of *species*, which is based on category theory and which was developed after the above three references by A. Joyal [3.23] and his collaborators. For further information on species, see [16]. Another category theory approach to the exponential formula was given by A. W. M. Dress and T. Müller [73]. The exponential formula has been frequently rediscovered in various guises; an interesting example is [251]. A q-analogue has been given by Gessel [117].

Let us turn to the applications of the exponential formula given in Section 2. Example 5.2.3 first appeared in [4.36, Example 6.6]. The generating functions (5.27) and (5.28) for total partitions and binary total partitions, as well as the explicit formula $b(n) = 1 \cdot 3 \cdot 5 \cdots (2n-3)$, are given by E. Schröder [291] as the fourth and third problems of his famous "vier combinatorische Probleme." (We will discuss the first two problems in Chapter 6.) A minor variation of the combinatorial proof given here of the formula for $b(n)$ appears in [82, Cor. 2], though there may be earlier proofs of a similar nature. See Exercise 43 for a generalization and further references. For further work related to Schröder's fourth problem, see the solution to Exercise 40. The generating functions and recurrence relations for $S_n(2)$ and $S_n^*(2)$ in Examples 5.2.7 and 5.2.8 were found (with a different proof from ours) by H. Gupta [133, (6.3), (6.4), (6.7), and (6.8)]. For a generalization, see R. Grimson [132]. Example 5.2.9 is due to I. Schur [293] and is also discussed in [253, Problem VII.45]. Schur considers some variants, one of which leads to the generating function for $T_n(2)$ given

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in Example 5.2.8. On the other hand, the generating function for $T_n^*(2)$ (equation (5.29)) essentially appears (again in a different context, discussed here in Exercise 23) in [77].

We already mentioned that equation (5.30) is due to Pólya (or possibly Frobenius or Hurwitz). It seems clear from the work of Touchard [329] that he was aware of the generating function $\exp \sum_{d|r} \frac{x^d}{d}$ of Example 5.2.10. The first explicit statement is due to Chowla, Herstein, and Scott [49], the earlier cases $r = 2$ and r prime having been investigated by Chowla, Herstein, and Moore [48] and by Jacobstahl [160], respectively. Comtet [2.3, Exer. 9, p. 257] discusses this subject and gives some additional references. For a significant generalization, see Exercise 13(a).

Example 5.2.11 was found in collaboration with I. Gessel. Similar arguments appear in Exercise 21 and in the paper [226] of Metropolis and Rota.

The concept of tree as a formal mathematical object goes back to Kirchhoff and von Staudt. Trees were first extensively investigated by Cayley, to whom the term "tree" is due. In particular, in [43] Cayley states the formula $t(n) = n^{n-2}$ for the number of free trees on an n-element vertex set, and he gives a vague idea of a combinatorial proof. Cayley points out, however, that an equivalent result had been proved earlier by Borchardt [28]. Moreover, this result appeared even earlier in a paper of Sylvester [324]. Undoubtedly Cayley and Sylvester could have furnished a complete, rigorous proof had they the inclination to do so. The first explicit combinatorial proof of the formula $t(n) = n^{n-2}$ is due to Prüfer [255], and is essentially the same as the case $k = 1$ of our first proof of Proposition 5.3.2. The second proof of Proposition 5.3.2 (or more precisely, the version given for trees at the beginning of the proof) is due to Joyal [3.23, Example 12, pp. 15-16]. The more general formula for $p_S(n)$ given in Proposition 5.3.2 was also stated by Cayley and is implicit in the work of Borchardt. Raney [259] uses a straightforward generalization of Prüfer sequences to give a formal solution to the functional equation

$$\sum_i A_i e^{B_i x} = x.$$

A less obvious generalization of Prüfer sequences was given by Knuth [170] and is also discussed in [231, Section 2.3].

The connection between Prüfer sequences and degree sequences of trees was observed by Neville [240]. It was also pointed out by Moon [229][230, p. 72] and Riordan [272], who noted that it implied the case $k = 1$ of Theorem 5.3.4. The second proof of Theorem 5.3.4 is based on the paper [189] of Labelle.

The enumeration of plane (or ordered) trees by degree sequences (the case $k = 1$ of Theorem 5.3.10) is due to Erdélyi and Etherington [81]; their

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basic tool is essentially the Lagrange inversion formula. (Erdélyi and Etherington work with "non-associative combinations" rather than trees, but in [84] Etherington points out the connection, known to Cayley, between non-associative combinations and plane trees.) The first combinatorial proof of Theorem 5.3.10, essentially the proof given here, is due to Raney [258, Thm. 2.2]. (Raney works with "words" or more generally "lists of words" rather than trees; his words are essentially the Łukasiewicz words of equation (5.50).) Raney used his result to give a combinatorial proof of the Lagrange inversion formula, as discussed below. The crucial combinatorial result on which the proof of Theorem 5.3.10 is based is Lemma 5.3.7. This result (including the statement after Example 5.3.8 that if $\phi(w) = -k$ then precisely k cyclic shifts of w belong to $\mathcal{B}^*$) is part of a circle of results known as the "Cycle Lemma." The first such result (which includes the case $\mathcal{A} = \{x_0, x_{-1}\}$ of Lemma 5.3.7) is due to Dvoretzky and Motzkin [76]. For further information and references, see [68]. For further information on the extensively developed subject of tree enumeration, see for instance [125] [171, Section 2.3][230][231].

The Lagrange inversion formula (Theorem 5.4.2) is due, logically enough, to Lagrange [66]. His proof is the same as our first proof. This proof is repeated by Bromwich [30, Ch. VIII, §55.1], who gives many interesting applications (see our Exercises 53, 54, and 57). The first combinatorial proof is due to Raney [258]. His proof is essentially the same as our second proof, though as mentioned earlier he worked entirely with words and only implicitly with plane trees and forests. Streamlined versions of Raney's proof appear in Schützenberger [300] and Lothaire [4.21, Ch. 11]. Our third proof of Theorem 5.4.2 is essentially the same as that of Labelle [189]. For some further references, see [2.3, pp. 148-149] and [115].

There have been many generalizations of the Lagrange inversion formula. For fascinating surveys of multivariable Lagrange inversion formulas and their interconnections, see Gessel [119] and Henrici [148]. Gessel gives a combinatorial proof which generalizes our third proof of Theorem 5.4.2. There has also been considerable work on q-analogues of the Lagrange inversion formula. Special cases were found by Jackson and Carlitz, followed by more general versions and/or applications due to Andrews, Cigler, Garsia, Garsia and Remmel, Gessel, Gessel and Stanton, Hofbauer, Krattenthaler, Paule, et al. A survey of these results is given by Stanton [321]. A subsequent unified approach to q-Lagrange inversion was given by Singer [306]. Finally, Gessel [115] gives a generalization of Lagrange inversion to noncommutative power series (as well as a q-analogue).

Exponential structures (Definition 5.5.1) were first considered by Stanley [315]. Their original motivation was to "explain" the formula

$\mu_n = (-1)^n E_{2n-1}$ of Example 5.5.7, which had earlier been obtained by G. Sylvester [325] by *ad hoc* reasoning. (An equivalent result, though not stated in terms of posets and Möbius functions, had earlier been given by Rosen [280, Lemma 3].) Exponential structures are closely related to the exponential prefabs of Bender and Goldman [3.3]; see [315] for further information.

We have already encountered the function $H(n,r)$ of Corollaries 5.5.9 and 5.5.11 in Section 4.6 (where it was denoted $H_n(r)$). In that section we were concerned with the behavior of $H(n,r)$ for fixed n, while here we are concerned with fixed r. Corollary 5.5.11 was first proved by Anand, Dumir, and Gupta [4.1, Sect. 8] using a different technique (viz., first obtaining a recurrence relation). The approach we have taken here first appeared in [4.36, Example 6.11].

The characterization of Eulerian digraphs given by Theorem 5.6.1 is a result of Good [124], while the fundamental connection between oriented subtrees and Eulerian tours in a balanced digraph that was used to prove Theorem 5.6.2 was shown by van Aardenne-Ehrenfest and de Bruijn [333, Thm. 5a]. This result is sometimes called the BEST Theorem, after de **B**ruijn, van Aardenne-**E**hrenfest, **S**mith, and **T**utte. However, Smith and Tutte were not involved in the original discovery. (In [308] Smith and Tutte give a determinantal formula for the number of Eulerian tours in a special class of balanced digraphs. Van Aardenne-Ehrenfest and de Bruijn refer to the paper of Smith and Tutte in a footnote added in proof.) The determinantal formula for the number of oriented subtrees of a directed graph (Theorem 5.6.4) is due to Tutte [331, Thm. 3.6]. The Matrix-Tree Theorem (Theorem 5.6.8) was first proved by Borchardt [28] in 1860, though a similar result had earlier been published by Sylvester [324] in 1857. Cayley [41, p. 279] in fact in 1856 referred to the not-yet-published work of Sylvester. For further historical information on the Matrix-Tree theorem, see [231, p. 42]. Typically the Matrix-Tree theorem is proved using the Binet–Cauchy formula (a formula for the determinant of the product of an $m \times n$ matrix and an $n \times m$ matrix); see [231, §5.3] for such a proof. Additional information on the eigenvalues of the adjacency matrix and Laplacian matrix of a graph may be found in [51][62][61].

The fundamental reason underlying the simple product formula for $c(C_n)$ given by equation (5.85) is that the graph C_n has a high degree of symmetry, namely, it is a *Cayley graph* of the abelian group $\Gamma = (\mathbb{Z}/2\mathbb{Z})^n$. This is equivalent to the statement that Γ acts regularly on the vertices of C_n, that is, Γ is transitive and only the identity element fixes a vertex. At the end of Example 5.6.10 we said that a bijective proof of equation (5.85) is not known. On the other hand, two combinatorial proofs (but not direct bijections) are given by O. Bernardi [19].

For the complexity of an arbitrary Cayley graph of a finite abelian group, see Exercise 68. In general, it follows from group representation theory that the automorphism group of a graph G "induces" a factorization of the characteristic polynomial of the adjacency matrix of G; see for example [62, Ch. 5] for an exposition. For further aspects of Cayley graphs of $(\mathbb{Z}/2\mathbb{Z})^n$, see [70].

The de Bruijn sequences of Example 5.6.12 are named after Nicolaas Govert de Bruijn, who published his work on this subject in 1946 [63]. However, it was found by Stanley in 1975 that the problem of enumerating de Bruijn sequences had been posed by de Rivière [67] and solved by Flye Sainte-Marie in 1894 [90]. See [65] for an acknowledgment of this discovery. De Bruijn sequences have a number of interesting applications to the design of switching networks and related topics. For further information, see [123]. Additional references to de Bruijn sequences may be found in [338, p. 92]. For generalizations of de Bruijn sequences see Chung, Diaconis, and Graham [52], and for some applications to magic see Diaconis and Graham [71].

Exercises for Chapter 5

1. (a) [2–] Each of n (distinguishable) telephone poles is painted red, white, blue, or yellow. An odd number are painted blue and an even number yellow. In how many ways can this be done?

   (b) [2] Suppose now the colors orange and purple are also used. The number of orange poles plus the number of purple poles is even. Now how many ways are there?

2. (a) [3–] Write

$$1 + \sum_{n \geq 1} f_n x^n = \exp \sum_{n \geq 1} h_n \frac{x^n}{n},$$

   where $h_n \in \mathbb{Q}$ (or any field of characteristic 0). Show that the following four conditions are equivalent for fixed $N \in \mathbb{P}$:

   (i) $f_n \in \mathbb{Z}$ for all $n \in [N]$.

   (ii) $h_n \in \mathbb{Z}$ and $\displaystyle\sum_{d|n} h_d \, \mu(n/d) \equiv 0 \pmod{n}$ for all $n \in [N]$, where μ denotes the ordinary number-theoretic Möbius function.

   (iii) $h_n \in \mathbb{Z}$ for all $n \in [N]$, and $h_n \equiv h_{n/p} \pmod{p^r}$, whenever $n \in [N]$ and p is a prime such that $p^r | n$, $p^{r+1} \nmid n$, $r \geq 1$.

   (iv) There exists a polynomial $P(t) = \Pi_1^N (t - \alpha_i) \in \mathbb{Z}[t]$ (where $\alpha_i \in \mathbb{C}$) such that $h_n = \sum_{i=1}^N \alpha_i^n$ for all $n \in [N]$.

   (b) [2+] (basic knowledge of finite fields required) Let $\mathcal{S}$ be a set of polynomial equations in the variables $x_1, \ldots, x_k$ over the field $\mathbb{F}_q$. Let N_n denote the number of solutions $(\alpha_1, \ldots, \alpha_k)$ to the equations such that each $\alpha_i \in \mathbb{F}_{q^n}$. Show that the generating function

$$Z(x) = \exp \sum_{n \geq 1} N_n \frac{x^n}{n}$$

   has integer coefficients.

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(c) [3–] Show that if $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ and $\sum_{i=1}^{N} \alpha_i^n \in \mathbb{Z}$ for all $n \in \mathbb{P}$, then $\prod_{i=1}^{N}(t-\alpha_i) \in \mathbb{Z}[t]$.

3. (a) [2–] Let $f(n) = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ and $g(n) = 2^n n!$. Show that $E_g(x) = E_f(x)^2$.
   (b) [3–] Give a combinatorial proof based on Proposition 5.1.1.
4. (a) [2] A *threshold graph* is a simple (i.e., no loops or multiple edges) graph which may be defined inductively as follows:
   (i) The empty graph is a threshold graph.
   (ii) If G is a threshold graph, then so is the disjoint union of G with a one-vertex graph.
   (iii) If G is a threshold graph, then so is the (edge) complement of G.
   Let $t(n)$ denote the number of threshold graphs with vertex set $[n]$ (with $t(0) = 1$), and let $s(n)$ denote the number of such graphs with no isolated vertex (so $s(0) = 1, s(1) = 0$). Set

$$T(x) = E_t(x) = 1 + x + 2\frac{x^2}{2!} + 8\frac{x^3}{3!} + 46\frac{x^4}{4!} + \cdots$$

$$S(x) = E_s(x) = 1 + \frac{x^2}{2!} + 4\frac{x^3}{3!} + 23\frac{x^4}{4!} + \cdots .$$

   Show that $T(x) = e^x S(x)$ and $T(x) = 2S(x) + x - 1$ to deduce

$$\begin{aligned} T(x) &= e^x(1-x)/(2-e^x) \\ S(x) &= (1-x)/(2-e^x). \end{aligned} \tag{5.86}$$

   (b) [2] Let $c(n)$ denote the number of ordered partitions (or preferential arrangements) of $[n]$, so by Example 3.18.10 $E_c(x) = 1/(2-e^x)$. It follows from (5.86) that $s(n) = c(n) - nc(n-1)$. Give a direct combinatorial proof.
   (c) [3–] Let $\mathcal{T}_n$ denote the set of all hyperplanes $x_i + x_j = 0$, $1 \le i < j \le n$, in $\mathbb{R}^n$. The hyperplane arrangement $\mathcal{T}_n$ is called the *threshold arrangement*. Show that the number of regions of $\mathcal{T}_n$ (i.e., the number of connected components of the space $\mathbb{R}^n - \bigcup_{H \in \mathcal{T}_n} H$) is equal to $t(n)$.
   (d) [3–] Let L_n be the intersection poset of $\mathcal{T}_n$, as defined in Chapter 3.11.2. Show that the characteristic polynomial of L_n is given by

$$\sum_{n \ge 0} (-1)^n \chi(L_n, -q) \frac{x^n}{n!} = (1-x) \left(\frac{e^x}{2-e^x} \right)^{\frac{q+1}{2}}.$$

   This result generalizes (c), since by Theorem 3.11.7 the number of regions of $\mathcal{L}_n$ is equal to $|\chi(L_n, -1)|$.

5. [2+] Let $b_k(n)$ be the number of bipartite graphs (without multiple edges) with k edges on the vertex set $[n]$. For instance, $b_0(3) = 1$, $b_1(3) = 3$, $b_2(3) = 3$, and $b_3(3) = 0$. Show that

$$\sum_{n\geq 0}\sum_{k\geq 0} b_k(n) q^k \frac{x^n}{n!} = \left[\sum_{n\geq 0}\left(\sum_{i=0}^{n}(1+q)^{i(n-i)}\binom{n}{i}\right)\frac{x^n}{n!}\right]^{1/2}.$$

6. [2] Let $\chi(K_{mn}, q)$ denote the chromatic polynomial (as defined in Exercise 3.108) of the complete bipartite graph K_{mn}. Show that

$$\sum_{m,n\geq 0}\chi(K_{mn}, q)\frac{x^m}{m!}\frac{y^n}{n!} = (e^x + e^y - 1)^q.$$

7. In this exercise we develop the rudiments of the theory of "combinatorial trigonometry." Let E_n be the number of alternating permutations π of $[n]$, as discussed in Chapter 1.6.1. Thus $\pi = a_1 a_2 \cdots a_n$, where $a_1 > a_2 < a_3 > \cdots a_n$.
   (a) [2] Using the fact (Propostion 1.6.1) that

   $$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \tan x + \sec x,$$

   give a *combinatorial* proof that $1 + \tan^2 x = \sec^2 x$.
   NOTE. This identity is equivalent to $\sin^2 x + \cos^2 x = 1$, which in turn is equivalent to the Pythagorean theorem. Thus we have a combinatorial proof of the Pythagorean theorem. Of the hundreds of known proofs of this result, our combinatorial proof is perhaps the worst.
   (b) [2+] Do the same for the identity

   $$\tan(x+y) = \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)}. \tag{5.87}$$

8. (a) [2] The *central factorial numbers* $T(n, k)$ are defined for $n, k \in \mathbb{N}$ by

   $$T(0,0) = 1,\ T(n,0) = T(0,k) = 0 \text{ for } n,k \geq 1,\ T(1,1) = 1$$

   $$T(n,k) = k^2 T(n-1,k) + T(n-1,k-1), \text{ for } (n,k) \in \mathbb{P}^2 - \{(1,1)\}.$$

   Show that

   $$T(n,k) = 2\sum_{j=1}^{k}\frac{j^{2n}(-1)^{k-j}}{(k-j)!\,(k+j)!}$$

   and

   $$\sum_{n\geq 0} T(n,k)\frac{x^{2n}}{(2n)!} = \frac{1}{(2k)!}\left(2\sinh\left(\frac{x}{2}\right)\right)^{2k}. \tag{5.88}$$

(b) [2] Show that

$$\sum_{n\geq 0} T(n,k)x^n = \frac{x^k}{(1-1^2x)(1-2^2x)\cdots(1-k^2x)}.$$

(c) [2] Show that $T(n,k)$ is equal to the number of partitions of the set $\{1,1',2,2',\ldots,n,n'\}$ into k blocks, such that for every block B, if i is the least integer for which $i\in B$ or $i'\in B$, then both $i\in B$ and $i'\in B$.

(d) [2+] The *Genocchi numbers* G_n are defined by

$$\begin{aligned}\frac{2x}{e^x+1} &= \sum_{n\geq 1} G_n \frac{x^n}{n!}\\ &= x-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{3x^6}{6!}+\frac{17x^8}{8!}-\frac{155x^{10}}{10!}+\frac{2073x^{12}}{12!}-\cdots.\end{aligned}$$

Show that $G_{2n+1}=0$ if $n\geq 1$, and that $(-1)^nG_{2n}$ is an odd positive integer. (Sometimes $(-1)^nG_{2n}$ is called a Genocchi number.) Note also that

$$x\tan\frac{x}{2} = \sum_{n\geq 1}(-1)^n G_{2n}\frac{x^{2n}}{(2n)!}.$$

(e) [3] Show that

$$G_{2n+2} = \sum_{i=1}^{n}(-1)^{i+1}(i!)^2 T(n,i).$$

(f) [3] Show that $(-1)^nG_{2n}$ counts the following:

(i) the number of permutations $\pi\in\mathfrak{S}_{2n-2}$ such that $1\leq\pi(2i-1)\leq 2n-2i$ and $2n-2i\leq\pi(2i)\leq 2n-2$.

(ii) The number of permutations $\pi\in\mathfrak{S}_{2n-1}$ with descents after even numbers and ascents after odd numbers, for example, 2143657 and 3564217. (Such permutations must end with $2n-1$.)

(iii) The number of pairs $(a_1,a_2,\ldots,a_{n-1})$ and $(b_1,b_2,\ldots,b_{n-1})$ such that $a_i,b_i\in[i]$ and every $j\in[n-1]$ occurs at least once among the a_i's and b_i's.

(iv) The number of reverse alternating permutations $a_1<a_2>a_3<a_4>\cdots>a_{2n-1}$ of $[2n-1]$ whose inversion table (as defined in Chapter 1.3) has only even entries. For example, for $n=3$ we have the three permutations $45231, 34251, 24153$ with inversion tables $42200, 42000, 20200$.

9. Let S be a "structure" that can be put on a finite set by choosing a partition of S and putting a "connected" structure on each block, so that the exponential formula (Corollary 5.1.6) is applicable. Let $f(n)$ be the number of

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structures that can be put on an n-set, and let $F(x) = E_f(x)$, the exponential generating function of f.

(a) [2–] Let $g(n)$ be the number of structures that can be put on an n-set so that every connected component has even cardinality. Show that

$$E_g(x) = \sqrt{F(x)F(-x)}.$$

(b) [2] Let $e(n)$ be the number of structures that can be put on an n-set so that the number of connected components is even. Show that

$$E_e(x) = \frac{1}{2}\left(F(x) + \frac{1}{F(x)}\right).$$

10. (a) [2–] Let $k > 2$. Give a generating function proof that the number $f_k(n)$ of permutations $\pi \in \mathfrak{S}_n$ all of whose cycle lengths are divisible by k is given by

$$1^2 \cdot 2 \cdot 3 \cdots (k-1)(k+1)^2(k+2)\cdots(2k-1)(2k+1)^2(2k+2)\cdots(n-1)$$

if $k|n$, and is 0 otherwise.

(b) [2] Give a combinatorial proof of (a).

(c) [2] Let $k \in \mathbb{P}$. Give a generating function proof that the number $g_k(n)$ of permutations $\pi \in \mathfrak{S}_n$ none of whose cycle lengths is divisible by k is given by

$$1 \cdot 2 \cdots (k-1)^2(k+1)\cdots(2k-2)(2k-1)^2(2k+1)\cdots(n-1)n, \quad \text{if } k \nmid n$$

$$1 \cdot 2 \cdots (k-1)^2(k+1)\cdots(2k-2)(2k-1)^2(2k+1)\cdots(n-2)(n-1)^2, \quad \text{if } k|n.$$

(d) [3–] Give a combinatorial proof of (c).

11. (a) [2] Let $a(n)$ be the number of permutations w in $\mathfrak{S}_n$ that have a square root, that is, there exists $u \in \mathfrak{S}_n$ satisfying $u^2 = w$. Show that

$$\sum_{n\geq 0} a(n)\frac{x^n}{n!} = \left(\frac{1+x}{1-x}\right)^{1/2} \prod_{k\geq 1} \cosh(x^{2k}/2k).$$

(b) [2–] Deduce from (a) that $a(2n+1) = (2n+1)a(2n)$. Is there a simple combinatorial proof?

12. [2+] Let $f(n)$ be the number of pairs (u, v) of permutations in $\mathfrak{S}_n$ satisfying $u^2 = v^2$. Find the exponential generating function $F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$.

13. (a) [2+] Let G be a finitely generated group, and let $\mathrm{Hom}(G, \mathfrak{S}_n)$ denote the set of homomorphisms $G \to \mathfrak{S}_n$. Let $j_d(G)$ denote the number of subgroups of G of index d. Show that

$$\sum_{n\geq 0} \#\mathrm{Hom}(G, \mathfrak{S}_n)\frac{x^n}{n!} = \exp\left(\sum_{d\geq 1} j_d(G)\frac{x^d}{d}\right).$$

Note that equation (5.31) is equivalent to the case $G = \mathbb{Z}/r\mathbb{Z}$.

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(b) [1+] Let F_s denote the free group on s generators. Deduce from (a) that

$$\sum_{n\geq 0} n!^{s-1}x^n = \exp\left(\sum_{d\geq 1} j_d(F_s)\frac{x^d}{d}\right). \tag{5.89}$$

(c) [3–] With G as above, let $u_d(G)$ denote the number of conjugacy classes of subgroups of G of index d. In particular, if every subgroup of G of index d is normal (e.g., if G is abelian) then $u_d(G) = j_d(G)$. Show that

$$\sum_{n\geq 0} \#\mathrm{Hom}(G\times\mathbb{Z},\mathfrak{S}_n)\frac{x^n}{n!} = \prod_{d\geq 1}\left(1-x^d\right)^{-u_d(G)}. \tag{5.90}$$

(d) [1+] Let $c_m(n)$ be the number of commuting m-tuples $(u_1,\ldots,u_m)\in\mathfrak{S}_n^m$, that is, $u_iu_j = u_ju_i$ for all i and j. Deduce from (c) that

$$\sum_{n\geq 0} c_m(n)\frac{x^n}{n!} = \prod_{d\geq 1}\left(1-x^d\right)^{-j_d\left(\mathbb{Z}^{m-1}\right)}.$$

(e) [3–] Let $h_k(n)$ be the number of graphs (with multiple edges allowed) on the vertex set $[n]$ with edges colored $1,2,\ldots,k-1$ satisfying the following properties:

i. For each i, the edges colored i have no vertices in common.
ii. For each $i < k-1$, every connected component of the (spanning) subgraph consisting of all edges colored i and $i+1$ is either a single vertex, a path of length two, a two-cycle (that is, an edge colored i and an edge colored $i+1$ with the same vertices), or a six-cycle.
iii. For each i,j such that $j-i\geq 2$, every connected component of the subgraph consisting of all edges colored i and j is either a single vertex, a single edge (colored either i or j), a two-cycle, or a four-cycle.

Show that

$$\sum_{n\geq 0} h_k(n)\frac{x^n}{n!} = \exp\left(\sum_{d\mid k!} j_d(\mathfrak{S}_k)\frac{x^d}{d}\right).$$

14. (a) [2–] Let $A_n(t)$ denote an Eulerian polynomial, as defined in Chapter 1.3, and set $y = \sum_{n\geq 1} A_n(t)\frac{x^n}{n!}$. Show that y is the unique power series for which there exists a power series z satisfying the two formulas

$$1+y = \exp(tx+z)$$
$$1+t^{-1}y = \exp(x+z).$$

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(b) [2+] Show that the power series z of (a) is given by

$$z = \sum_{n \geq 2} A_{n-1}(t) \frac{x^n}{n!}.$$

(c) [2+] Set $(1+y)^q = \sum_{n \geq 0} B_n(q,t) \frac{x^n}{n!}$. Show that

$$B_n(q,t) = \sum_{w \in \mathfrak{S}_n} q^{m(w)} t^{1+d(w)},$$

where $m(w)$ denotes the number of left-to-right minima of w, and $d(w)$ denotes the number of descents of w.

(d) [2–] Deduce that the coefficient of $\frac{x^n}{n!}$ in $(1+y)^{q/t}$ is a polynomial in q and t with integer coefficients.

15. [2] For each of the following sets of graphs, let $f(n)$ be the number of graphs G on the vertex set $[n]$ such that every connected component of G is isomorphic to some graph in the set. Find for each set $E_f(x) = \sum_{n \geq 0} f(n) x^n / n!$. (Set $f(0) = 1$.)
    (a) cycles C_i of length $i \geq k$ (for some fixed $k \geq 3$)
    (b) stars K_{1i}, $i \geq 1$ (K_{rs} denotes a complete bipartite graph)
    (c) wheels W_i with $i \geq 4$ vertices (W_i is obtained from C_{i-1} by adding a new vertex joined to every vertex of C_{i-1})
    (d) paths P_i with $i \geq 1$ vertices (so P_1 is a single vertex and P_2 is a single edge).
16. Let G be a simple graph (i.e., no loops or multiple edges) on the vertex set $[n]$. The (ordered) *degree sequence* of G is defined to be $d(G) = (d_1, \ldots, d_n)$, where d_i is the degree (number of incident edges) of vertex i. Let $f(n)$ be the number of *distinct* degree sequences of simple graphs on the vertex set $[n]$. For instance, all eight graphs on $[3]$ have different degree sequences, so $f(3) = 8$. On the other hand, there are three graphs on $[4]$ with degree sequence $(1,1,1,1)$, so $f(4) < 2^{\binom{4}{2}} = 64$. (In fact, $f(4) = 54$.)
    (a) [3+] Show that

$$f(n) = \sum_X \max\left\{1, 2^{c(X)-1}\right\}, \tag{5.91}$$

where X ranges over all graphs on $[n]$ such that every connected component is either a tree or has a single cycle, and all cycles of X are of odd length; and where $c(X)$ denotes the number of (odd) cycles of X.

(b) [3–] Let

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$$

$$= 1 + x + 2\frac{x^2}{2!} + 8\frac{x^3}{3!} + 54\frac{x^4}{4!} + 533\frac{x^5}{5!} + 6944\frac{x^6}{6!} + \cdots .$$

Assuming (a), show that

$$F(x) = \frac{1}{2}\left[\left(1 + 2\sum_{n\geq 1} n^n \frac{x^n}{n!}\right)^{1/2} \left(1 - \sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right) + 1\right] e^{\sum_{n\geq 1} n^{n-2}x^n/n!},$$

where we set $0^0 = 1$ in the term $n = 1$ of the second sum on the right.

17. (a) [2] Fix $k, n \in \mathbb{P}$. In how many ways may n people form exactly k lines? (In other words, how many ways are there of partitioning the set $[n]$ into k blocks, and then linearly ordering each block?) Give a simple combinatorial proof.

(b) [2–] Deduce that

$$1 + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1} x^k \frac{u^n}{n!} = \exp\frac{xu}{1-u}.$$

(c) [2+] Let $a \in \mathbb{P}$. Extend the argument of (a) to deduce that

$$1 + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{n!}{k!}\binom{n+(a-1)k-1}{n-k} x^k \frac{u^n}{n!} = \exp\frac{xu}{(1-u)^a} \tag{5.92}$$

and

$$1 + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{n!}{k!}\binom{ak}{n-k} x^k \frac{u^n}{n!} = \exp xu(1+u)^a. \tag{5.93}$$

NOTE. Since these identities hold for all $a \in \mathbb{P}$, they also hold if a is an indeterminate.

(d) [2] Fix $k, n, \alpha \in \mathbb{N}$. Let A be a set of cardinality α disjoint from $[n]$. In how many ways can we choose a subset S of $[n]$, then choose a partition π of S into exactly k blocks, then linearly order each block of π, and finally choose an injection $f : \bar{S} \to \bar{S} \cup A$, where $\bar{S} = [n] - S$? Give a simple combinatorial proof.

(e) [2–] Deduce that

$$\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}(\alpha+n)_{n-k}\,x^k\frac{u^n}{n!}=(1-u)^{-\alpha-1}\exp\frac{xu}{1-u}.$$

(Note that we obtain (b) by setting $\alpha=-1$.)

18. [2] Call two permutations $\pi,\sigma\in\mathfrak{S}_n$ *equivalent* if every cycle C of π is a power D^j (where j depends on C) of some cycle D of σ. Clearly this is an equivalence relation; let $e(n)$ be the number of equivalence classes (with $e(0)=1$). Show that

$$\sum_{n\geq 0}e(n)\frac{x^n}{n!}=\exp\sum_{n\geq 1}\frac{x^n}{n\phi(n)},$$

where ϕ is Euler's phi-function.

19. [3–] Define polynomials $K_n(a)$ by

$$\sum_{n\geq 0}K_n(a)\frac{u^n}{n!}=\exp\left(au+\frac{u^2}{2}\right).$$

Thus it follows from Example 5.2.10 that

$$K_n(a)=\sum_{\pi}a^{c_1(\pi)}, \tag{5.94}$$

where π ranges over all involutions (i.e., $\pi^2=1$) in $\mathfrak{S}_n$, and $c_1(\pi)$ is the number of 1-cycles (fixed points) of π. Using (5.94), give a combinatorial proof of the identity

$$\sum_{n\geq 0}K_n(a)K_n(b)\frac{x^n}{n!}=(1-x^2)^{-1/2}\exp\left[\frac{abx+\frac{1}{2}(a^2+b^2)x^2}{1-x^2}\right]. \tag{5.95}$$

20. (a) [2+] A *block* is a finite connected graph B (allowing multiple edges but not loops) with at least two vertices such that the removal of any vertex v and all edges incident to v leaves a connected graph. Let $\mathcal{B}$ be a collection of nonisomorphic blocks. Let $b(n)$ be the number of blocks on the vertex set $[n]$ which are isomorphic to some block in $\mathcal{B}$. In other words, if $\operatorname{Aut}B$ denotes the automorphism group of the block B, then

$$b(n)=\sum_{B}\frac{n!}{\#(\operatorname{Aut}B)},$$

summed over all n-vertex blocks B in $\mathcal{B}$. Call a graph G a $\mathcal{B}$-*graph* if it is connected and its maximal blocks (i.e., maximal induced subgraphs which are blocks) are all isomorphic to members of $\mathcal{B}$. For $n\geq 2$, let $f(n)$ be the number of *rooted* $\mathcal{B}$-graphs on an n-element vertex set V (i.e., a $\mathcal{B}$-graph with a vertex chosen as a root). Set $f(0) = 0$ and $f(1) = 1$, and put

$$B(x) = E_b(x) = \sum_{n\geq 2} b(n)\frac{x^n}{n!}$$

$$F(x) = E_f(x) = \sum_{n\geq 1} f(n)\frac{x^n}{n!}.$$

Show that

$$F(x) = xe^{B'(F(x))}, \tag{5.96}$$

and hence

$$\sum_{n\geq 1} b(n+1)\frac{x^n}{n!} = \log\left(\frac{x}{F^{\langle -1\rangle}(x)}\right). \tag{5.97}$$

For instance, if $\mathcal{B}$ contains only the single block consisting of one edge, then a $\mathcal{B}$-graph is a (free) tree. Hence $f(n)$ is the number of rooted trees on n vertices, $B(x) = x^2/2!$, and $F(x) = xe^{F(x)}$ (agreeing with Proposition 5.3.1).

(b) [2] Let $g(n)$ be the total number of blocks without multiple edges on an n-element vertex set. Show that

$$\sum_{n\geq 1} g(n+1)\frac{x^n}{n!} = \log\left(\frac{x}{G(x)^{\langle -1\rangle}}\right),$$

where

$$G(x) = \frac{\displaystyle\sum_{n\geq 1} 2^{\binom{n}{2}}\frac{x^n}{(n-1)!}}{\displaystyle\sum_{n\geq 0} 2^{\binom{n}{2}}\frac{x^n}{n!}}.$$

21. [3–] Find a combinatorial proof of equation (4.41). More specifically, using the notation of Chapter 4.7.4, given a pair (π, u), where $\pi \in \mathfrak{S}_n$ and $u \in \mathcal{B}_n^*$, associate with it in bijective fashion a permutation $\sigma \in \mathfrak{S}_n$ with a cyclic shift v_C of an element of $\mathcal{B}_k^*$ attached to each k-cycle C of σ. The multiset of letters in u should coincide with those in all the v_C's so that the bijection is weight-preserving.
22. [2] Let $L(n)$ be the function of Example 5.2.9, so in particular $L(n)$ is the number of graphs on the vertex set $[n]$ for which every component is a

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cycle (including loops and double edges). Give a direct combinatorial proof that

$$L(n+1) = (n+1)L(n) - \binom{n}{2}L(n-2), \ n \geq 2.$$

23. [2] Let Δ be a set $\{\delta_1, \dots, \delta_n\}$ of n straight lines in the plane lying in general position (i.e., no two are parallel and no three meet at a point). Let P be the set of their points $\delta_i \cap \delta_j$ of intersection, so $\#P = \binom{n}{2}$. A *cloud* is an n-subset of P containing no three collinear points. Find a bijection between clouds and regular graphs on $[n]$ (without loops and multiple edges) of degree two. Hence by (5.29) if $c(n)$ is the number of clouds for $\#\Delta = n$, then

$$\sum_{n \geq 0} c(n) \frac{x^n}{n!} = (1-x)^{-1/2} \exp\left(-\frac{x}{2} - \frac{x^2}{4}\right).$$

24. (a) [2+] Let Σ_n be the convex polytope of all $n \times n$ *symmetric* doubly-stochastic matrices. Show that the extreme points (vertices) of Σ_n consist of all matrices $\frac{1}{2}(P + P^t)$, where P is a permutation matrix corresponding to a permutation with no cycles of even length ≥ 4.
(b) [2+] Let $M(n)$ be the number of vertices of Σ_n. Show that

$$\sum_{n \geq 0} M(n) \frac{x^n}{n!} = \left(\frac{1+x}{1-x}\right)^{1/4} \exp\left(\frac{x}{2} + \frac{x^2}{2}\right). \tag{5.98}$$

(c) [2+] Find polynomials $p_0(n), \dots, p_3(n)$ such that

$$M(n+1) = p_0(n)M(n) + p_1(n)M(n-1) + p_2(n)M(n-2) + p_3(n)M(n-3),$$

for all $n \geq 3$.
(d) [2+] Find a direct combinatorial proof of (c), analogous to Exercise 22?

25. (a) [2+] Let Σ_n^* be the convex polytope of all $n \times n$ *symmetric substochastic* matrices (i.e., the entries are ≥ 0, and all line sums are ≤ 1). Show that the vertices of Σ_n^* are obtained from those of Σ_n (defined in the previous exercise) by replacing some 1's on the main diagonal by 0's.
(b) [2] Let $M^*(n)$ be the number of vertices of Σ_n^*. Show that

$$\sum_{n \geq 0} M^*(n) \frac{x^n}{n!} = e^x \sum_{n \geq 0} M(n) \frac{x^n}{n!},$$

where $M(n)$ is defined in Exercise 24.
(c) [2] Find polynomials $p_0^*(n), \dots, p_3^*(n)$ such that

$$M^*(n+1) = p_0^*(n)M^*(n) + p_1^*(n)M^*(n-1) + p_2^*(n)M^*(n-2) + p_3^*(n)M^*(n-3).$$

26. [2+] Let $f(n)$ be the number of sets S of nonempty subsets of $[n]$ (including $S = \emptyset$) such that any two elements of S are either disjoint or comparable (with respect to inclusion). Let $g(n)$ be the number of such sets S which contain $[n]$, with $g(0) = 0$. Set

$$\begin{aligned} F(x) = E_f(x) &= 1 + 2x + 8\frac{x^2}{2!} + 64\frac{x^3}{3!} + 832\frac{x^4}{4!} + 15104\frac{x^5}{5!} + \cdots \\ G(x) = E_g(x) &= x + 4\frac{x^2}{2!} + 32\frac{x^3}{3!} + 416\frac{x^4}{4!} + 7552\frac{x^5}{5!} + \cdots . \end{aligned}$$

Show that $F(x) = 1 + 2G(x)$ and $F(x) = e^{x+G(x)}$. Hence [why?]

$$G(x) = (\log(1 + 2x) - x)^{\langle -1 \rangle} \tag{5.99}$$

$$F(x) - 1 = \left(\log(1 + x) - \frac{x}{2}\right)^{\langle -1 \rangle} .$$

27. [2] Find the number $e(n)$ of trees with $n + 1$ unlabelled vertices and n labelled *edges*. Give a simple bijective proof.
28. [2+] Let $k \in \mathbb{P}$. A *k-edge colored tree* is a tree whose edges are colored from a set of k colors such that any two edges with a common vertex have different colors. Show that the number $T_k(n)$ of k-edge colored trees on the vertex set $[n]$ is given by

$$T_k(n) = k(nk - n)(nk - n - 1) \cdots (nk - 2n + 3) = k(n - 2)! \binom{nk - n}{n - 2}.$$

29. (a) [2] Let P_n be the set of all planted forests on $[n]$. Let uv be an edge of a forest $F \in P_n$ such that u is closer than v to the root of its component. Define F to *cover* the rooted forest F' if F' is obtained by removing the edge uv from F, and rooting the new tree containing v at v. This definition of cover defines the covering relation of a partial order on P_n. Under this partial order P_n is graded of rank $n - 1$. The rank of a forest F in P_n is its number of edges. Show that an element F of P_n of rank i covers i elements and is covered by $(n - i - 1)n$ elements.
    (b) [2] By counting in two ways the number of maximal chains of P_n, deduce that the number $r(n)$ of rooted trees on $[n]$ is equal to n^{n-1}.
    (c) [2+] Let $\bar{P}_n$ be P_n with a $\hat{1}$ adjoined. Show that

    $$\mu(\hat{0}, \hat{1}) = (-1)^n (n - 1)^{n-1},$$

    where μ denotes the Möbius function of $\bar{P}_n$.
30. [2+] Let $R = \{1, 2, \ldots, r\}$ and $S = \{1', 2', \ldots, s'\}$ be disjoint sets of cardinalities r and s, respectively. A *free bipartite tree* with vertex bipartition

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(R,S) is a free tree T on the vertex set $R \cup S$ such that every edge of T is incident to a vertex in R and a vertex in S. By modifying the two proofs of Theorem 5.3.4, give two combinatorial proofs that

$$\sum_T \left(\prod_{i\in R} x_i^{\deg i}\right)\left(\prod_{j'\in S} y_j^{\deg j'}\right) =$$

$$(x_1\cdots x_r)(y_1\cdots y_s)(x_1+\cdots+x_r)^{s-1}(y_1+\cdots+y_s)^{r-1}, \quad (5.100)$$

summed over all free bipartite trees T with vertex bipartition (R,S). In particular, the total number of such trees (i.e., the *complexity* $c(K_{rs})$ of the complete bipartite graph K_{rs}) is $r^{s-1}s^{r-1}$, agreeing with the computation at the end of the solution to Exercise 2.27(c).

31. (a) [1+] Let S and T be finite sets, and for each $t \in T$ let x_t be an indeterminate. Show that

$$\sum_{f:S\to T}\prod_{s\in S} x_{f(s)} = \left(\sum_{t\in T} x_t\right)^{\#S},$$

where the first sum ranges over all functions $f : S \to T$.

(b) [3–] By considering the case $S = [n]$ and $T = [n+2]$, show that

$$(x_1+\cdots+x_{n+2})^n = \sum_{A\subseteq[n]} x_{n+1}\left(x_{n+1}+\sum_{i\in A} x_i\right)^{\#A-1}\left(x_{n+2}+\sum_{i\in A'} x_i\right)^{n-\#A},$$

where $A' = [n] - A$. Note that when $A = \emptyset$, we have

$$x_{n+1}\left(x_{n+1}+\sum_{i\in A} x_i\right)^{\#A-1} = 1.$$

(c) [2–] Deduce from (b) that

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k},$$

where x, y, z are indeterminates. Note that the case $z = 0$ is the binomial theorem.

(d) [2–] Deduce from (c) the identity

$$\sum_{n\geq 0}(n+1)^n\frac{x^n}{n!} = \left(\sum_{n\geq 0} n^n\frac{x^n}{n!}\right)\left(\sum_{n\geq 0}(n+1)^{n-1}\frac{x^n}{n!}\right),$$

where we set $0^0 = 1$.

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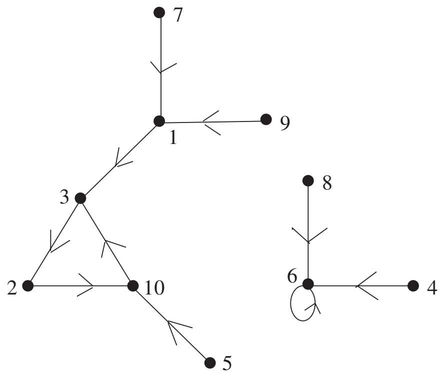

Figure 5.18 The digraph D_f of a function $f : [10] \to [10]$

32. (a) [2+] Let $f : [n] \to [n]$, and let D_f denote the digraph of f, that is, the directed graph on the vertex set $[n]$ with an arrow from i to j if $f(i) = j$. Thus every connected component of D_f contains a unique cycle, and every vertex i of this cycle is the root of a rooted tree (possibly consisting of the single point i) directed toward i. Let $w_f(i) = t_{jk}$ (an indeterminate) if vertex i is at distance k from a j-cycle of D_f. Let

$$w(f) = \prod_{i=1}^{n} w_f(i).$$

For instance, if D_f is given by Figure 5.18, then $w_f(1) = t_{31}, w_f(2) = t_{30}, w_f(3) = t_{30}, w_f(4) = t_{11}, w_f(5) = t_{31}, w_f(6) = t_{10}, w_f(7) = t_{32}, w_f(8) = t_{11}, w_f(9) = t_{32}, w_f(10) = t_{30}$, so

$$w(f) = t_{30}^3 t_{31}^2 t_{32}^2 t_{10} t_{11}^2.$$

The (augmented) *cycle index* or *cycle indicator* $\tilde{Z}_n(t_{jk})$ of the *symmetric semigroup* $\Lambda_n = [n]^{[n]}$ of all functions $f : [n] \to [n]$ is the polynomial defined by

$$\tilde{Z}_n(t_{jk}) = \sum_{f \in \Lambda_n} w(f).$$

For instance,

$$\tilde{Z}_2 = t_{10}^2 + t_{20}^2 + 2t_{10}t_{11}.$$

Note that

$$\tilde{Z}_n(t_{jk})|_{t_{jk}=0 \text{ for } k>0} = \tilde{Z}(\mathfrak{S}_n, t_{10}, t_{20}^2, t_{30}^3, \dots),$$

where $\tilde{Z}(\mathfrak{S}_n)$ is defined in Example 5.2.10.

Show that

$$\sum_{n\geq 0} \tilde{Z}_n(t_{jk})\frac{x^n}{n!} = \exp \sum_{j\geq 1} \frac{1}{j}\left[t_{j0}xe^{t_{j1}xe^{t_{j2}xe^{\cdot^{\cdot^{\cdot}}}}} \right]^j. \tag{5.101}$$

(b) [1+] Put each $t_{jk} = 1$ to deduce (with $0^0 = 1$) that

$$\begin{aligned} \sum_{n\geq 0} n^n \frac{x^n}{n!} &= \left[1 - xe^{xe^{xe^{\cdot^{\cdot^{\cdot}}}}} \right]^{-1} \\ &= \left(1 - \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!} \right)^{-1}. \end{aligned}$$

(c) [2] Fix $a, b \in \mathbb{P}$. Let $g(n)$ denote the number of functions $f : [n] \to [n]$ satisfying $f^a = f^{a+b}$ (exponents denote functional composition). Show that

$$\sum_{n\geq 0} g(n)\frac{x^n}{n!} = \exp \sum_{j|b} \frac{1}{j}\left[\underbrace{xe^{xe^{xe^{\cdot^{\cdot^{\cdot^{xe^x}}}}}}}_{a\ e\text{'s}} \right]^j \tag{5.102}$$

In particular, if $a = 1$ then

$$\sum_{n\geq 0} g(n)\frac{x^n}{n!} = \exp \sum_{j|b} \frac{1}{j}(xe^x)^j. \tag{5.103}$$

(d) [2] Deduce from (a) or (c) that the number $h(n)$ of functions $f : [n] \to [n]$ satisfying $f = f^{1+b}$ for *some* $b \in \mathbb{P}$ is given by

$$h(n) = \sum_{k=1}^{n} k^{n-k}(n)_k, \tag{5.104}$$

while the number $g(n)$ of idempotent functions $f : [n] \to [n]$ (i.e., $f^2 = f$) is given by

$$g(n) = \sum_{k=1}^{n} k^{n-k}\binom{n}{k}. \tag{5.105}$$

(e) [2–] How many functions $f : [n] \to [n]$ satisfy $f^a = f^{a+1}$ for some $a \in \mathbb{P}$?

(f) [1+] How many functions $f : [n] \to [n]$ have no fixed points?

33. [2] Find the flaw in the following argument. Let $c(n)$ be the total number of chains $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$ in Π_n. Thus from Chapter 3.6,

$$c(n) = (2-\zeta)^{-1}(\hat{0},\hat{1}),$$

where ζ is the zeta function of Π_n. Since

$$(2-\zeta)(x,y) = \begin{cases} 1, & x = y \\ -1, & x < y, \end{cases}$$

we have

$$E_{2-\zeta}(x) = x - \sum_{n\geq 2} \frac{x^n}{n!} = 1 + 2x - e^x.$$

Thus by Theorem 5.1.11, the generating function

$$y := E_c(x) = \sum_{n\geq 1} c(n)\frac{x^n}{n!}$$

satisfies

$$1 + 2y - e^y = x.$$

Equivalently,

$$y = (1 + 2x - e^x)^{\langle -1\rangle},$$

which is the same as (5.27).

34. (a) [2] Fix $k \in \mathbb{P}$, and for $n \in \mathbb{N}$ define Ψ_n to be the subposet of Π_{kn+1} consisting of all partitions whose block sizes are $\equiv 1 \pmod{k}$. Thus Ψ_n is graded of rank n with rank function given by $\rho(\pi) = n - \frac{1}{k}(|\pi| - 1)$. Note that if $k = 1$, then $\Psi_n = \Pi_{n+1}$. It is easy to see that if $\sigma \leq \pi$ in Ψ_n, then

$$[\sigma,\pi] \cong \Psi_0^{a_0} \times \Psi_1^{a_1} \times \cdots \times \Psi_n^{a_n}$$

for certain a_i satisfying $\sum ia_i = \rho(\sigma,\pi)$ (= the length of the interval $[\sigma,\pi]$) and $\sum a_i = |\pi|$. As in equations (5.11) and (5.12) we can define a *multiplicative function* $f : \mathbb{P} \to K$ on $\boldsymbol{\Psi} = (\Psi_0, \Psi_1, \ldots)$, and the product (convolution) fg of two multiplicative functions. Lemma 5.1.10 remains true, so the multiplicative functions $f : \mathbb{P} \to K$ on $\boldsymbol{\Psi}$ form a monoid $M(\boldsymbol{\Psi}) = M(\boldsymbol{\Psi}, K)$.

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As in Theorem 5.1.11, define a map $\varphi : M(\Psi) \to xK[[x]]$ by

$$\varphi(f) = \sum_{n\geq 0} f(n)\frac{x^{kn+1}}{(kn+1)!}.$$

Show that φ is an anti-isomorphism of monoids, so $\varphi(fg) = \varphi(g)(\varphi(f))$ (power series composition).

(b) [1+] Let $q_n = \#\Psi_n$ and $\mu_n = \mu_{\Psi_n}(\hat{0}, \hat{1})$. Show that

$$\sum_{n\geq 0} q_n \frac{x^{kn+1}}{(kn+1)!} = e_k(e_k(x))$$

$$\sum_{n\geq 0} \mu_n \frac{x^{kn+1}}{(kn+1)!} = e_k^{\langle -1\rangle}(x),$$

where $e_k(x) = \sum_{n\geq 0} x^{kn+1}/(kn+1)!$. In particular, when $k = 2$, $e_k(x) = \sinh x$.

(c) [2] Let $\chi_n(t)$ denote the characteristic polynomial of Ψ_n (as defined in Chapter 3.10). Show that

$$\sum_{n\geq 0} \chi_n(t)\frac{x^{kn+1}}{(kn+1)!} = t^{-1/k} e_k(t^{1/k} e_k^{\langle -1\rangle}(x)). \tag{5.106}$$

Deduce that when $k = 2$,

$$\chi_n(t) = (t-1^2)(t-3^2)\cdots(t-(2n-1)^2). \tag{5.107}$$

In particular, $\mu_n = (-1)^n(1\cdot 3\cdot 5\cdots(2n-1))^2$.

35. In this exercise we develop a noncrossing analogue of the exponential formula (Corollary 5.1.6) and its interpretation in terms of incidence algebras (Theorem 5.1.11).

(a) [2+] Show that the number of noncrossing partitions of $[n]$ (the elements of the poset $P_{1,n}$ of Exercise 3.158) of type $s_1, \dots, s_n$ (i.e., with s_i blocks of size i) is equal to $(n)_{k-1}/s_1!\cdots s_n!$, where $k = \sum s_i$.

(b) [2+] Let NC_n denote the poset (actually a lattice) of noncrossing partitions of $[n]$, as defined in Exercise 3.158 (where $P_{1,n}$ is used instead of NC_n). Let K be a field. Given a function $f : \mathbb{P} \to K$, define a new function $h : \mathbb{P} \to K$ by

$$h(n) = \sum_{\pi=\{B_1,\dots,B_k\}\in \mathrm{NC}_n} f(\#B_1)f(\#B_2)\cdots f(\#B_k).$$

Let $F(x) = 1 + \sum_{n\geq 1} f(n)x^n$ and $H(x) = 1 + \sum_{n\geq 1} h(n)x^n$. Show that

$$xH(x) = \left(\frac{x}{F(x)}\right)^{\langle -1\rangle}. \tag{5.108}$$

(c) [3–] Let $\mathbf{NC} = (\mathrm{NC}_2, \mathrm{NC}_3, \dots)$. For each $n \geq 2$, let $f_n \in I(\mathrm{NC}_n, K)$, the incidence algebra of NC_n. It is easy to see that every interval $[\sigma, \pi]$ of NC_n has a canonical decomposition

$$[\sigma, \pi] \cong \mathrm{NC}_2^{a_2} \times \mathrm{NC}_3^{a_3} \times \cdots \times \mathrm{NC}_n^{a_n}, \tag{5.109}$$

where $|\sigma| - |\pi| = \sum (i-1)a_i$. Suppose that the sequence $f = (f_2, f_3, \dots)$ satisfies the following property: there is a function (also denoted f) $f : \mathbb{P} \to K$ such that if $\sigma \leq \pi$ in NC_n and $[\sigma, \pi]$ satisfies (5.109), then

$$f_n(\sigma, \pi) = f(2)^{a_2} f(3)^{a_3} \cdots f(n)^{a_n}.$$

We then call f a *multiplicative function* on $\mathbf{NC}$. (This definition is in exact analogy with the definition of a multiplicative function on $\boldsymbol{\Pi}$ following Corollary 5.1.9.)

Let $M(\mathbf{NC})$ denote the set of all multiplicative functions on $\mathbf{NC}$. Define the convolution fg of $f, g \in M(\mathbf{NC})$ analogously to (5.12). It is not hard to see that $fg \in M(\mathbf{NC})$. Given $f \in M(\mathbf{NC})$, set $f(1) = 1$ and define

$$\Gamma_f(x) = \frac{1}{x}\left(\sum_{n\geq 1} f(n)x^n\right)^{\langle -1\rangle}.$$

Show that $\Gamma_{fg} = \Gamma_f \Gamma_g$ for all $f, g \in M(\mathbf{NC})$. (In particular, $M(\mathbf{NC})$ is a *commutative* monoid. This fact also follows by reasoning as in Exercise 3.155 and using the fact that every interval of NC_n is self-dual.)

36. (a) [2+] Find the coefficients of the power series

$$y = \left(\tfrac{1}{2}(1 + 2x - e^x)\right)^{\langle -1\rangle} - (\log(1+2x) - x)^{\langle -1\rangle}.$$

(b) [1+] Let $t(n)$ be the number of total partitions of n, as defined in Example 5.2.5. Let $g(n)$ have the same meaning as in Exercise 26. Deduce from (a) that $g(n) = 2^n t(n)$ for $n \geq 1$.

(c) [2+] Give a simple combinatorial proof of (b).

37. (a) [2+] Let $1 = p_0(x), p_1(x), \dots$ be a sequence of polynomials (with coefficients in some field K of characteristic 0), with $\deg p_n = n$ for all $n \in \mathbb{N}$, and with $p_n(0) = \delta_{0n}$. Show that the following four conditions are equivalent:

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(i) $p_n(x+y) = \sum_{k\geq 0} \binom{n}{k} p_k(x) p_{n-k}(y)$, for all $n \in \mathbb{N}$.

(ii) There exists a power series $f(u) = a_1 u + a_2 u^2 + \cdots \in K[[u]]$ such that

$$\sum_{n\geq 0} p_n(x) \frac{u^n}{n!} = \exp x f(u). \tag{5.110}$$

NOTE: The hypothesis that $\deg p_n = n$ implies that $a_1 \neq 0$.

(iii) $\displaystyle\sum_{n\geq 0} p_n(x) \frac{u^n}{n!} = \left[\sum_{n\geq 0} p_n(1) \frac{u^n}{n!} \right]^x$.

(iv) There exists a K-linear operator Q on the vector space $K[x]$ of all polynomials in x, with the following properties:

- Qx is a nonzero constant
- Q is a *shift-invariant operator*, that is, for all $a \in K$, Q commutes with the *shift operator* E^a defined by $E^a p(x) = p(x+a)$.
- We have

$$Qp_n(x) = np_{n-1}(x), \text{ for all } n \in \mathbb{P}. \tag{5.111}$$

NOTE: A sequence $p_0, p_1, \ldots$ of polynomials satisfying the above conditions is said to be of *binomial type*. The operator Q is called a *delta operator*, and the (unique) sequence $1 = p_0(x), p_1(x), \ldots$ satisfying (5.111) is called a *basic sequence* for Q.

(b) [3–] Show that the following sequences are of binomial type (with $p_0(x) = 1$ and with $n \geq 1$ below):

$$\begin{aligned} p_n(x) &= x^n \\ p_n(x) &= (x)_n = x(x-1)\cdots(x-n+1) \\ p_n(x) &= x^{(n)} = x(x+1)\cdots(x+n-1) \end{aligned}$$

$$p_n(x) = x(x-an)^{n-1}, \text{ for fixed } a \in K \text{ (Abel polynomials)}$$

$$p_n(x) = \sum_{k=1}^{n} S(n,k) x^k \text{ (exponential polynomials)}$$

$$p_n(x) = \sum_{k=1}^{n} \frac{n!}{k!} \binom{n+(a-1)k-1}{n-k} x^k, \text{ for fixed } a \in K$$

(Laguerre polynomials at $-x$, for $a = 1$)

$$p_n(x) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k} x^k.$$

In each case, find the power series $f(u)$ of (a)(ii) above. What is the operator Q of (a)(iv)?

(c) [2+] Let T be a shift-invariant operator, and let Q be a delta operator with basic sequence $p_n(x)$. Show that

$$T = \sum_{n \geq 0} a_n \frac{Q^n}{n!},$$

where

$$a_n = [Tp_n(x)]_{x=0}.$$

(d) [2+] Let Q be a delta operator with basic polynomials $p_n(x)$. Show that there exists a unique power series $q(u) = b_1 u + \cdots$ ($b_1 \neq 0$) satisfying $q(D) = Q$, where D is the shift-invariant operator $\frac{d}{dx}$. Show also that the power series $f(u)$ of (5.110) is given by $f(u) = q^{\langle -1 \rangle}(u)$.

(e) [2+] Suppose that $1 = p_0, p_1, \ldots$ is a sequence of polynomials of binomial type. Let

$$q_n(x) = \frac{x}{x + \alpha n} p_n(x + \alpha n),\ n \geq 0,$$

where α is a parameter. Show that the sequence $q_0, q_1, q_2, \ldots$ is also a sequence of polynomials of binomial type.

38. (a) [2–] Let P be a binomial poset with factorial function $B(n)$, and let $Z_n(x)$ be the zeta polynomial of an n-interval of P. (See Chapters 3.11 and 3.15 for definitions.) Show that $n!Z_n(x)/B(n), n \geq 0$, is a sequence of polynomials of binomial type, as defined in the previous exercise.

(b) [2–] Let $\boldsymbol{Q} = (Q_1, Q_2, \ldots)$ be an exponential structure with denominator sequence $(M(1), M(2), \ldots)$, and let $P_n(r, t)$ be the polynomial (in t) of equation (5.74). Set $M(0) = 1$. Show that for fixed $r \in \mathbb{Z}$ (or even r an indeterminate), the sequence of polynomials $P_n(r, x)/M(n), n \geq 0$, is a sequence of polynomials of binomial type. Note the special cases $r = 1$ (equation (5.72)) and $r = 0$ (equation (5.75)).

39. [2+] Let $f(n)$ be the number of partial orderings of $[n]$ which are isomorphic to posets P that can be obtained from a one-element poset by successive iterations of the operations $+$ (disjoint union) and $\oplus$ (ordinal

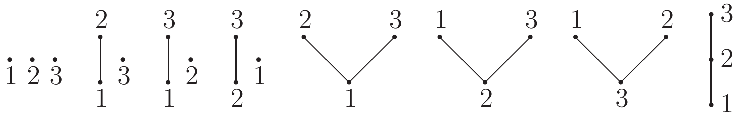

Figure 5.19 The eight inequivalent series-parallel posets on [3]

sum). Such posets are called *series-parallel* posets. For instance, all 19 partial orderings of [3] are counted by $f(3)$. Let

$$F(x) = \sum_{n\geq 1} f(n)\frac{x^n}{n!} = x+3\frac{x^2}{2!}+19\frac{x^3}{3!}+195\frac{x^4}{4!}+2791\frac{x^5}{5!}+51303\frac{x^6}{6!}+\cdots.$$

Show that

$$1+F(x) = \exp\left[x+\frac{F(x)^2}{1+F(x)}\right]. \tag{5.112}$$

Hence

$$\begin{aligned} F(x) &= \left(\log(1+x) - \frac{x^2}{1+x}\right)^{\langle -1\rangle} \\ &= \left(x - \frac{3}{2}x^2 + \frac{4}{3}x^3 - \frac{5}{4}x^4 + \frac{6}{5}x^5 - \cdots\right)^{\langle -1\rangle}. \end{aligned}$$

40. (a) [2+] Suppose that in the previous exercise we consider $P_1 \oplus P_2$ and $P_2 \oplus P_1$ to be equivalent. This induces an equivalence relation on the set of series-parallel posets on $[n]$. The equivalence classes are equivalent to what are called *series-parallel networks*. (The elements of the poset P correspond to the *edges* of a series-parallel network.) Figure 5.19 shows the eight inequivalent series-parallel posets on [3]. Let $s(n)$ be the number of equivalence classes of series-parallel posets on $[n]$ (or the number of series-parallel networks on n labelled edges), and set

$$\begin{aligned} S(x) &= \sum_{n\geq 1} s(n)\frac{x^n}{n!} \\ &= x+2\frac{x^2}{2!}+8\frac{x^3}{3!}+52\frac{x^4}{4!}+472\frac{x^5}{5!}+5504\frac{x^6}{6!}+\cdots. \end{aligned}$$

Show that

$$1+S(x) = \exp\left(\frac{1}{2}(x+S(x))\right). \tag{5.113}$$

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Hence

$$\begin{aligned} S(x) &= (2\log(1+x) - x)^{\langle -1\rangle} \\ &= \left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5 - \frac{1}{3}x^6 + \cdots\right)^{\langle -1\rangle}. \end{aligned}$$

(b) [3–] Two graphs G_1 and G_2 (without loops or multiple edges) on the vertex set $[n]$ are said to be *switching equivalent* if G_2 can be obtained from G_1 by choosing a subset X of $[n]$ and interchanging adjacency and non-adjacency between X and its complement $[n] - X$, leaving all edges within or outside X unchanged. Let $t(n)$ be the number of switching equivalence classes E of graphs on $[n]$ such that no graph in E contains an induced pentagon (5-cycle). Show that $t(n) = s(n-1)$.

(c) [3–] A (real) *vector lattice* is a real vector space V with the additional structure of a lattice such that

$$x \le y \Longrightarrow x + z \le y + z, \text{ for all } x, y, z \in V$$

$$x \ge 0 \Longrightarrow \alpha x \ge 0, \text{ for all } x \in V, \alpha \in \mathbb{R}^+.$$

There is an obvious notion of isomorphism of vector lattices. Show that the number of nonisomorphic n-dimensional vector lattices is equal to the number of nonisomorphic *unlabelled* equivalence classes (as defined in (a)) of n-element series-parallel posets.

41. (a) [2+] A tree on a linearly ordered vertex set is *alternating* (or *intransitive*) if for every vertex i the vertices adjacent to i are either all smaller than i or all larger than i. Let $f(n)$ denote the number of alternating trees on the vertex set $\{0, 1, \ldots, n\}$, and set

$$\begin{aligned} F(x) &= \sum_{n\ge 0} f(n)\frac{x^n}{n!} \\ &= 1 + x + 2\frac{x^2}{2!} + 7\frac{x^3}{3!} + 36\frac{x^4}{4!} + 246\frac{x^5}{5!} + 2104\frac{x^6}{6!} + 21652\frac{x^7}{7!} + \cdots. \end{aligned}$$

Show that $F(x)$ satisfies the functional equation

$$F(x) = \exp\left(\frac{x}{2}(F(x)+1)\right).$$

(Compare the similar but apparently unrelated (5.113).)

(b) [2] Deduce that

$$f(n) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{n-1}.$$

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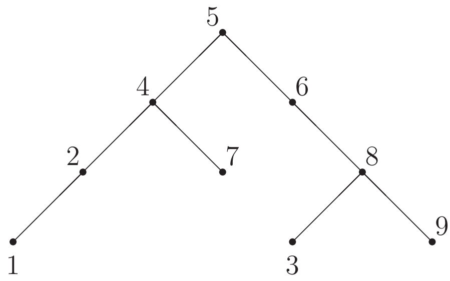

Figure 5.20 A local binary search tree.

(c) [2] Let $f_k(n)$ denote the number of alternating trees on $\{0, 1, \ldots, n\}$ such that vertex 0 has degree k. Set

$$P_n(q) = \sum_{k=1}^{n} f_k(n)q^k.$$

For instance,

$$P_0(q) = 1,\ \ P_1(q) = q,\ \ P_2(q) = q^2 + q,\ \ P_3(q) = q^3 + 3q^2 + 3q.$$

Show that

$$\sum_{n\geq 0} P_n(q)\frac{x^n}{n!} = F(x)^q.$$

(d) [2+] Show that

$$P_n(q) = \frac{q}{2^n}\sum_{k=0}^{n}\binom{n}{k}(q+k)^{n-1}.$$

(e) [3] Show that if z is a complex number for which $P_n(z) = 0$, then either $z = 0$ or $\Re(z) = -n/2$, where $\Re$ denotes real part.

(f) [2] Deduce from (e) that if $Q_n(q) = P_n(q)/q$, then

$$Q_n(q) = (-1)^{n-1}Q_n(-q-n).$$

(g) [3–] A *local binary search tree* is a (plane) binary tree, say with vertex set $[n]$, such that every left child of a vertex is less than its parent, and every right child is greater than its parent. An example of such a tree is shown in Figure 5.20. Show that $f(n)$ is equal to the number of local binary search trees with vertex set $[n]$.

(h) [3] Let $\mathcal{L}_n$ denote the set of all hyperplanes $x_i - x_j = 1$, $1 \leq i < j \leq n$, in $\mathbb{R}^n$. Show that the number of regions of $\mathcal{L}_n$ (i.e., the number of connected components of the space $\mathbb{R}^n - \bigcup_{H\in\mathcal{L}_n} H$) is equal to $f(n)$.

(i) [3] Let L_n be the intersection poset of $\mathcal{L}_n$, as defined in Exercise 3.56. Show that the characteristic polynomial of L_n is given by

$$\chi(L_n, q) = (-1)^n P_n(-q).$$

This result generalizes (h), since by Exercise 3.56(a) the number of regions of $\mathcal{L}_n$ is equal to $|\chi(L_n, -1)|$.

(j) [3–] An *alternating graph* on $[n]$ is a graph (without loops or multiple edges) on the vertex set $[n]$ such that every vertex is either smaller than all its neighbors or greater than all its neighbors. Let $g_k(n)$ denote the number of alternating graphs on $[n]$ with k edges. Show that

$$\sum_{n\geq 0}\sum_{k\geq 0} g_k(n) q^k \frac{x^n}{n!} = e^{-x} \sum_{n\geq 0} \left(\sum_{k=0}^{n} \binom{\boldsymbol{n}}{\boldsymbol{k}}_{q+1} \right) \frac{x^n}{n!},$$

where $\binom{\boldsymbol{n}}{\boldsymbol{k}}_{q+1}$ denotes the q-binomial coefficient $\binom{\boldsymbol{n}}{\boldsymbol{k}}$ with the variable q replaced by $q+1$.

(k) [2+] An *edge labelled alternating tree* is a tree, say with $n+1$ vertices, whose edges are labelled $1, 2, \ldots, n$ such that no path contains three consecutive edges whose labels are increasing. How many edge labelled alternating trees have $n+1$ vertices?

42. (a) [2] Let $y = R(x) = \sum_{n\geq 1} n^{n-1} \frac{x^n}{n!}$. Show from $y = xe^y$ that

$$(1 - R(x))^{-1} = 1 + \sum_{n\geq 1} n^n \frac{x^n}{n!}.$$

(b) [2+] Give a combinatorial proof, based on the fact that n^{n-1} is the number of rooted trees and n^n the number of double rooted trees on $[n]$.

43. [2] Generalize the bijection of Example 5.2.6 to show the following. Fix a sequence $(r_1, r_2, \ldots)$, with $r_i \in \mathbb{N}$ and $\sum ir_i = n < \infty$. Let $k = n + 1 - \sum r_i$. Then the number of (unordered) rooted trees with $n+1$ vertices and k leaves (or endpoints), whose leaves are labelled with the integers $1, 2, \ldots, k$, and with r_i nonleaf vertices of degree (= number of successors) i, is equal to the number of partitions of the set $[n]$ into $n+1-k$ blocks, with r_i blocks of cardinality i.

44. [3–] Let $a_1, a_2, \ldots, a_k$ be positive integers summing to n. Let $f(a_1, \ldots, a_k)$ be the number of permutations $w_1 w_2 \cdots w_n$ of the multiset $\{1^{a_1}, \ldots, k^{a_k}\}$ such that if there is a subsequence of the form $xyyx$, then there must be an x between the two y's. More precisely, if $r < s < t < u$, $w_r = w_u$,

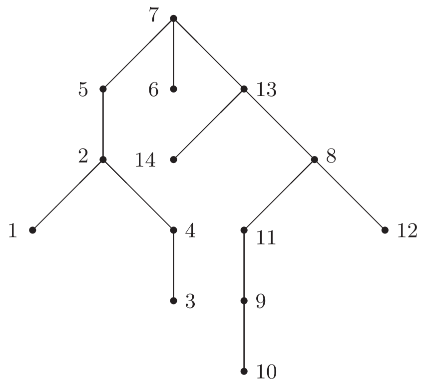

Figure 5.21 A recursively labelled tree

and $w_s = w_t \neq w_r$, then there is a $s < v < t$ with $w_r = w_v$. Show that $f(a_1, \ldots, a_k) = n!/(n-k+1)!$.

45. [2+] A *recursively labelled tree* is a tree on the vertex set $[n]$, regarded as a poset with root $\hat{1}$, such that the vertices of every principal order ideal consist of consecutive integers. See Figure 5.21 for an example. Similarly define a *recursively labelled forest*. Let t_n (respectively, f_n) denote the number of recursively labelled trees (respectively, forests) on the vertex set $[n]$. Show that

$$t_n = \frac{1}{n}\binom{3n-2}{n-1}, \quad f_n = \frac{1}{2n+1}\binom{3n}{n}.$$

Note that by Theorem 5.3.10 or Proposition 6.6.2.2, f_n is the number of plane ternary trees with $3n+1$ vertices (or, by removing the endpoints, the number of ternary trees with n vertices). Similarly it is not hard to see that t_n is the number of ternary trees on n vertices except that the root has only two (linearly ordered) subtrees (rather than three). Equivalently, t_n is the number of ordered pairs of ternary trees with a total of $n-1$ vertices.

46. [2+] A tree on a linearly ordered vertex set is called *noncrossing* if ik and jl are not both edges whenever $i < j < k < l$. Show that the number $f(n)$ of noncrossing trees on $[n]$ is equal to $\frac{1}{2n-1}\binom{3(n-1)}{n-1}$, which by Theorem 5.3.10 or Proposition 6.6.2.2 is the number of ternary trees with $n-1$ vertices.

47. (a) [2+] Show that the number of ways to write the cycle $(1, 2, \ldots, n) \in \mathfrak{S}_n$ as a product of $n-1$ transpositions (the minimum possible) is n^{n-2}. For instance (multiplying right-to-left), $(1,2,3) = (1,2)(2,3) = (2,3)(1,3) = (1,3)(1,2)$.

(b) [3–] Define two factorizations of $(1,2,\dots,n)$ into $n-1$ transpositions to be *equivalent* if one can be obtained from the other by allowing transpositions with no common elements to commute. Thus the three factorizations of $(1,2,3)$ are all inequivalent, while the factorization $(1,5)(2,4)(2,3)(1,4)$ of $(1,2,3,4,5)$ is equivalent to itself and $(2,4)(1,5)(2,3)(1,4)$, $(1,5)(2,4)(1,4)(2,3)$, $(2,4)(1,5)(1,4)(2,3)$, and $(2,4)(2,3)(1,5)(1,4)$. Show that the number $g(n)$ of equivalence classes is equal to the number of noncrossing trees on the vertex set $[n]$, as defined in Exercise 46, and hence is equal to $\frac{1}{2n-1}\binom{3(n-1)}{n-1}$.

(c) [3] Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n, and let w be a permutation of $1,2,\dots,n$ of cycle type λ. Let $f(\lambda)$ be the number of ways to write $w = t_1 t_2 \cdots t_k$ where the t_i's are transpositions that generate all of $\mathfrak{S}_n$, and where k is minimal with respect to the condition on the t_i's. (It is not hard to see that $k = n + \ell(\lambda) - 2$, where $\ell(\lambda)$ denotes the number of parts of λ.) Show that (writing ℓ for $\ell(\lambda)$)

$$f(\lambda) = (n+\ell-2)!\, n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i+1}}{\lambda_i!}.$$

48. (a) [3–] Let τ be a rooted tree with vertex set $[n]$ and root 1. An *inversion* of τ is a pair (i,j) such that $1 < i < j$ and the unique path in τ from 1 to i passes through j. For instance, the tree τ of Figure 5.22 has the inversions $(3,4), (2,4), (2,6)$, and $(5,6)$. Let $i(\tau)$ denote the number of inversions of τ. Define

$$I_n(t) = \sum_{\tau} t^{i(\tau)}, \tag{5.114}$$

summed over all n^{n-2} trees on $[n]$ with root 1. For instance,

$$\begin{aligned}
I_1(t) &= 1\\
I_2(t) &= 1\\
I_3(t) &= 2+t\\
I_4(t) &= 6+6t+3t^2+t^3\\
I_5(t) &= 24+36t+30t^2+20t^3+10t^4+4t^5+t^6\\
I_6(t) &= 120+240t+270t^2+240t^3+180t^4+120t^5+70t^6+35t^7\\
&\quad +15t^8+5t^9+t^{10}.
\end{aligned}$$

Show that

$$t^{n-1} I_n(1+t) = \sum_G t^{e(G)},$$

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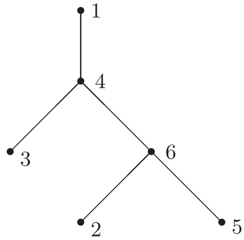

Figure 5.22 A tree with four inversions

summed over all *connected* graphs G (without loops or multiple edges) on the vertex set $[n]$, where $e(G)$ is the number of edges of G.

It follows by a simple application of the exponential formula (Corollary 5.1.6) that

$$\sum_{n\geq 0}(1+t)^{\binom{n}{2}}\frac{x^n}{n!}=\exp\sum_{n\geq 1}t^{n-1}I_n(1+t)\frac{x^n}{n!}, \tag{5.115}$$

so

$$\sum_{n\geq 1}I_n(t)\frac{x^n}{n!}=(t-1)\log\sum_{n\geq 0}t^{\binom{n}{2}}(t-1)^{-n}\frac{x^n}{n!}.$$

(b) [2] Deduce from (5.115) that

$$\sum_{n\geq 0}I_{n+1}(t)(t-1)^n\frac{x^n}{n!}=\frac{\sum_{n\geq 0}t^{\binom{n+1}{2}}\frac{x^n}{n!}}{\sum_{n\geq 0}t^{\binom{n}{2}}\frac{x^n}{n!}}.$$

49. (a) [2] There are n parking spaces $1,2,\ldots,n$ (in that order) on a one-way street. Cars $C_1,\ldots,C_n$ enter the street in that order and try to park. Each car C_i has a preferred space a_i. A car will drive to its preferred space and try to park there. If the space is already occupied, the car will park in the next available space. If the car must leave the street without parking, then the process fails. If $\alpha=(a_1,\ldots,a_n)$ is a sequence of preferences that allows every car to park, then we call α a *parking function*. Show that a sequence $(a_1,\ldots,a_n)\in[n]^n$ is a parking function if and only if the increasing rearrangement $b_1\leq b_2\leq\cdots\leq b_n$ of $a_1,a_2,\ldots,a_n$ satisfies $b_i\leq i$. In other words, $\alpha=(a_1,\ldots,a_n)$ is a parking function if and only if the sequence $(a_1-1,\ldots,a_n-1)$ is a permutation of the inversion table of a permutation $\pi\in\mathfrak{S}_n$, as defined in Chapter 1.3.

(b) [2+] Regard the elements of the group $G=\mathbb{Z}/(n+1)\mathbb{Z}$ as being the integers $0,1,\ldots,n$. Let H be the (cyclic) subgroup of order $n+1$ of

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the group G^n generated by $(1, 1, \ldots, 1)$. Show that each coset of H contains exactly one parking function. Hence the number $P(n)$ of parking functions of length n is given by

$$P(n) = (n+1)^{n-1}. \tag{5.116}$$

(c) [3–] Let $\mathcal{P}_n$ denote the set of all parking functions $\alpha = (a_1, \ldots, a_n)$ of length n, and write $|\alpha| = a_1 + \cdots + a_n$. Show that

$$\sum_{\alpha \in \mathcal{P}_n} t^{|\alpha|} = t^{\binom{n+1}{2}} I_{n+1}(1/t),$$

where $I_n(t)$ is defined in equation (5.114). Try to give a bijective proof. (Note also that putting $t = 1$ yields (5.116).)

(d) [2] Let $\alpha = (a_1, \ldots, a_n)$ be a parking function. Suppose that when the cars $C_1, \ldots, C_n$ park according to α, then C_i occupies space $w(i)$. Hence w is a permutation of $1, 2, \ldots, n$, which we denote by $w(\alpha)$. For instance, $w(3, 1, 3, 5, 1, 3) = 314526$. Given $u = u_1 \cdots u_n \in \mathfrak{S}_n$, let $\nu(u)$ be the number of parking functions α for which $w(\alpha) = u$. For $1 \le j \le n$, define

$$\tau(u, j) = 1 + \max\{k : j-1, j-2, \ldots, j-k \text{ precede } j \text{ in } u\},$$

and set $\tau(u) = (\tau(u, 1), \ldots, \tau(u, n))$. For instance, $\tau(314526) = (1, 2, 1, 2, 3, 6)$. Show that

$$\nu(u) = \tau(u, 1) \cdots \tau(u, n).$$

(e) [3–] Given $\sigma \in \mathbb{P}^n$, let

$$T_\sigma = \{u \in \mathfrak{S}_n : \tau(u) = \sigma\}.$$

For instance, $T_{(1,2,1,2,1)} = \{53412, 35412, 53142, 35142, 31542, 51342, 15342, 13542\}$. Suppose that $\sigma = (s_1, \ldots, s_n) = \tau(u)$ for some $u \in \mathfrak{S}_n$. (For the characterization and enumeration of the sequences $\tau(u)$, $u \in \mathfrak{S}_n$, see Exercise 6.19(z).) Define

$$t_i = \max\{j : s_{i+r} \le r \text{ for } 1 \le r \le j\}.$$

(If $s_{i+1} > 1$ then set $t_i = 0$.) Show that

$$\#T_\sigma = \frac{n!}{(s_1 + t_1)(s_2 + t_2) \cdots (s_n + t_n)}.$$

(f) [3–] A parking function $\alpha = (a_1, \ldots, a_n)$ is said to be *prime* if for all $1 \le j \le n-1$, at least $j+1$ cars want to park in the first j places. (Equivalently, if we remove some term of α equal to 1, then we still have a parking function.) Show that the number $Q(n)$ of prime parking functions of length n is equal to $(n-1)^{n-1}$.

50. (a) [3–] Let $\mathcal{S}_n$ denote the set of all hyperplanes $x_i - x_j = 0, 1$ $(1 \le i < j \le n)$ in $\mathbb{R}^n$. (Hence $\#\mathcal{S}_n = n(n-1)$.) Show bijectively that the number of regions of $\mathcal{S}_n$ (i.e., the number of connected components of the space $\mathbb{R}^n - \bigcup_{H \in \mathcal{S}_n} H$) is equal to $(n+1)^{n-1}$.

(b) [3–] Let $L_{\mathcal{S}_n}$ denote the intersection poset of $\mathcal{S}_n$. Use the finite field method of Chapter 3.11.4 to show that

$$\chi(L_{\mathcal{S}_n}, q) = q(q-n)^{n-1}.$$

By Theorem 3.11.7 this result generalizes (a).

(c) [3] Let R_0 be the region of $\mathcal{S}_n$ defined by $x_i - 1 < x_j < x_i$ for all $i < j$. For any region R of $\mathcal{S}_n$, let $d(R)$ be the number of hyperplanes $H \in \mathcal{S}_n$ that *separate* R from R_0, that is, R and R_0 lie on different sides of H. Define the polynomial

$$J_n(q) = \sum_R q^{d(R)},$$

summed over all regions of $\mathcal{S}_n$. Show that

$$J_n(q) = q^{\binom{n}{2}} I_{n+1}(1/q),$$

where $I_n(t)$ is defined in equation (5.114).

(d) [2+] Show that (c) is equivalent to the following result. Given a permutation $\pi \in \mathfrak{S}_n$, let $P_\pi = \{(i,j) : 1 \le i < j \le n,\ \pi(i) < \pi(j)\}$. Define a partial ordering on P_π by $(i,j) \le (k,l)$ if $k \le i < j \le l$. Let $F(J(P_\pi), q)$ denote the rank-generating function of the lattice of order ideals of P_π. (For instance, if $\pi = n, n-1, \ldots, 1$, then $P_\pi = \emptyset$ and $F(J(P_\pi), q) = 1$.) Then

$$\sum_{\pi \in \mathfrak{S}_n} F(J(P_\pi), q) = I_{n+1}(q).$$

(e) [3–] Show that the number of elements of rank k in the intersection poset $L_{\mathcal{S}_n}$ is equal to the number of ways to partition the set $[n]$ into $n-k$ blocks, and then linearly order each block. (It is easy to see that this number is given by $\frac{n!}{(n-k)!}\binom{n-1}{k}$; see Exercise 17.)

51. [2+] Let $A(x) = ax + \cdots$, $B(x) = bx + \cdots$, $C(x) = c + \cdots \in K[[x]]$ with $abc \neq 0$ and $[x]C(x) \neq 0$. Show that the following two formulas are equivalent:

$$\begin{aligned} &\text{(i)} \quad & A(x)^{\langle -1 \rangle} &= C(x)B(x)^{\langle -1 \rangle} \\ &\text{(ii)} \quad & \frac{x}{C(A(x))} &= [xC(B(x))]^{\langle -1 \rangle}. \end{aligned}$$

52. (a) [2] Let $F(x) = x + \sum_{n\geq 2} f_n \frac{x^n}{n!} \in K[[x]]$. Given $k \in \mathbb{P}$, let

$$F^{\langle k\rangle}(x) = x + \sum_{n\geq 2} \varphi_n(k) \frac{x^n}{n!}. \tag{5.117}$$

Show that for fixed n, the function $\varphi_n(k)$ is a polynomial in k (whose coefficients are polynomials in $f_2, \ldots, f_n$). For instance,

$$\varphi_2(k) = f_2 k$$

$$\varphi_3(k) = f_3 k + 3f_2^2 \binom{k}{2}$$

$$\varphi_4(k) = f_4 k + \left(10 f_2 f_3 + 3 f_2^3\right)\binom{k}{2} + 18 f_2^3 \binom{k}{3}$$

$$\begin{aligned}\varphi_5(k) = f_5 k &+ \left(15 f_2 f_4 + 10 f_3^2 + 25 f_2^2 f_3\right)\binom{k}{2} \\ &+ \left(130 f_2^2 f_3 + 75 f_2^4\right)\binom{k}{3} + 180 f_2^4 \binom{k}{4}.\end{aligned}$$

(b) [2] Since $\varphi_n(k)$ is a polynomial in k, it can be defined for any $k \in K$ (or for k an indeterminate). Thus (5.117) allows us to define $F^{\langle k\rangle}(x)$ for any k. Show that for all $j, k \in K$, we have

$$F^{\langle j+k\rangle}(x) = F^{\langle j\rangle}(F^{\langle k\rangle}(x)),$$

$$F^{\langle jk\rangle}(x) = (F^{\langle j\rangle})^{\langle k\rangle}(x).$$

In particular, the two ways of defining $F^{\langle -1\rangle}(x)$ (viz., by putting $k = -1$ in (5.117), or as the compositional inverse of $F(x)$) agree.

(c) [5–] Investigate the combinatorial significance of "fractional composition." For instance, setting

$$\begin{aligned}(e^x - 1)^{\langle 1/2\rangle} &= \sum_{n\geq 1} a_n \frac{x^n}{n!} \\ &= x + \frac{1}{2}\frac{x^2}{2!} + \frac{1}{2^3}\frac{x^3}{3!} + \frac{1}{2^5}\frac{x^5}{5!} - \frac{7}{2^7}\frac{x^6}{6!} + \frac{1}{2^7}\frac{x^7}{7!} + \frac{159}{2^8}\frac{x^8}{8!} \\ &\quad - \frac{843}{2^8}\frac{x^9}{9!} - \frac{1231}{2^{12}}\frac{x^{10}}{10!} + \frac{2359233}{2^{14}}\frac{x^{11}}{11!} - \frac{13303471}{2^{14}}\frac{x^{12}}{12!} \\ &\quad - \frac{271566005}{2^{15}}\frac{x^{13}}{13!} + \frac{10142361989}{2^{16}}\frac{x^{14}}{14!} \\ &\quad + \frac{126956968965}{2^{18}}\frac{x^{15}}{15!} - \frac{10502027401553}{2^{18}}\frac{x^{16}}{16!} + \cdots,\end{aligned}$$

do the coefficients a_n have a simple combinatorial interpretation? (Unfortunately they are not integers, nor do their signs seem predictable.)

53. [2+] Find the sum of the first n terms in the binomial expansion of

$$\left(1-\frac{1}{2}\right)^{-n} = 1 + \frac{1}{2}n + \frac{1}{4}\binom{n+1}{2} + \cdots .$$

For instance, when $n = 3$ we get $1 + \frac{3}{2} + \frac{6}{4} = 4$. (Use the Lagrange inversion formula.)

54. [2+] For each of the following four power series $F(x)$, find for all $n \in \mathbb{P}$ the coefficient of $1/x$ in the Laurent expansion about 0 of $F(x)^{-n}$: $\sin x$, $\tan x$, $\log(1+x)$, $1 + x - \sqrt{1+x^2}$.

55. (a) [2] Find the unique power series $F_1(x) \in \mathbb{Q}[[x]]$ such that for all $n \in \mathbb{N}$, we have $[x^n]F_1(x)^{n+1} = 1$.
 (b) [2+] Find the unique power series $F_2(x) \in \mathbb{Q}[[x]]$ such that for all $n \in \mathbb{N}$, we have $[x^n]F_2(x)^{2n+1} = 1$.
 (c) [2+] Let $k \in \mathbb{P}$, $k \geq 3$. What difficulty arises in trying to find an explicit expression for the unique power series $F_k(x) \in \mathbb{Q}[[x]]$ such that for all $n \in \mathbb{N}$, we have $[x^n]F_k(x)^{kn+1} = 1$?

56. (a) [2+] Let $F(x) = a_1x + a_2x^2 + \cdots \in K[[x]]$ with $a_1 \neq 0$, and let $n \in \mathbb{P}$. Show that

$$n[x^n] \log \frac{F^{\langle -1 \rangle}(x)}{x} = [x^n]\left(\frac{x}{F(x)}\right)^n. \tag{5.118}$$

 (This formula may be regarded as the "correct" case $k=0$ of equation (5.53).)
 (b) [2] Find the unique power series $G(x) = 1 + x - \frac{1}{2}x^2 + \cdots$ satisfying $[x]G(x) = 1$ and $[x^n]G(x)^n = 0$ for $n > 1$.

57. [2] Show that the coefficient of x^{n-1} in the power series expansion of the rational function $(1+x)^{2n-1}(2+x)^{-n}$ is equal to $1/2$. Equivalently, the unique power series $J(x) \in \mathbb{Q}[[x]]$ satisfying

$$[x^{n-1}]\frac{J(x)^n}{1+x} = \frac{1}{2}, \text{ for all } n \in \mathbb{P},$$

is given by $J(x) = (1+x)^2/(2+x)$.

58. [3–] Let $f(x)$ and $g(x)$ be power series with $g(0) = 1$. Suppose that

$$f(x) = g(xf(x)^\alpha), \tag{5.119}$$

where α is a parameter (indeterminate). Show that

$$(t + \alpha n)[x^n]f(x)^t = t[x^n]g(x)^{t+\alpha n},$$

as a polynomial identity in the two variables t and α.

59. [2+] Let $f(x) \in K[[x]]$ with $f(0) = 0$. Let $F(x,y) \in K[[x,y]]$, and suppose that f satisfies the functional equation $f = F(x,f)$. Show that for every $k \in \mathbb{P}$,

$$f(x)^k = \sum_{n\geq 1} \frac{k}{n}[y^{n-k}]F(x,y)^n.$$

60. (a) [2] Let $A(x) = 1 + \sum_{n\geq 1} a_n x^n \in K[[x]]$. For fixed $k \in \mathbb{N}$, define for $n \in \mathbb{Z}$

$$q_k(n) = [x^k]A(x)^n.$$

Show that $q_k(n)$ is a polynomial in n of degree $\leq k$.

(b) [2] Let $F(x) = x + \sum_{n\geq 2} f_n \frac{x^n}{n!} \in K[[x]]$ (where char $K = 0$). Define functions $p_k(n)$ by

$$e^{tF(x)} = \sum_{n\geq 0}\sum_{k\geq 0} p_k(n)t^n \frac{x^{n+k}}{(n+k)!}.$$

Show that $p_k(n)$ is a polynomial in n (of degree $\leq 2k$).

(c) [2+] Let $F(x)$ and $p_k(n)$ be as in (b). Since $p_k(n)$ is a polynomial in n, it is defined for all $n \in \mathbb{Z}$. Show that

$$e^{tF^{\langle -1\rangle}(x)} = \sum_{n\geq 0}\sum_{k\geq 0}(-1)^k p_k(-n-k)t^n \frac{x^{n+k}}{(n+k)!}.$$

(d) [2] What are $p_k(n)$ and $p_k(-n-k)$ in the special case $F(x) = e^x - 1$?

(e) [2] Find $p_k(n)$ when $F(x) = x/(1-x)$. What does (c) tell us about $p_k(n)$?

(f) [2] Find $p_k(n)$ when $F(x) = xe^{-x}$. Deduce a formula for

$$\exp t(xe^{-x})^{\langle -1\rangle} = \exp t \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}.$$

61. (a) [2] Let P and Q be finite posets with $\hat{1}$'s. For any poset T let $\overline{T} = T \cup \{\hat{0}\}$, and for any finite poset T with $\hat{0}$ and $\hat{1}$ let $\mu(T) = \mu_T(\hat{0},\hat{1})$. Show that

$$-\mu(\overline{P \times Q}) = \mu(\overline{P} \times \overline{Q}) = \mu(\overline{P})\mu(\overline{Q}).$$

(b) [2] Use (a) and Corollary 5.5.5 to give a direct proof of equation (5.78).

62. (a) [2+] Let $f_r(n)$ be the number of $n \times n$ $\mathbb{N}$-matrices A with every row and column sum equal to r and with at most two nonzero entries in every row (and hence in every column [why?]). Find

$$\sum_{n \geq 0} f_r(n) \frac{x^n}{n!^2}.$$

(b) [1] Use (a) to find $f_3(n)$ explicitly.

63. [2+] Let $N_k(n)$ denote the number of sequences $(P_1, P_2, \ldots, P_{2k})$ of $n \times n$ permutation matrices P_i such that each entry of $P_1 + P_2 + \cdots + P_{2k}$ is $0, k$, or $2k$. Show that

$$\sum_{n \geq 0} N_k(n) \frac{x^n}{n!^2} = (1-x)^{-\binom{2k-1}{k}} \exp\left[x\left(1 - \binom{2k-1}{k}\right)\right].$$

64. (a) [2+] Let $\mathcal{D}_n$ be the set of all $n \times n$ matrices of $+1$'s and -1's. For $k \in \mathbb{P}$ let

$$f_k(n) = 2^{-n^2} \sum_{M \in \mathcal{D}_n} (\det M)^k$$

$$g_k(n) = 2^{-n^2} \sum_{M \in \mathcal{D}_n} (\operatorname{per} M)^k,$$

where per denotes the permanent function, defined by

$$\operatorname{per}[m_{ij}] = \sum_{\pi \in \mathfrak{S}_n} m_{1,\pi(1)} m_{2,\pi(2)} \cdots m_{n,\pi(n)}.$$

Find $f_k(n)$ and $g_k(n)$ explicitly when k is odd or $k = 2$.

(b) [3–] Show that $f_4(n) = g_4(n)$, and show that

$$\sum_{n \geq 0} f_4(n) \frac{x^n}{n!^2} = (1-x)^{-3} e^{-2x}. \tag{5.120}$$

Hint. We have

$$\sum_M (\det M)^4 = \sum_M \left(\sum_{\pi \in \mathfrak{S}_n} \pm m_{1,\pi(1)} \cdots m_{n,\pi(n)} \right)^4.$$

Interchange the order of summation and use Exercise 63.

(c) [2+] Show that $f_{2k}(n) < g_{2k}(n)$ if $k \geq 3$ and $n \geq 3$.

(d) [3–] Let $\mathcal{D}'_n$ be the set of all $n \times n$ 0–1 matrices. Let $f'_k(n)$ and $g'_k(n)$ be defined analogously to $f_k(n)$ and $g_k(n)$. Show that $f'_k(n) = 2^{-kn} f_k(n+1)$.

Show also that

$$\begin{aligned} g_1'(n) &= 2^{-n}n! \\ g_2'(n) &= 4^n n!^2\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right). \end{aligned}$$

65. (a) [3–] Let $f(m,n)$ be the number of $m\times n$ $\mathbb{N}$-matrices with every row and column sum at most two. For instance, $f(1,n)=1+2n+\binom{n}{2}$. Show that

$$\begin{aligned} F(x,y) &:= \sum_{m,n\geq 0} f(m,n)\frac{x^m y^n}{m!\,n!} \\ &= (1-xy)^{-\frac{1}{2}}\exp\left[\frac{\frac{1}{2}xy(3-xy)+\frac{1}{2}(x+y)(2-xy)}{1-xy}\right]. \end{aligned} \tag{5.121}$$

(b) [2+] Deduce from (a) that

$$\sum_{n\geq 0} f(n,n)\frac{t^n}{n!^2} = (1-t)^{-\frac{1}{2}}e^{\frac{t(3-t)}{2(1-t)}}\sum_{n\geq 0}\frac{t^n}{n!^2}\left(\frac{1-\frac{1}{2}t}{1-t}\right)^{2n}.$$

The latter sum may be rewritten as $J_0\left(\sqrt{-t}(2-t)/(1-t)\right)$, where J_0 denotes the Bessel function of order zero.

66. [2+] Let $\boldsymbol{L}=\boldsymbol{L}(K_{rs})$ be the Laplacian matrix of the complete bipartite graph K_{rs}.
(a) Find a simple upper bound on rank$(\boldsymbol{L}-r\boldsymbol{I})$. Deduce a lower bound on the number of eigenvalues of $\boldsymbol{L}$ equal to r.
(b) Assume $r\neq s$, and do the same as (a) for s instead of r.
(c) Find the remaining eigenvalues of $\boldsymbol{L}$.
(d) Use (a)–(c) to compute $c(K_{rs})$, the number of spanning trees of K_{rs}.

67. (a) [3] Let q be a prime power, and let $\mathbb{F}_q$ denote the finite field with q elements. Given $f:\binom{[n]}{2}\to\mathbb{F}_q$ and a free tree T on the vertex set $[n]$, define $f(T)=\prod_e f(e)$, where e ranges over all edges (regarded as two-element subsets of $[n]$) of T. Let $P_n(q)$ denote the number of maps f for which

$$\sum_T f(T)\neq 0 \quad (\text{in } \mathbb{F}_q),$$

where T ranges over all n^{n-2} free trees on the vertex set $[n]$. Show that

$$\begin{aligned} P_{2m}(q) &= q^{m(m-1)}(q-1)(q^3-1)\cdots(q^{2m-1}-1) \\ P_{2m+1}(q) &= q^{m(m+1)}(q-1)(q^3-1)\cdots(q^{2m-1}-1). \end{aligned}$$

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(b) [2+] How many simple (i.e., no loops or multiple edges) graphs on the vertex set $[n]$ have an odd number of spanning trees?

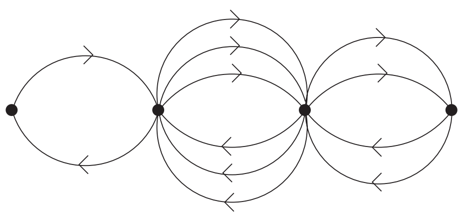
68. [3–] This exercise assumes a basic knowledge of the character theory of finite abelian groups. Let Γ be a finite abelian group, written additively. Let $\hat{\Gamma}$ denote the set of (irreducible) characters $\chi : \Gamma \to \mathbb{C}$ of Γ, with the trivial character denoted by χ_0. Let $\sigma : \Gamma \to K$ be a weight function (where K is a field of characteristic zero). Define $D = D_\sigma$ to be the digraph on the vertex set Γ with an edge $u \to u + v$ of weight $\sigma(v)$ for all $u, v \in \Gamma$. Note that D is balanced as a weighted digraph (every vertex has indegree and outdegree equal to $\sum_{u\in\Gamma} \sigma(u)$). If T is any spanning subgraph of D, then let $\sigma(T) = \prod_e \sigma(e)$, where e ranges over all edges of T. Define

$$c_\sigma(D) = \sum_T \sigma(T),$$

where T ranges over all oriented (spanning) subtrees of D with a fixed root. Show that

$$c_\sigma(D) = \frac{1}{|\Gamma|} \prod_{\substack{\chi\in\hat{\Gamma} \\ \chi\neq\chi_0}} \left[\sum_{v\in\Gamma} \sigma(v)(1 - \chi(v))\right].$$

69. Choose positive integers $a_1, \ldots, a_{p-1}$. Let $D = D(a_1, \ldots, a_{p-1})$ be the digraph defined as follows. The vertices of D are $v_1, \ldots, v_p$. For each $1 \le i \le p - 1$, there are a_i edges from x_i to x_{i+1} and a_i edges from x_{i+1} to x_i. For instance, $D(1, 3, 2)$ looks like



(a) [2–] Find by a direct argument (no determinants) the number $\tau(D, v)$ of oriented subtrees with a given root v.

(b) [2–] Find the number $\epsilon(D, e)$ of Eulerian tours of D whose first edge is e.

70. [2] Let $d > 1$. A *d-ary de Bruijn sequence* of degree n is a sequence $A = a_1a_2 \cdots a_{d^n}$ whose entries a_i belong to $\{0, 1, \ldots, d-1\}$ such that every d-ary sequence $b_1b_2 \cdots b_n$ of length n occurs exactly once as a circular factor of A. Find the number of d-ary de Bruijn sequences of degree n that begin with n 0's.

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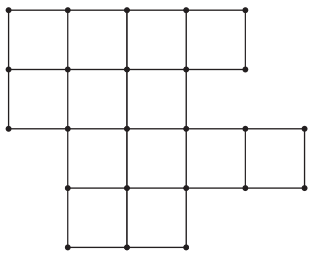
71. [2+] Let G be a regular graph with no loops or multiple edges. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ be nonzero real numbers such that for all $\ell \geq 1$, the number $W(\ell)$ of closed walks in G of length ℓ is given by

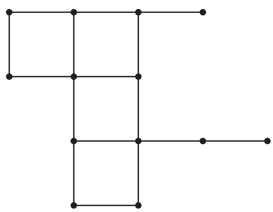
$$W(\ell) = \sum_{j=1}^{m} \lambda_j^{\ell}. \tag{5.122}$$

Find the number $c(G)$ of spanning trees of G in terms of the given data.

72. [3–] Let V be the subset of $\mathbb{Z} \times \mathbb{Z}$ on or inside some simple closed polygonal curve whose vertices belong to $\mathbb{Z} \times \mathbb{Z}$, such that every line segment that makes up the curve is parallel to either the x-axis or y-axis. Draw an edge e between any two points of V at distance one apart, provided e lies on or inside the boundary curve. We obtain a planar graph G, an example being



Let G' be the dual graph G^* with the "outside" vertex deleted. (The vertices of G^* are the regions of G. For every edge e of G there is an edge e^* of G^* connecting the two regions that have e on their boundary.) For the above example, G' is given by



Let $\lambda_1, \ldots, \lambda_p$ denote the eigenvalues of G' (i.e., of the adjacency matrix $\boldsymbol{A}(G')$). Show that

$$c(G) = \prod_{i=1}^{p} (4 - \lambda_i).$$

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73. [5–] Let $\mathcal{B}(n)$ be the set of (binary) de Bruijn sequences of degree n, and let $\mathcal{S}_n$ be the set of all binary sequences of length 2^n. According to Corollary 5.6.15 we have $(\#\mathcal{B}(n))^2 = \#\mathcal{S}(n)$. Find an explicit bijection $\mathcal{B}(n) \times \mathcal{B}(n) \to \mathcal{S}(n)$.
74. Let D be a digraph with p vertices, and let ℓ be a fixed positive integer. Suppose that for every pair u, v of vertices of D, there is a unique (directed) walk of length ℓ from u to v.
    (a) [2+] What are the eigenvalues of the (directed) adjacency matrix $\boldsymbol{A} = \boldsymbol{A}(D)$?
    (b) [2] How many loops (v, v) does D have?
    (c) [3–] Show that D is connected and balanced.
    (d) [1] Show that all vertices have the same outdegree d. (By (c), all vertices then also have indegree d.) Find a simple formula relating p, d, and ℓ.
    (e) [2] How many Eulerian tours does D have starting with a given edge e?
    (f) [5–] What more can be said about D? Must D be a de Bruijn graph (the graphs used to solve Exercise 70)?

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Solutions to Exercises for Chapter 5

1. (a) Let $h(n)$ be the desired number. By Proposition 5.1.3, we have

$$\begin{aligned} E_h(x) &= \left(\sum_{n\geq 0}\frac{x^n}{n!}\right)^2 \left(\sum_{n\geq 0}\frac{x^{2n+1}}{(2n+1)!}\right)\left(\sum_{n\geq 0}\frac{x^{2n}}{(2n)!}\right) \\ &= e^{2x}\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}\right) \\ &= \tfrac{1}{4}\left(e^{4x}-1\right) \\ &= \sum_{n\geq 1} 4^{n-1}\frac{x^n}{n!}, \end{aligned}$$

whence $h(n) = 4^{n-1}$.

(b) Pick a set S of $2k$ poles to be either orange or purple, and pick a subset of S to be orange in 2^{2k} ways. Thus we obtain an extra factor of

$$\sum_{n\geq 0} 2^{2n}\frac{x^{2n}}{(2n)!} = \tfrac{1}{2}\left(e^{2x}+e^{-2x}\right).$$

Hence

$$\begin{aligned} E_h(x) &= \tfrac{1}{4}\left(e^{4x}-1\right)\cdot\tfrac{1}{2}\left(e^{2x}+e^{-2x}\right) \\ &= \tfrac{1}{8}\left(e^{6x}-e^{-2x}\right), \end{aligned}$$

so $h(n) = \frac{1}{8}(6^n - (-2)^n)$.

2. (a) By Exercise 1.40, there are unique numbers a_i such that

$$1 + \sum_{n\geq 1} f_n x^n = \prod_{i\geq 1}(1-x^i)^{-a_i}.$$

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It is easily seen that $f_n \in \mathbb{Z}$ for all $n \in [N]$ if and only if $a_i \in \mathbb{Z}$ for all $i \in [N]$. Now by the solution to Exercise 1.40

$$h_n = \sum_{d|n} d a_d,$$

so by the classical Möbius inversion formula,

$$a_n = \frac{1}{n} \sum_{d|n} h_d\, \mu(n/d),$$

and the equivalence of (i) and (ii) follows.
Now let $p|n$, and let S be the set of distinct primes other than p which divide n. If $T \subseteq S$ then write $\Pi(T) = \prod_{q \in T} q$. Then

$$A_n := \sum_{d|n} h_d\, \mu(n/d) = \sum_{T \subseteq S} (-1)^{\#T} \left(h_{n/\Pi(T)} - h_{n/p\cdot\Pi(T)}\right). \tag{5.123}$$

Hence if (iii) holds for *all* $n \in [N]$, then by (5.123) we have $p^r | A_n$. Thus $A_n \equiv 0 \pmod{n}$. Conversely, if (ii) holds for all $n \in [N]$ then (iii) follows from (5.123) by an easy induction on n.
Finally observe that

$$\exp \sum_{n \geq 1} \left(\sum_{i=1}^{N} \alpha_i^n \right) \frac{x^n}{n} = \frac{1}{(1-\alpha_1 x)\cdots(1-\alpha_N x)}.$$

From this it is immediate that (iv) $\Rightarrow$ (i). Conversely, if (i) holds then let

$$\left(1 + \sum_{n \geq 1} f_n x^n\right)^{-1} = 1 + \sum_{n \geq 1} e_n x^n.$$

Clearly $e_n \in \mathbb{Z}$ for $n \in [N]$. Set

$$1 + \sum_{n=1}^{N} e_n t^n = \prod_{1}^{N} (1 - \alpha_i t).$$

Then $h_n = \sum_{i=1}^N \alpha_i^n$ for all $n \in [N]$, as desired.
The equivalence of (ii) and (iv) goes back to W. Jänischen, *Sitz. Berliner Math. Gesellschaft* **20** (1921), 23–29. The condition (iii) is due to I. Schur, *Comp. Math.* **4** (1937), 432–444, who obtains several related results. The equivalence of (i) and (ii) in the case $N \to \infty$ appears in L. Carlitz, *Proc. Amer. Math. Soc.* **9** (1958), 32–33. Additional references are J. S. Frame, *Canadian J. Math.* **1** (1949), 303–304; G. Almkvist, The integrity of ghosts, preprint; A. Dold, *Inv. math.* **74** (1983), 419–435.

(b) Let us say that a solution $\alpha = (\alpha_1, \ldots, \alpha_k)$ has *degree n* if n is the smallest integer for which $\alpha \in \mathbb{F}_{q^n}^k$. By a simple Möbius inversion argument, the number M_n of solutions of degree n is given by $M_n = \sum_{d|n} N_d\, \mu(n/d)$. Write $\alpha^j = (\alpha_1^j, \ldots, \alpha_k^j)$. If α is a solution of degree n, then the k-tuples $\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{n-1}}$ are distinct solutions of degree n. Hence M_n is divisible by n. Now use the equivalence of (i) and (ii) in (a). See for instance K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, second ed., Springer-Verlag, New York/Berlin/Heidelberg, 1990 (§11.1).

A considerably deeper result, first proved by B. Dwork, *Amer. J. Math.* **82** (1959), 631–648, is that the generating function $Z(x)$ (known as the *zeta function* of the algebraic variety defined by the equations) is rational. A nice exposition of this result appears in N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, second ed., Springer-Verlag, New York/Heidelberg/Berlin, 1984 (Ch. V). For further information on zeta functions, see for example R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York/Heidelberg/Berlin, 1977 (App. C).

(c) See A. A. Jagers and I. Gessel (independently), Solution to E2993, *American Math. Monthly* **93** (1986), 483–484.

3. (a) Since $1 \cdot 3 \cdot 5 \cdots (2n-1) = (2n)!/2^n n!$, we have

$$\begin{aligned} E_f(x) &= \sum_{n\geq 0} 2^{-n} \binom{2n}{n} x^n \\ &= (1-2x)^{-1/2}, \end{aligned}$$

by Exercise 1.4(a). Hence

$$\begin{aligned} E_f(x)^2 &= (1-2x)^{-1} \\ &= \sum_{n\geq 0} 2^n n! \frac{x^n}{n!} \\ &= E_g(x). \end{aligned}$$

(b) **First proof.** $f(n)$ is the number of 1-factors (i.e., graphs whose components are all single edges) on $2n$ vertices, while $g(n)$ is the number of permutations π of $[n]$ with each element of $[n]$ labelled $+$ or $-$. Hence given a labelled permutation π, we want to construct a pair (G, H), where G is a 1-factor on a set of $2k$ vertices labelled by i and i', where i ranges over some subset S of $[n]$, while H is a 1-factor on the $2(n-k)$ complementary vertices j and j', where $j \in T = [n] - S$. Define

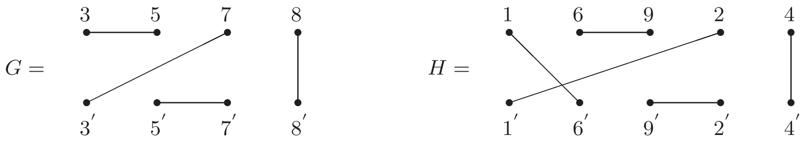

Figure 5.23 A pair of 1-factors

S (respectively, T) to consist of all i such that the cycle of π containing i has least element labelled $+$ (respectively, $-$). If $\pi(a) = b$, then draw an edge from either a or a' to either b or b', as follows: If a is the least element of its cycle and $a \neq b$, then draw an edge from a to b (respectively, b') if b is labelled $+$ (respectively, $-$). If neither a nor b is the least element of its cycle, then inductively assume that an edge is incident to either a or a'. Draw a new edge from the vertex a or a' without an edge to b (respectively b') if b is labelled $+$ (respectively $-$). Finally if b is the least element of its cycle then only two vertices remain for the last edge—it goes from a or a' (whichever has no edge) to b'. This procedure recursively defines G and H.

Example. Let

$$\pi = \left(\begin{array}{cccc} 1 & 6 & 9 & 2 \\ - & - & + & - \end{array} \right) \left(\begin{array}{ccc} 3 & 5 & 7 \\ + & + & - \end{array} \right) \left(\begin{array}{c} 4 \\ - \end{array} \right) \left(\begin{array}{c} 8 \\ + \end{array} \right)$$

Then G and H are given by Figure 5.23.

This bijection is based on work of D. Dumont, *Univ. Beograd. Publ. Elektrotechn. Fak.*, ser. Mat. Fiz., no. 634 – no. 677 (1979), pp. 116–125 (Prop. 3).

Second proof (I. Gessel). It is easy to see that the number of permutations $a_1 a_2 \cdots a_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ with no subsequence of the form bab with $a < b$ is equal to $f(n)$. (Write down two 1's in one way, then two consecutive 2's in three ways relative to the 1's, then two consecutive 3's in five ways relative to the 1's and 2's, etc.) Hence by Proposition 5.1.1, $E_f(x)^2$ is the exponential generating function for pairs (π, σ), where π is a permutation of some multiset $M = \{i_1^2, \ldots, i_k^2\} \subseteq \{1^2, 2^2, \ldots, n^2\}$ and σ is a permutation of $\{1^2, 2^2, \ldots, n^2\} - M$; and where both π and σ satisfy the above condition on subsequences bab. But to obtain π and σ we can place the two 1's in two ways (i.e., in either π or σ), then the two 2's in four ways, etc., for a total of $2 \cdot 4 \cdots (2n) = 2^n n!$ ways.

4. (a) To obtain a threshold graph G on $[n]$, choose a subset I of $[n]$ to be the set of isolated vertices of G, and choose a threshold graph without isolated vertices on $[n] - I$. From Proposition 5.1.1 it follows that $T(x) = e^x S(x)$.

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A threshold graph G with $n \geq 2$ vertices has no isolated vertices if and only if the complement $\bar{G}$ has an isolated vertex. Hence $t(n) = 2s(n)$ for $n \geq 2$. Since $t(0) = s(0) = 1,\ t(1) = 1,\ s(1) = 0$, it follows that $T(x) = 2S(x) + x - 1$.

These results, as well as others related to the enumeration of labelled threshold graphs, are essentially due to J. S. Beissinger and U. N. Peled, *Graphs and Combinatorics* **3** (1987), 213–219. For further information on threshold graphs, see N. V. R. Mahadev and U. N. Peled, *Threshold Graphs and Related Topics*, Annals of Discrete Mathematics, vol. 56, North-Holland, Amsterdam, 1995.

(b) Let G be a threshold graph on $[n]$ with no isolated vertices. Define an ordered partition $(B_1, \ldots, B_k)$ of $[n]$ as follows. Let B_1 be the set of isolated vertices of $\bar{G}$, so $\bar{G} = B_1 + G_1$, where G_1 is threshold graph with no isolated vertices. Let B_2 be the set of isolated vertices of $\bar{G}_1$. Iterate this procedure until reaching $\bar{G}_{k-1} = B_k$. We obtain in this way every ordered partition $(B_1, \ldots, B_k)$ of $[n]$ satisfying $\#B_k > 1$. Since there are clearly $nc(n-1)$ ordered partitions $(B_1, \ldots, B_k)$ of $[n]$ satisfying $\#B_k = 1$, it follows that $s(n) = c(n) - nc(n-1)$.

(c) The polytope $\mathcal{P}$ of Exercise 4.63 is called a *zonotope* with generators $v_1, \ldots, v_k$. Let Z_n be the zonotope generated by all vectors $e_i + e_j$, $1 \leq i < j \leq n$, where e_i is the ith unit coordinate vector in $\mathbb{R}^n$. The zonotope Z_n is called the *polytope of degree sequences*. By a well-known duality between hyperplane arrangements and zonotopes (see e.g. A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler, *Oriented Matroids*, Cambridge University Press, Cambridge, 1993 (Proposition 2.2.2)), the number of regions of $\mathcal{T}_n$ is equal to the number of vertices of Z_n. The number of vertices of Z_n was computed by J. S. Beissinger and U. N. Peled, *Graphs and Combinatorics* **3** (1987), 213–219. Further properties of Z_n appear in R. Stanley, in *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Volume **4**, American Mathematical Society, 1991, pp. 555–570.

(d) Special case of Exercise 3.112(c).

5. Let $c_k(n)$ be the number of ways to choose a connected bipartite graph on $[n]$ with k edges. Let $f_k(n)$ (respectively, $g_k(n)$) be the number of ways to choose a weak ordered partition (A, B) of $[n]$ into two parts, and then choose a bipartite graph (respectively, connected bipartite graph) with k edges on $[n]$ such that every edge goes from A to B. Let

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$$B(x) = \sum_{n\geq 0}\sum_{k\geq 0} b_k(n)q^k \frac{x^n}{n!},$$

and similarly for $C(x)$, $F(x)$, and $G(x)$. (The sums for $C(x)$ and $G(x)$ start at $n = 1$.) It is easy to see that

$$F(x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n}(1+q)^{i(n-i)}\binom{n}{i}\right)\frac{x^n}{n!}$$

$$F(x) = \exp G(x), \quad B(x) = \exp C(x), \quad G(x) = 2C(x),$$

and the proof follows.

6. It suffices to assume that $q \in \mathbb{P}$. Let K_{mn} have vertex bipartition (A, B). By an obvious extension of Proposition 5.1.3, the coefficient of $x^m y^n/m!\,n!$ in $(e^x + e^y - 1)^q$ is the number of q-tuples $\pi = (S_1, \dots, S_q)$ where each S_i is a (possibly empty) subset of A or of B, the S_i's are pairwise disjoint, and $\bigcup S_i = A \cup B$. Color the vertices in S_i with the color i. This yields a bijection with the q-tuples π and the q-colorings of K_{mn}, and the proof follows. Note that there is a straightforward extension of this result to the complete multipartite graph $K_{n_1,\dots,n_k}$, yielding the formula

$$\sum_{n_1,\dots,n_k\geq 0} \chi(K_{n_1,\dots,n_k}, q)\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_k^{n_k}}{n_k!} = \left(e^{x_1} + \cdots + e^{x_k} - k + 1\right)^q.$$

7. (a) Let $\mathcal{A}_n$ (respectively, $\mathcal{B}_n$) be the set of all pairs (π, σ) such that π and σ are alternating permutations of some complementary subsets S and $[2n] - S$ of $[2n]$ of odd (respectively, even) cardinality. Proposition 5.1.1 shows that the identity $1 + \tan^2 x = \sec^2 x$ follows from giving a bijection from $\mathcal{A}_n$ to $\mathcal{B}_n$ for $n \geq 1$. Suppose that $\pi = a_1a_2\cdots a_k$ and $\sigma = b_1b_2\cdots b_{2n-k}$. Then exactly one of the pairs $(a_1a_2\cdots a_k b_{2n-k}, b_1b_2\cdots b_{2n-k-1})$ and $(a_1a_2\cdots a_{k-1}, b_1b_2\cdots b_{2n-k}a_k)$ belongs to $\mathcal{B}_n$, and this establishes the desired bijection.

   (b) The identity (5.87) is equivalent to

$$\sum_{\substack{m,n\geq 0\\ m+n \text{ odd}}} E_{m+n}\frac{x^m}{m!}\frac{y^n}{n!} = \sum_{k\geq 0}\left[\left(\tan^k x\right)\left(\tan^{k+1} y\right) + \left(\tan^{k+1} x\right)\left(\tan^k y\right)\right].$$

   Let $m, n \geq 0$ with $m + n$ odd, and let π be an alternating permutation of $[m+n]$. Then either $n = 0$, or else π can be uniquely factored (as a word $a_1a_2\cdots a_{m+n}$) in the form

$$\pi = e_1\bar{o}_1 o_1\bar{o}_2 o_2\cdots\bar{o}_k o_k\bar{o}_{k+1}e_2,$$

where (i) e_1 is an alternating permutation (possibly empty) of some subset of $[m]$ of even cardinality, (ii) e_2 is a reverse alternating permutation (possibly empty) of some subset of $[m]$ of even cardinality, (iii) each o_i is a reverse alternating permutation of some subset of $[m]$ of odd cardinality, and (iv) each $\bar{o}_i$ is an alternating permutation of some subset of $[m+1, m+n]$ of odd cardinality. Using the bijection of (a) (after reversing e_2), we can transform the pair (e_1, e_2) into a pair (e'_1, e'_2) where the e'_i's are alternating permutations of sets of odd cardinality, unless both e_1 and e_2 are empty. It follows that

$$\sum_{\substack{m,n\geq 0 \\ m+n \text{ odd}}} E_{m+n} \frac{x^m}{m!} \frac{y^n}{n!} = \tan x + \sum_{k\geq 0} \left(1 + \tan^2 x\right) \left(\tan^k x\right) \left(\tan^{k+1} y\right)$$

$$= \sum_{k\geq 0} \left[\left(\tan^k x\right) \left(\tan^{k+1} y\right) + \left(\tan^{k+1} x\right) \left(\tan^k y\right)\right].$$

8. For further information on central factorial numbers (where our $T(n,k)$ is denoted $T(2n, 2k)$), see J. Riordan, *Combinatorial Identities*, John Wiley & Sons, New York, 1968 (Section 6.5). Part (e) is equivalent to a conjecture of J. M. Gandhi, *Amer. Math. Monthly* **77** (1970), 505–506. This conjecture was proved by L. Carlitz, *K. Norske Vidensk. Selsk. Sk.* **9** (1972), 1–4, and by J. Riordan and P. R. Stein, *Discrete Math.* **5** (1973), 381–388. A combinatorial proof of Gandhi's conjecture was given by J. Françon and G. Viennot, *Discrete Math.* **28** (1979), 21–35. The basic combinatorial property (f)(i) of Genocchi numbers is due to D. Dumont, *Discrete Math.* **1** (1972), 321–327, and *Duke Math. J.* **41** (1974), 305–318. For many further properties of Genocchi numbers, see the survey by G. Viennot, *Séminaire de Théorie des Nombres*, 1981/82, Exp. No. 11, 94 pp., Univ. Bordeaux I, Talence, 1982. A further reference is D. Dumont and A. Randrianarivony, *Discrete Math.* **132** (1994), 37–49.
9. (a) If $C(x)$ is the exponential generating function for the number of connected structures on an n-set, then Corollary 5.1.6 asserts that $F(x) = \exp C(x)$. Hence

$$E_g(x) = \exp \frac{1}{2}(C(x) + C(-x)) = \sqrt{F(x)F(-x)}.$$

(b) Let $c_k(n)$ be the number of k-component structures that can be put on an n-set, and let

$$F(x,t) = \sum_{n\geq 0} \sum_{k\geq 0} c_k(n) t^k \frac{x^n}{n!}.$$

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By Example 5.2.2 we have $F(x,t) = F(x)^t$, so

$$E_e(x) = \frac{1}{2}(F(x,1) + F(x,-1)) = \frac{1}{2}\left(F(x) + \frac{1}{F(x)}\right).$$

This formula was first noted by H. S. Wilf (private communication, 1997).

10. (a) Put

$$t_i = \begin{cases} 1, & \text{if } k|i \\ 0, & \text{if } k \not| \, i \end{cases}$$

in (5.30) to get

$$\begin{aligned} \sum_{n\geq 0} f_k(n)\frac{x^n}{n!} &= \exp \sum_{i\geq 1} \frac{x^{ki}}{ki} \\ &= \exp \frac{1}{k}\log(1-x^k)^{-1} \\ &= (1-x^k)^{-1/k} \\ &= \sum_{n\geq 0} \binom{-1/k}{n}(-1)^n x^{kn}. \end{aligned}$$

Hence $f_k(kn) = (kn)!\binom{-1/k}{n}(-1)^n$, which simplifies to the stated answer.

(b) Suppose $k|n$. We have $n-1$ choices for $\pi(1)$, then $n-2$ choices for $\pi^2(1)$, down to $n-k+1$ choices for $\pi^{k-1}(1)$. For $\pi^k(1)$ we have $n-k+1$ choices, since $\pi^k(1) = 1$ is possible. If $\pi^k(1) \neq 1$ we have $n-k-1$ choices for $\pi^{k+1}(1)$, while if $\pi^k(1) = 1$ we again have $n-k-1$ choices for $\pi(i)$, where i is the least element of $[n]$ not in the cycle $(1, \pi(1), \ldots, \pi^{k-1}(1))$. Continuing this line of reasoning, for our jth choice we always have $n-j$ possibilities if $k \not| \, j$ and $n-j+1$ possibilities if $k|j$, yielding the stated answer.

(c) Put

$$t_i = \begin{cases} 0, & \text{if } k|i \\ 1, & \text{if } k \not| \, i \end{cases}$$

in (5.30) to get

$$\begin{aligned} \sum_{n\geq 0} g_k(n)\frac{x^n}{n!} &= \exp\left(\sum_{i\geq 1}\frac{x^i}{i} - \sum_{i\geq 1}\frac{x^{ki}}{ki}\right) \\ &= \exp\left[\log(1-x)^{-1} - \frac{1}{k}\log(1-x^k)^{-1}\right] \\ &= (1-x^k)^{1/k}(1-x)^{-1} \end{aligned}$$

$$= \quad (1+x+\cdots+x^{k-1})(1-x^k)^{\frac{1-k}{k}}$$
$$= \quad (1+x+\cdots+x^{k-1})\sum_{n\geq 0}\binom{\frac{1-k}{k}}{n}(-1)^n x^{kn},$$

etc. (Compare Exercise 1.173(a), the case $k = 2$ of the present exercise.) Note that

$$\left(\sum_{n\geq 0} f_k(n)\frac{x^n}{n!}\right)\left(\sum_{n\geq 0} g_k(n)\frac{x^n}{n!}\right) = \sum_{n\geq 0} n!\frac{x^n}{n!} = \frac{1}{1-x},$$

since every cycle of a permutation either has length divisible by k or length not divisible by k.

This result appears in P. Erdős and P. Turan, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 151–163 (Lemma 1).

(d) See E. D. Bolker and A. M. Gleason, *J. Combinatorial Theory (A)* **29** (1980), 236–242, and E. A. Bertram and B. Gordon, *European J. Combinatorics* **10** (1989), 221–226. A combinatorial proof of a generalization of the case $k = 2$ different from (c) appears in R. P. Lewis and S. P. Norton, *Discrete Math.* **138** (1995), 315–318.

11. (a) A permutation $w \in \mathfrak{S}_n$ has a square root if and only if the number of cycles of each even length $2i$ is even. A simple variant of Example 5.2.10 yields

$$\sum_{n\geq 0} a(n)\frac{x^n}{n!} = e^x\left(\cosh\frac{x^2}{2}\right)e^{\frac{x^3}{3}}\left(\cosh\frac{x^4}{4}\right)e^{\frac{x^5}{5}}\cdots$$
$$= \left(\frac{1+x}{1-x}\right)^{1/2}\prod_{k\geq 1}\cosh(x^{2k}/2k).$$

This result appears in J. Blum, *J. Combinatorial Theory (A)* **17** (1974), 156–161 (equation (5)), and [1.6, §9.2] (but in this latter reference with the factors $\cosh(x^{2k}/2k)$ misstated as $\cosh(x^{2k}/k)$). These authors are concerned with the asymptotic properties of $a(n)$.

(b) Let $F(x) = \sum_n a(n)\frac{x^n}{n!}$. Then by (a) $F(x)/(1+x)$ is even, and the result follows. See J. Blum, *ibid.* (Theorem 1).

12. Let $a = u$ and $b = uv^{-1}$. Then $u = a$ and $v = b^{-1}a$, so a and b range over $\mathfrak{S}_n$ as u and v do. Note that $u^2v^{-2} = (aba^{-1})b$. Since every element of $\mathfrak{S}_n$ is conjugate to its inverse, the multiset of elements $aba^{-1}b$ ($a, b \in \mathfrak{S}_n$) coincides with the multiset of elements $aba^{-1}b^{-1}$. Thus $f(n)$ is equal to the number of pairs $(a, b) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that $ab = ba$. (See Exercise 7.69(h).)

Since the number $k(a)$ of conjugates of a is the index $[\mathfrak{S}_n : C(a)]$ of the centralizer of a, we have

$$\begin{aligned} f(n) &= \sum_{a \in \mathfrak{S}_n} \#C(a) \\ &= \sum_{a \in \mathfrak{S}_n} \frac{n!}{k(a)} \\ &= n!\,p(n), \end{aligned}$$

where $p(n)$ is the number of partitions of n (the number of conjugacy classes of $\mathfrak{S}_n$). Hence

$$F(x) = \prod_{i \geq 1} (1 - x^i)^{-1}.$$

A less conceptual proof can also be given by considering the possible cycle types of u and v.

Note that the above argument shows the following more general results. First, for any finite group G,

$$\#\{(u, v) \in G \times G : uv = vu\} = k(G) \cdot |G|,$$

where $k(G)$ denotes the number of conjugacy classes in G. (This result was known to P. Erdős and P. Turan, *Acta Math. Hung.* **19** (1968), 413–435 (Theorem IV, proved on p. 431)). Second (using the observation that if $aba^{-1}b = 1$, then b is conjugate to b^{-1}), we have

$$\#\{(u, v) \in G \times G : u^2 = v^2\} = |G| \cdot \iota(G), \tag{5.124}$$

where $\iota(G)$ is the number of "self-inverse" conjugacy classes of G, that is, conjugacy classes K for which $w \in K \Leftrightarrow w^{-1} \in K$. This result can also be proved using character theory, as done in Exercise 7.69(h) for a situation overlapping the present one when $G = \mathfrak{S}_n$. The problem of computing the left-hand side of (5.124) was posed by R. Stanley, Problem 10654, *Amer. Math. Monthly* **105** (1998), 272.

13. (a) A homomorphism $f : G \to \mathfrak{S}_n$ defines an action of G on $[n]$. The orbits of this action form a partition $\pi \in \Pi_n$. By the exponential formula (Corollary 5.1.6), we have

$$\sum_{n \geq 0} \#\mathrm{Hom}(G, \mathfrak{S}_n) \frac{x^n}{n!} = \exp\left(\sum_d g_d \frac{x^d}{d!} \right),$$

where g_d is the number of *transitive* actions of G on $[d]$. Such an action is obtained by choosing a subgroup H of index d to be the subgroup of G fixing a letter (say 1), and then choosing in $(d-1)!$ ways the letters $1 \neq i \in [d]$ corresponding to the proper cosets of H. Hence $g_d = (d-1)!j_d$, and the proof follows.

This result first appeared (though not stated in generating function form) in I. Dey, *Proc. Glasgow Math. Soc.* **7** (1965), 61–79. The proof given here appears in K. Wohlfahrt, *Arch. Math.* **29** (1977), 455–457. For some ramifications and generalizations, see T. Müller, *J. London Math. Soc. (2)* **44** (1991), 75–94; *Invent. math.* **126** (1996), 111–131, and *Adv. Math.* **153** (2000), 118–154; as well as T. Müller and J. Shareshian, *Adv. Math.* **171** (2002), 276–331, and [73, §3.1]. A general survey of the function $j_d(G)$ is given by A. Lubotzky, in *Proceedings of the International Congress of Mathematicians* (Zürich, 1994), vol. 1, Birkhäuser, Basel/Boston/Berlin, 1995, pp. 309–317.

(b) If F_s has generators $x_1, \dots, x_s$, then a homomorphism $\varphi : F_s \to \mathfrak{S}_n$ is determined by any choice of the $\varphi(x_i)$'s. Hence $\#\mathrm{Hom}(F_s, \mathfrak{S}_n) = n!^s$, and the proof follows from (a). A recurrence equivalent to (5.89) was found by M. Hall, Jr., *Canad. J. Math.* **1** (1949), 187–190, and also appears as Theorem 7.2.9 in M. Hall, Jr., *The Theory of Groups*, Macmillan, New York, 1959. Equation (5.89) itself first appeared in [4.55, eqn. (21)]. From (5.89) E. Bender, *SIAM Rev.* **16** (1974), 485–515 (§5), has derived an asymptotic expansion for $j_d(F_s)$ for fixed s. For further combinatorial aspects of $j_d(F_2)$, see A. W. M. Dress and R. Franz, *Bayreuth Math. Schr.*, No. 20 (1985), 1–8, and T. Sillke, in *Séminaire Lotharingien de Combinatoire (Oberfranken, 1990)*, Publ. Inst. Rech. Math. Av. **413**, Univ. Louis Pasteur, Strasbourg, 1990, pp. 111–119.

(c) Let $m|d$. Choose a subgroup H of G of index m, and let $N(H)$ denote its normalizer. Choose an element $z \in N(H)/H$, and define a subgroup K of $G \times \mathbb{Z}$ by

$$K = \{(w, da/m) \in G \times \mathbb{Z} : w \in N(H),\ w = z^a \text{ in } N(H)/H\}.$$

Then $[G \times \mathbb{Z} : K] = d$, and every subgroup K of $G \times \mathbb{Z}$ of index d is obtained uniquely in this way. (This fact is a special case of the description of the subgroups of the direct product of any two groups. See for example M. Suzuki, *Group Theory I*, Springer-Verlag, Berlin/Heidelberg/New York, 1982 (p. 141); translated from the Japanese edition *Gunron*, Iwanami Shoten, Tokyo, 1977 and 1978.) Once we choose m and H, there are $[N(H) : H]$ choices for z. Since the number of conjugates of H is equal to the index $[G : N(H)]$, we see easily that

$$u_m(G) = \frac{1}{m} \sum_{[G:H]=m} [N(H) : H].$$

It follows that

$$j_d(G \times \mathbb{Z}) = \sum_{m|d} m u_m(G), \tag{5.125}$$

and the proof follows from (a) and Exercise 1.158.

NOTE. The numbers $u_d(F_s)$ (where F_s is the free group on s generators) were computed by V. Liskovets, *Dokl. Akad. Nauk BSSR* **15** (1971), 6–9 (in Russian), essentially by using equation (5.90). A messier formula for $u_d(F_s)$ appears in J. H. Kwak and J. Lee, *J. Graph Theory* **23** (1996), 105–109. Note that

$$\#\mathrm{Hom}(\mathbb{Z} \times F_s, \mathfrak{S}_n) = \sum_{w \in \mathfrak{S}_n} (\#C(w))^s,$$

where $C(w)$ denotes the centralizer of w in $\mathfrak{S}_n$ (whose cardinality is given explicitly by equation (7.7.17). Using (5.90) it is then not hard to obtain the formula

$$\prod_{i \geq 1} \left(\sum_{j \geq 0} (j!\, i^j)^{s-1} x^{ij} \right) = \prod_{d \geq 1} (1 - x^d)^{-u_d(F_s)},$$

which is equivalent to the formula of Liskovets for $u_d(F_s)$.

(d) Observe that

$$c_m(n) = \#\mathrm{Hom}(\mathbb{Z}^m, \mathfrak{S}_n).$$

Now use (c). An equivalent result (stated below) was first proved by J. Bryan and J. Fulman, *Ann. Comb.* **2** (1998), 1–6. NOTE. It is well-known (and an easy consequence of Exercise 3.126(c) or of equation (5.125)) that

$$\sum_{d \geq 1} j_d(\mathbb{Z}^{m-1}) d^{-s} = \zeta(s)\zeta(s-1) \cdots \zeta(s-m+2), \tag{5.126}$$

where ζ denotes the Riemann zeta function. For the history of this result, see L. Solomon, in *Relations between Combinatorics and Other Parts of Mathematics*, Proc. Symp. Pure Math. **34**, American Mathematical Society, 1979, pp. 309–330. By iterating (5.90) or by using (5.126) directly, we obtain the formula of Bryan and Fulman, namely,

$$\sum_{n \geq 0} c_m(n) \frac{x^n}{n!} = \prod_{i_1, \ldots, i_{m-1} \geq 1} \left(\frac{1}{1 - x^{i_1 \cdots i_{m-1}}} \right)^{i_1^{m-2} i_2^{m-3} \cdots i_{m-2}}.$$

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(e) By (a), we want to show that $h_k(n) = \#\mathrm{Hom}(\mathfrak{S}_k, \mathfrak{S}_n)$. Let $s_m = (m, m+1) \in \mathfrak{S}_k$, $1 \le m \le k-1$. Given a homomorphism $f : \mathfrak{S}_k \to \mathfrak{S}_n$, define a graph Γ_f on the vertex set $[n]$ by the condition that there is an edge colored m with vertices $a \neq b$ if $f(s_m)(a) = b$. One checks that the conditions (i)–(iii) are equivalent to the well-known Coxeter relations (e.g., J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990 (Section 1.9)) satisfied by the generators s_m of $\mathfrak{S}_k$.

14. (a) Take the logarithm of both formulas, subtract one from the other, and solve for y to get

$$1 + t^{-1}y = \frac{1-t}{e^{x(t-1)} - t}. \tag{5.127}$$

Comparing with Exercise 3.204(c) (after correcting a typographical error) shows that the only possible y is as claimed. Since the steps are reversible, the proof follows.

(b) While this result can easily be proved using the explicit formula (5.127) and the fact that $\frac{d}{dx}\sum_{n\ge 2} A_{n-1}(t)\frac{x^n}{n!} = y$, we prefer as usual a combinatorial proof. Define a *connected A-structure* on a finite subset S of $\mathbb{P}$ to consist of a permutation $w = a_1a_2\cdots a_j$ of S whose smallest element $\min a_i$ is a_1. Define the weight $f(w)$ of w by $f(w) = t^{1+d(w)}$. If $\#S = n$ then it is easy to see that

$$C_n(t) := \sum_w f(w) = \begin{cases} t, & n = 1 \\ A_{n-1}(t), & n > 1, \end{cases}$$

where w ranges over all connected A-structures on S. By the exponential formula (Corollary 5.1.6), we have

$$\exp\left(tx + \sum_{n\ge 2} A_{n-1}(t)\frac{x^n}{n!}\right) = \sum_{n\ge 0} \tilde{A}_n(t)\frac{x^n}{n!},$$

where

$$\tilde{A}_n(t) = \sum_{\pi=\{B_1,\dots,B_k\}\in\Pi_n} C_{\#B_1}(t)\cdots C_{\#B_k}(t).$$

Given an A-structure w_i on each block B_i of π, where the indexing is chosen so that $\min w_1 > \min w_2 > \cdots > \min w_k$, the concatenation $w = w_1w_2\cdots w_k$ is a permutation of $[n]$ such that

$$f(w_1)f(w_2)\cdots f(w_k) = t^{1+d(w)}.$$

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Conversely, given $w \in \mathfrak{S}_n$ we can uniquely recover $w_1, w_2, \dots, w_k$, since the elements $\min w_i$ are the left-to-right minima of w. (Compare the closely related bijection $\mathfrak{S}_n \xrightarrow{\wedge} \mathfrak{S}_n$ of Proposition 1.3.1.) Hence

$$\tilde{A}_n(t) = \sum_{w \in \mathfrak{S}_n} t^{1+d(w)} = A_n(t),$$

completing the proof.

(c) It follows from the argument above that the number of left-to-right minima of w is k, the number of blocks of π. The stated formula is now an immediate consequence of the discussion in Example 5.2.2. This result is due to L. Carlitz and R. A. Scoville, *J. Combinatorial Theory* **22** (1977), 129–145 (equation (1.13)), with a more computational proof than ours. Carlitz and Scoville state their result in terms of the number of cycles and excedances (which they call “ups”) of w, but the bijection $\mathfrak{S}_n \xrightarrow{\wedge} \mathfrak{S}_n$ of Proposition 1.3.1 shows that the two results are equivalent.

(d) If a_i is a left-to-right minimum of $w = a_1 a_2 \cdots a_n$, then either $i = 1$ or $i \in D(w)$. Hence $1 + d(w) - m(w) \geq 0$. By (c) we have

$$(1+y)^{q/t} = \sum_{n \geq 0} \left(\sum_{w \in \mathfrak{S}_n} q^{m(w)} t^{1+d(w)-m(w)} \right) \frac{x^n}{n!},$$

and the proof follows.

15. (a)

$$\begin{aligned} E_f(x) &= \exp \sum_{i \geq k} \tfrac{1}{2}(i-1)! \frac{x^i}{i!} \\ &= (1-x)^{-1/2} \exp\left(-\frac{x}{2} - \frac{x^2}{4} - \cdots - \frac{x^{k-1}}{2(k-1)} \right) \end{aligned}$$

(b)

$$\begin{aligned} E_f(x) &= \exp\left(\frac{x^2}{2!} + \sum_{i \geq 3} i \frac{x^i}{i!} \right) \\ &= \exp\left(-x - \frac{x^2}{2} + xe^x \right) \end{aligned}$$

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(c)

$$\begin{aligned} E_f(x) &= \exp\left(\frac{x^4}{4!} + \sum_{i\geq 5} \frac{i(i-2)!}{2}\frac{x^i}{i!}\right) \\ &= (1-x)^{-x/2}\exp\left(-\frac{1}{2}x^2 - \frac{1}{4}x^3 - \frac{1}{8}x^4\right) \end{aligned}$$

(d)

$$\begin{aligned} E_f(x) &= \exp\left(x + \sum_{i\geq 2} \frac{i!}{2}\frac{x^i}{i!}\right) \\ &= \exp\left(\frac{x}{2} + \frac{x}{2(1-x)}\right) \end{aligned}$$

16. (a) See R. Stanley, in *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Volume **4**, American Mathematical Society, 1991, pp. 555–570 (Cor. 3.4). It would be interesting to have a direct combinatorial proof of (5.91). For some work in this direction, see C. Chan, Ph.D. thesis, M.I.T., 1992 (§3).

(b) By Proposition 5.3.2 the number of rooted trees on n vertices is n^{n-1}, with exponential generating function

$$R(x) = \sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}.$$

Hence by Proposition 5.1.3 the exponential generating function for k-tuples of rooted trees is $R(x)^k$, and so for undirected k-cycles of rooted trees (i.e., graphs with exactly one cycle, which is of length $k \geq 3$) is $R(x)^k/2k$.

Let $h(j,n)$ be the number of graphs G on the vertex set $[n]$ such that every component has exactly one cycle, which is of odd length ≥ 3, and such that G has a total of j cycles. (Such graphs have exactly n edges.) Then by the exponential formula (Corollary 5.1.6) we have

$$\begin{aligned} \sum_{j,n\geq 0} h(j,n)\frac{t^j x^n}{n!} &= \exp \sum_{k\geq 1} \frac{t}{2(2k+1)} R(x)^{2k+1} \\ &= \exp \frac{t}{2}\left[\frac{1}{2}\left(\log(1-R(x))^{-1} - \log(1+R(x))^{-1}\right) - R(x)\right] \\ &= \left(\frac{1+R(x)}{1-R(x)}\right)^{t/4} e^{-tR(x)/2}. \end{aligned}$$

Thus,

$$1+\sum_{j,n\geq 1} 2^{j-1}h(j,n)\frac{x^n}{n!} = \frac{1}{2}\left[\left(\frac{1+R(x)}{1-R(x)}\right)^{1/2} e^{-R(x)} - 1\right]$$
$$= \frac{1}{2}\left[\left(-1+\frac{2}{1-R(x)}\right)^{1/2} e^{-R(x)} + 1\right].$$

It is easy to deduce from $R(x) = xe^{R(x)}$ that

$$\frac{1}{1-R(x)} = \sum_{n\geq 0} n^n \frac{x^n}{n!}, \quad e^{-R(x)} = 1 - \sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!} \tag{5.128}$$

(for the first of these formulas see Exercise 42, while the second follows from equation (5.67)), so we get

$$1+\sum_{j,n\geq 1} 2^{j-1}h(j,n)\frac{x^n}{n!} = \frac{1}{2}\left[\left(1+2\sum_{n\geq 1} n^n\frac{x^n}{n!}\right)^{1/2}\right.$$
$$\left.\left(1-\sum_{n\geq 1}(n-1)^{n-1}\frac{x^n}{n!}\right)+1\right].$$

Now by Propositions 5.1.1 and 5.1.6, the exponential generating function for the right-hand side of (5.91) is

$$\left(1+\sum_{j,n\geq 1} 2^{j-1}h(j,n)\frac{x^n}{n!}\right)\cdot e^{T(x)},$$

where $T(x) = \sum_{n\geq 1} n^{n-2}\frac{x^n}{n!}$ is the exponential generating function for free trees on the vertex set $[n]$, and the proof follows.

This result appears in R. Stanley, *ibid.*, Cor. 3.6.

17. (a) Line up all n persons in $n!$ ways. Break the line in $k-1$ of the $n-1$ places between two consecutive persons, in $\binom{n-1}{k-1}$ ways. This yields k lines, but the same k lines could have been obtained in any order, so we must divide by the $k!$ ways of ordering k lines. Thus there are $\frac{n!}{k!}\binom{n-1}{k-1}$ ways. (Exercise 1.31(b) is essentially the same as this one.)

(b) Put $f(n)=n!$ and $g(k)=x^k$ in Theorem 5.1.4 (or $f(n)=xn!$ in Corollary 5.1.6).

(c) We have

$$\left[\frac{u^r}{r!}\right]\frac{u}{(1-u)^a} = r!\left(\!\!\binom{r}{a-1}\!\!\right),$$

the number of ways to linearly order an r-element set, say $z_1, z_2, \ldots, z_r$, and then to place $a-1$ bars in the spaces between the z_i's or before z_1 (but not after z_r), allowing any number of bars in each space. On the other hand, we have $\binom{n+(a-1)k-1}{n-k} = \left(\!\binom{n-k+1}{ak-1}\!\right)$, the number of ways to place $ak-1$ bars $B_1, \ldots, B_{ak-1}$ (from left-to-right) in the spaces between a line of $n-k$ dots, or at the beginning and end of the line, allowing any number of bars in each space. Put a new bar B_0 at the beginning and a new bar B_{ka} at the end. Put a new dot just before the bar B_{ja} for $1 \le j \le k$. We now have n dots in all. Replace them with a permutation of $[n]$ in $n!$ ways. By considering the configuration between $B_{(j-1)a}$ and B_{ja} for $1 \le j \le k$, we see that our structure is equivalent to an *ordered* partition of $[n]$ into k blocks, such that each block has a linear ordering $z_1, \ldots, z_r$ together with $a-1$ bars in the spaces between the z_i's, allowing bars before z_1 but not after z_r. Since there are $k!$ ways of "unordering" the k blocks, equation (5.92) follows from Corollary 5.1.6 (the exponential formula). Equation (5.93) is proved similarly.

Essentially the same argument was found by C. A. Athanasiadis, H. Cohn, and L. W. Shapiro (independently). These identities are also easy to prove algebraically. For instance,

$$\begin{aligned}\exp \frac{xu}{(1-u)^a} &= \sum_{k\ge 0} \frac{u^k}{(1-u)^{ak}} \frac{x^k}{k!} \\ &= \sum_{k\ge 0} \left(\sum_{n\ge k} \left(\!\binom{ak}{n-k}\!\right) u^n \right) \frac{x^k}{k!},\end{aligned}$$

etc.

(d) Choose an $(n-k)$-subset T of $[n]$ in $\binom{n}{k}$ ways. Choose an injection $g : T \to [n] \cup A$ in $(\alpha+n)_{n-k}$ ways. We have $\binom{n}{k}(\alpha+n)_{n-k}$ ways of choosing in all. If $i \in [n]-T$ and i is not in the image of g, then define $\{i\}$ to be a block of π (which of course has a unique linear ordering). If $i \in [n]-T$ and $i = g(j)$ for some j, then there is a unique $m \in T$ for which $g^r(m) = i$ for some $r \ge 1$, and m is not in the image of g. Define a linearly ordered block of π by

$$m > g(m) > g^2(m) > \cdots > g^r(m) = i.$$

The remaining elements of $[n]$ (those not in some block of π) form the set $\bar{S}$, and the restriction of g to $\bar{S}$ defines f.

(e) Note that

$$(1-u)^{-\alpha-1} = \sum_{j\geq 0}(\alpha+j)_j \frac{u^j}{j!},$$

and that $(\alpha+j)_j$ is the number of injections $f : \bar{S} \to \bar{S} \cup A$, where $\#\bar{S} = j$. Now use Proposition 5.1.1 and (b). There is also an easy algebraic proof analogous to that given at the end of (c).

The polynomials

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(\alpha+n)_{n-k}(-x)^k$$

are the *Laguerre polynomials*. The combinatorial approach used here is due to D. Foata and V. Strehl, in *Enumeration and Design* (D. M. Jackson and S. A. Vanstone, eds.), Academic Press, Toronto/Orlando (1984), 123–140. They derive many additional properties of Laguerre polynomials by similar combinatorial reasoning. Combinatorial approaches toward other classical sequences of polynomials have been undertaken by a number of researchers; see for example X. G. Viennot, *Une théorie combinatoire des polynômes orthogonaux*, lecture notes, Université de Québec à Montréal, Dépt. de Maths., 1984, 215 pp.; various papers in Springer Lecture Notes in Math., vol. 1171, Springer-Verlag, Berlin, 1985 (especially pp. 111–157); J. Labelle and Y.-N. Yeh, *Studies in Applied Math.* **80** (1989), 25–36. See also Exercise 19 for a further example of this type of reasoning. Additional references appear in [16, p. xiv].

18. If C is a cycle of length n, then the number of distinct cycles which are powers of C is $\phi(n)$ (since the distinct cycles are C^j where $1 \leq j \leq n$ and $(j,n) = 1$). Hence if π has cycles $C_1, C_2, \ldots, C_k$, then the number of permutations equivalent to π is $\prod_{i=1}^{k}\phi(\#C_i)$. Therefore

$$e(n) = \sum_{\pi\in\mathfrak{S}_n}\left(\prod_i \phi(\#C_i)^{-1}\right),$$

where the C_i's are the cycles of π. Now use Corollary 5.1.9.

This result was proposed as a problem by R. Stanley, *Amer. Math. Monthly* **80** (1973), 949, with a solution published due to A. Nijenhius, **82** (1975), 86–87.

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19. We have

$$K_n(a)K_n(b) = \sum_{\pi,\sigma} a^{c_1(\pi)} b^{c_1(\sigma)},$$

summed over pairs (π,σ) of involutions in $\mathfrak{S}_n$. Represent (π,σ) by a graph $G(\pi,\sigma)$ on the vertex set $[n]$ by putting a red (respectively, blue) edge between i and j if (i,j) is a cycle of π (respectively, σ). If $\pi(i) = i$ (respectively, $\sigma(i) = i$) then we put a red (respectively, blue) loop on the vertex i. (Thus if $\pi(i) = i$ and $\sigma(i) = i$, then there are two loops on i, one red and one blue.) There are three types of components of $G(\pi,\sigma)$:

(i) A path with a loop at each end and with $2k+1 \geq 1$ vertices, with red and blue edges alternating. There are $(2k+1)!$ such paths, and all have one red and one blue loop. Thus each contribute a factor ab to the term $a^{c_1(\pi)}b^{c_1(\sigma)}$.

(ii) A path as in (i) with $2k \geq 2$ vertices. There are $\frac{1}{2}(2k)!$ paths before we color the edges. One coloring produces two red loops and the other two blue loops, thus contributing a^2 and b^2, respectively, to $a^{c_1(\pi)}b^{c_1(\sigma)}$.

(iii) A cycle of length $2k \geq 2$ with red and blue edges alternating. There are $(2k-1)!$ such cycles, and all have no loops. Thus a cycle contributes a factor of 1 to $a^{c_1(\pi)}b^{c_1(\sigma)}$.

It follows from Corollary 5.1.6 (the exponential formula) that

$$\begin{aligned}
\sum_{n\geq 0} K_n(a)K_n(b)\frac{x^n}{n!} &= \exp\left[ab\sum_{k\geq 0}\frac{(2k+1)!x^{2k+1}}{(2k+1)!}\right. \\
&\qquad \left. + \frac{1}{2}(a^2+b^2)\sum_{k\geq 1}\frac{(2k)!x^{2k}}{(2k)!} + \sum_{k\geq 1}\frac{(2k-1)!x^{2k}}{(2k)!}\right] \\
&= (1-x^2)^{-1/2}\exp\left[\frac{abx+\frac{1}{2}(a^2+b^2)x^2}{1-x^2}\right].
\end{aligned}$$

The *Hermite polynomials* $H_n(a)$ may be defined by

$$1 + \sum_{n\geq 1} H_n(a)\frac{x^n}{n!} = \exp(2ax - x^2). \tag{5.129}$$

(Sometimes a different normalization is used, so the right-hand side of (5.129) becomes $\exp(ax - \frac{x^2}{2})$.) In terms of the Hermite polynomials, the identity (5.95) becomes

$$\sum_{n\geq 0} H_n(a)H_n(b)\frac{x^n}{n!} = (1-4x^2)^{-1/2}\exp\left[\frac{4abx - 4(a^2+b^2)x^2}{1-4x^2}\right].$$

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This identity is known as *Mehler's formula*. M.-P. Schützenberger suggested finding a combinatorial proof, and essentially the above proof was given by D. Foata, *J. Combinatorial Theory (A)* **24** (1978), 367–376. For further results along these lines, see D. Foata, *Advances in Applied Math.* **2** (1981), 250–259, and D. Foata and A. M. Garsia, *Proc. Symp. Pure Math.* (D. K. Ray-Chaudhuri, ed.), vol. 34, American Mathematical Society, Providence, RI, 1979, pp. 163–179.

20. (a) We want to interpret $xe^{B'(F(x))}$ as the exponential generating function (e.f.g.) for rooted $\mathcal{B}$-graphs on n vertices. By (5.20), $B'(x)$ is the e.f.g. for blocks on an $(n+1)$-element vertex set which are isomorphic to a block in $\mathcal{B}$. Thus by Theorem 5.1.4, $B'(F(x))$ is the e.f.g. for the following structure on an n-element vertex set V. Partition V, and then place a rooted $\mathcal{B}$-graph on each block. Add a new vertex v_0, and place on the set of root vertices together with v_0 a block in $\mathcal{B}$. This is equivalent to a $\mathcal{B}$-graph G on $n+1$ vertices, rooted at a vertex v_0 with the property that only one block of G contains v_0.

It follows from Corollary 5.1.6 that $e^{B'(F(x))}$ is the e.f.g. for the following structure on an n-set V. Choose a partition π of V. Add a root vertex v_A to each block A of π. Place on each set $A \cup \{v_A\}$ a $\mathcal{B}$-graph G_A such that v_A is contained in a single block.

If we identify all the vertices v_A to a single vertex v_*, then we obtain simply a $\mathcal{B}$-graph G on $V \cup \{v_*\}$. Moreover, given G we can uniquely recover the partition π and the graphs G_A by removing v_* from G, seeing the connected components which remain (whose vertex sets will be the A's), and adjoining v_A to each component connected in the same way that v_* was connected to that component. Thus $e^{B'(F(x))}$ is the e.f.g. for connected $\mathcal{B}$-graphs on $V \cup \{v_*\}$, where $\#V = n$.

Lastly it follows from (5.19) that $xe^{B'(F(x))}$ is the e.f.g. for the following structure on an n-set W. Choose an element $w \in W$, then add an element w_* to $W - \{w\}$ and place a connected $\mathcal{B}$-graph on $(W - \{w\}) \cup \{w_*\}$. This is equivalent to rooting W at w and placing a connected $\mathcal{B}$-graph on W. In other words, $xe^{B'(F(x))}$ is the e.f.g. for rooted connected $\mathcal{B}$-graphs on n vertices, and hence coincides with $F(x)$. To obtain (5.97), substitute $F^{\langle -1 \rangle}(x)$ for x in (5.96) and solve for $B'(x) = \sum_{n \geq 1} b(n+1)\frac{x^n}{n!}$.

See Figure 5.24 for an example of the decomposition of rooted connected $\mathcal{B}$-graphs described by $xe^{B'(F(x))}$. Equation (5.96) is known as the *block-tree theorem*, and is due to G. W. Ford and G. E. Uhlenbeck, *Proc. Nat. Acad. Sci.* **42** (1956), 122–128 (the case $y_0 = 1$ of (5.6)). Ford and Uhlenbeck in fact prove a more general result where they

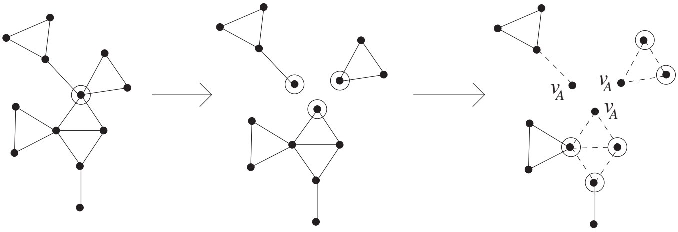

Figure 5.24 The block-tree decomposition

keep track of the number of occurrences of each block in a $\mathcal{B}$-graph G. They then use Lagrange inversion to obtain that the number of $\mathcal{B}$-graphs on an n-element vertex set with k_B blocks isomorphic to B is equal to

$$\frac{n! \cdot n^{\Sigma_B k_B - 1}}{\prod_B \left(\frac{|\mathrm{Aut}\, B|}{p_B}\right)^{k_B} k_B!}$$

where the block B has p_B vertices.

(b) Let $\mathcal{B}$ be the set of all blocks without multiple edges. A $\mathcal{B}$-graph is just a connected graph without multiple edges. Letting $F(x)$ and $B(x)$ be as in (a), by (5.37) and (5.21) we have

$$F(x) = x\frac{d}{dx}\log \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}.$$

Now use (5.97).

21. Let $u = u_1u_2\cdots u_n$, where $u_i \in \mathcal{A}$. Represent u as a row of n dots, and connect two adjacent dots if they belong to the same word of $\mathcal{B}$ when u is factored into words in $\mathcal{B}$. If $\pi = a_1a_2\cdots a_n$, then place a_i below the ith dot. For instance, if $u = u_1u_2\cdots u_9$ where $u_1 \cdot u_2u_3u_4 \cdot u_5 \cdot u_6u_7 \cdot u_8u_9$ represents the factorization of u into words in $\mathcal{B}$, and if $\pi = 529367148$, then we obtain the diagram

$$\begin{array}{ccccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 \\ \bullet & \bullet\!\!- & \!\!\bullet\!\!- & \!\!\bullet & \bullet & \bullet\!\!- & \!\!\bullet & \bullet\!\!- & \!\!\bullet \\ 5 & 2 & 9 & 3 & 6 & 7 & 1 & 4 & 8 \end{array}$$

Consider the subsequence ρ of π consisting of the labels of the *first* elements of each connected string. For the above example, we get $\rho = 52674$. Draw a bar before all left-to-right maxima (except the first) of the sequence ρ. For $\rho = 52674$, the left-to-right maxima are 5, 6, and 7. Thus we get

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$$
\begin{array}{cccc|c|cccc}
u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 \\
\bullet & \bullet & \!\!-\!\!\bullet\!\!-\!\! & \bullet & \bullet & \bullet & \!\!-\!\!\bullet & \bullet & \!\!-\!\!\bullet \\
5 & 2 & 9 & 3 & 6 & 7 & 1 & 4 & 8
\end{array}
$$

For each sequence v of u_i's separated by bars, write down the cyclic permutation of v whose first element corresponds to the largest possible element of π. Arrange in a cycle the elements of π which are below v. For our example, we obtain:

$$
\begin{array}{rl}
u_3u_4u_1u_2 & \quad (5293) \\
u_5 & \quad (6) \\
u_9u_6u_7u_8 & \quad (7148)
\end{array}
$$

We leave the reader to verify that this procedure establishes the desired bijection.

This bijection is due to I. Gessel.

22. To form a graph with every component a cycle on the vertex set $[n+1]$, first choose such a graph G on the vertex set $[n]$ (in $L(n)$ ways). Then insert the vertex $n+1$ into it, either as an isolated vertex (one way) or by choosing an edge e of G and inserting $n+1$ in the middle of it (n ways). Every allowable graph on $[n+1]$ will arise exactly once, except that the two ways of inserting $n+1$ into a 2-cycle (double edge) result in the same graph. There are $\binom{n}{2}$ possible edges, and $L(n-2)$ graphs which contain a given one of them. Hence $L(n+1) = (n+1)L(n) - \binom{n}{2}L(n-2)$, as desired.
This result was first proved by I. Schur, *Arch. Math. Phys. Series 3* **27** (1918), 162, in a less combinatorial fashion. See also [253, Problem VII.45].
23. Let N be a cloud. Identify the line δ_i with the node i, and the intersection $\delta_i \cap \delta_j \in N$ with the edge $\{i,j\}$.
This exercise is taken from [2.3, pp. 273–277]. The connection between clouds and graphs goes back to W. A. Whitworth, *Choice and Chance*, Bell, 1901 (reprinted by Hafner, 1965), Exercise 160 on p. 269. Whitworth erroneously claimed that $c(n) = \frac{1}{2}(n-1)!$. His error was corrected by Robin Robinson, *Amer. Math. Monthly* **58** (1951), 462–469, who obtained the recurrence for $c(n)$ given in Example 5.2.8 (where $T_n^*(2)$ is used instead of $c(n)$) by simple combinatorial reasoning. The generating function (5.29) was derived from the recurrence in an editorial note [77], and was used to complete an asymptotic formula for $c(n)$ partially found by Robinson. Some congruence properties of $c(n)$ were later given by L. Carlitz in *Amer. Math. Monthly* **61** (1954), 407–411, and **67** (1960), 961–966.

24. (a) By Example 4.6.16(b), the vertex set $V(\Sigma_n)$ of Σ_n satisfies

$$V(\Sigma_n) \subseteq \left\{ \frac{1}{2}(P + P^t) : \ P \text{ is an } n \times n \text{ permutation matrix} \right\}.$$

It is fairly straightforward to check which matrices $\frac{1}{2}(P + P^t)$ are actually vertices. See M. Katz, *J. Combinatorial Theory* **8** (1970), 417–423 (Theorem 1).

(b) Let $\frac{1}{2}(P+P^t) \in V(\Sigma_n)$. Suppose that P corresponds to the permutation π of $[n]$. Define a graph $G = G(P)$ on the vertex set $[n]$ by drawing an edge between i and j if $\pi(i) = j$ or $\pi(j) = i$. By (a), the components of G are single vertices with one loop, single edges, or odd cycles of length ≥ 3. Moreover, every such G corresponds to a unique vertex of Σ_n (though not necessarily to a unique P). There is one way to place a loop on one vertex or an edge on two vertices, and $\frac{1}{2}(2i)!$ ways to place a cycle on $2i + 1 \geq 3$ vertices. Hence

$$\begin{aligned}
\sum_{n\geq 0} M(n)\frac{x^n}{n!} &= \exp\left(x + \frac{x^2}{2} + \sum_{i\geq 1} \frac{1}{2}(2i)!\frac{x^{2i+1}}{(2i+1)!} \right) \\
&= \exp\left(\frac{x}{2} + \frac{x^2}{2} + \frac{1}{2}\sum_{i\geq 0} \frac{x^{2i+1}}{2i+1} \right) \\
&= \exp\left(\frac{x}{2} + \frac{x^2}{2} + \frac{1}{4}\left(\log(1-x)^{-1} - \log(1+x)^{-1} \right) \right) \\
&= \left(\frac{1+x}{1-x} \right)^{1/4} \exp\left(\frac{x}{2} + \frac{x^2}{2} \right).
\end{aligned}$$

An equivalent result (but not stated in terms of generating functions) appears in M. Katz, *ibid.* (Theorem 2).

(c) Take the logarithm of (5.98) and differentiate to get

$$\begin{aligned}
\sum_{n\geq 0} M(n+1)\frac{x^n}{n!} = \left(\sum_{n\geq 0} M(n)\frac{x^n}{n!} \right) \frac{d}{dx}\Bigg[\frac{1}{4}\log(1+x) \\
-\frac{1}{4}\log(1-x) + \frac{x}{2} + \frac{x^2}{2} \Bigg] \\
= \left(\sum_{n\geq 0} M(n)\frac{x^n}{n!} \right) \left(\frac{1}{2(1-x^2)} + \frac{1}{2} + x \right).
\end{aligned}$$

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Multiply by $2(1-x^2)$ and take the coefficient of $x^n/n!$ on both sides to obtain

$$M(n+1) = M(n)+n^2M(n-1)-\binom{n}{2}M(n-2)-n(n-1)(n-2)M(n-3).$$

This recurrence first appeared (with a misprint) in [317, Example 2.8].

(d) A solution was found by the Cambridge Combinatorics and Coffee Club, February 2000.

25. (a) This result is stated without proof (in a more complicated but equivalent form) by M. Katz, *J. Combinatorial Theory* **8** (1970), 417–423, and proved by the same author in *J. Math. Anal. Appl.* **37** (1972), 576–579.

(b) Arguing as in the solution to Exercise 24(b), the graph G corresponding to a matrix now can have as a component a single vertex with no loop. (Removing a 1 from the main diagonal converts a loop to a loopless vertex.) Thus when applying the exponential formula as in Exercise 24(b), we obtain an additional factor of e^x. (An erroneous generating function appears in [317, Example 2.8].)

(c) As in Exercise 24(c), we obtain

$$\sum_{n\geq 0} M^*(n+1)\frac{x^n}{n!} = \left(\sum_{n\geq 0} M^*(n)\frac{x^n}{n!}\right)\left(\frac{1}{2(1-x^2)}+\frac{3}{2}+x\right),$$

from which there follows

$$M^*(n+1) = 2M^*(n) + n^2M^*(n-1) - 3\binom{n}{2}M^*(n-2) - n(n-1)(n-2)M^*(n-3).$$

Is there a combinatorial proof, analogous to Exercise 22?

26. Given a set X, let $\mathcal{D}(X)$ denote the set of all subsets S of $2^X - \{\emptyset\}$ such that any two elements of S are either disjoint or comparable. Write $\mathcal{D}(n)$ for $\mathcal{D}([n])$. Since for $n \geq 1$ we have $S \in \mathcal{D}(n)$ and $[n] \notin S$ if and only if $[n] \notin S$ and $S \cup \{[n]\} \in \mathcal{D}(n)$, it follows that $F(x) = 1 + 2G(x)$. Now let $S \in \mathcal{D}(n)$, and regard S as a poset ordered by inclusion. It is not hard to see that S is a disjoint union of rooted trees, with the successors of any vertex being disjoint subsets of $[n]$. Hence S can be uniquely obtained as follows. Choose a partition $\pi = \{B_1, \ldots, B_k\}$ of $[n]$. For each block B_i of π, choose a set $S_i \in \mathcal{D}(B_i)$ such that $B_i \in S_i$ (in $g(\#B_i)$ ways). If $\#B_i = 1$, then we can also choose to have $B_i \notin S_i$. Finally let $S = \bigcup S_i$. Since there are $g(\#B_i)$ choices for each B_i and one extra choice when $\#B_i = 1$, it follows from Corollary 5.1.6 that $F(x) = e^{x+G(x)}$.
This exercise is due to I. Gessel.

27. Given an edge-labelled tree T with n edges, choose a vertex of T in $n+1$ ways and label it 0. Then "push" each edge label to the vertex of that edge farthest from 0. We obtain a bijection between (a) the $(n+1)e(n)$ ways to choose T and the vertex 0, and (b) the $(n+1)^{n-1}$ ways to choose a labelled tree on $n+1$ vertices. Hence $e(n) = (n+1)^{n-2}$. Essentially this bijection (though not an explicit statement of the formula $e(n) = (n+1)^{n-2}$) appears in J. Riordan, *Acta Math.* **97** (1957), 211–225 (see equation (17)), though there may be much earlier references.
28. Suppose that the tree T on the vertex set $[n]$ has ordered degree sequence $(d_1, \ldots, d_n)$ (i.e., vertex i has d_i adjacent vertices), where necessarily $\sum d_i = 2n-2$. Choose a vertex of degree one (endpoint), and adjoin vertices one at a time to the graph already constructed, keeping the graph connected. Color each edge as it is added to the graph. For the first edge we have k choices of colors. If one edge of a vertex of degree d has been colored, then there are $(d-1)!\binom{k-1}{d-1}$ ways to color the others. It follows easily from Theorem 5.3.4 that the number of free trees with ordered degree sequence $(d_1, \ldots, d_n)$ is equal to the multinomial coefficient
$$\binom{n-2}{d_1-1, d_2-1, \ldots, d_n-1}.$$
Hence the total number of k-edge colored trees is given by
$$\begin{aligned} T_k(n) &= k(n-2)! \sum_{d_1+\cdots+d_n=2n-2} \prod_{i=1}^{n} \binom{k-1}{d_i-1} \\ &= k(n-2)![x^{n-2}]\left((1+x)^{k-1}\right)^n \\ &= k(n-2)!\binom{(k-1)n}{n-2}. \end{aligned}$$
This result is due to I. Gessel (private communication). A bijective proof based on Prüfer codes is due to the Cambridge Combinatorics and Coffee Club, December 1999.
29. (a) If $F \in P_n$ has rank i, then any of the i edges of F can be removed from F to obtain an element that F covers. Hence F covers i elements. To obtain an element that covers F, choose a vertex v of F in n ways, and then choose a connected component T of F not containing v in $(n-i-1)$ ways. Attach the root of T below v. Thus F is covered by $(n-i-1)n$ elements.
    (b) Let $M(n)$ denote the number of maximal chains in P_n. We obtain a maximal chain by choosing a maximal element of P_n in $r(n)$ ways, then

an element that it covers in $n-1$ ways, etc. Hence $M(n) = r(n)(n-1)!$. On the other hand, we can choose a maximal chain by starting at $\hat{0}$, choosing an element u covering $\hat{0}$ in $(n-1)n$ ways, then an element covering u in $(n-2)n$ ways, etc. Hence $M(n) = n^{n-1}(n-1)!$, so $r(n) = n^{n-1}$.

This elegant proof appears in J. Pitman, *J. Combin. Theory Ser. A* **85** (1999), 165–193. The same reasoning can be used to compute the number $p_k(n)$ of planted forests on $[n]$ with k components (i.e., the number of elements of P_n of rank $n-k$), as was done by other methods in the text (Proposition 5.3.2 and Example 5.4.4). Note also that P_n, with a $\hat{1}$ adjoined, is a *triangular poset* in the sense of Exercise 3.201 (except for not having all maximal chains of infinite length). Further results on P_n and related posets are given by D. N. Kozlov,

(c) The poset P_n is *simplicial*, that is, every interval $[\hat{0}, t]$ is isomorphic to a boolean algebra. (In fact, P_n is the face poset of a simplicial complex.) It follows from Example 3.8.3 and the recurrence (3.14) defining the Möbius function that

$$\mu_n := \mu(\hat{0}, \hat{1}) = -p_n(n) + p_{n-1}(n) - \cdots \pm p_1(n),$$

where $p_k(n)$ denotes the number of planted forests on $[n]$ with k components. If $R(x)$ denotes the exponential generating function for rooted trees (defined in Section 5.3), then by the exponential formula (Corollary 5.1.6) we have

$$\sum_{n\geq 1} \mu_n \frac{x^n}{n!} = 1 - e^{-R(-x)}.$$

Now use the second formula of equation (5.128).

30. **First solution**. Linearly order $R \cup S$ by $1 \leq \cdots \leq r \leq 1' \leq \cdots \leq s'$. Given T, define a sequence $T_1, T_2, \ldots, T_{r+s-2}$ as follows: set $T_1 = T$. If $i \leq r+s-2$ and T_i has been defined, then define T_{i+1} to be the tree obtained from T_i by removing its *largest endpoint* v_i (and the edge incident to v_i). For each i we also define a pair (u_i, u'_i) of sequences (or words) $u_i \in R^*$ and $u'_i \in S^*$ as follows. Set $(u_0, u'_0) = (\emptyset, \emptyset)$, where $\emptyset$ denotes the empty word. Let t_i be the unique vertex of T_i adjacent to v_i. If $t_i \in R$ then set $(u_i, u'_i) = (u_{i-1}t_{i-1}, u'_{i-1})$. If $t_i \in S$ then set $(u_i, u'_i) = (u_{i-1}, u'_{i-1}t_i)$. Thus for the tree T we obtain a pair of words $(u, u') = (u_{r+s-2}, u'_{r+s-2})$, where $u_{r+s-2} \in R^*_{s-1}$, $u'_{r+s-2} \in S^*_{r-1}$. As in the first proof of Proposition 5.3.2, the correspondence $T \mapsto (u, u')$ is a bijection between free bipartite trees on (R, S) and the set $R^*_{s-1} \times S^*_{r-1}$. Moreover, a vertex t appears in u and u' one fewer times than its degree, from which (5.100) follows.

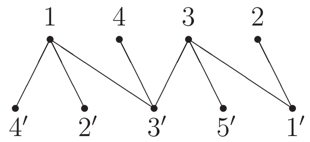

Figure 5.25 A labelled bipartite tree

Example. For the tree T of Figure 5.25, we have $(u, u') = (3113, 3'1'3')$, and T_7 consists of a single edge connecting 1 and $3'$.

Second solution. There are $r^s s^r$ functions $f : R \cup S \to R \cup S$ satisfying $f(R) \subseteq S$ and $f(S) \subseteq R$. Let D_f denote the digraph of such a function f. The "cyclic part" of D_f corresponds to a permutation π of some subset $R_1 \cup S_1$ of $R \cup S$, where $\pi(R_1) = S_1$ and $\pi(S_1) = R_1$. Linearly order $R_1 \cup S_1$ as $a'_1 < a_1 < a'_2 < a_2 < \cdots < a'_j < a_j$, where $a'_1 < a'_2 < \cdots < a'_j$ and $a_1 < a_2 < \cdots < a_j$ as integers. This linear ordering allows π to be written as a word $w = b_1 b'_1 b_2 b'_2 \cdots b_j b'_j$, where $\pi(a'_i) = b_i$, $\pi(a_i) = b'_i$. Regard the word w as a path P in a (bipartite) graph. Circle the endpoints b_1 and b'_j. Attach to each vertex t of P the tree that is attached to t in D_f (with the arrows removed from each edge), yielding a bipartite tree T on (R, S) with a root in R and a root in S. As in the second proof of Proposition 5.3.2, the map $f \mapsto T$ is a bijection between functions $f : R \cup S \to R \cup S$ with $f(R) \subseteq S$ and $f(S) \subseteq R$, and "bi-rooted" bipartite trees on (R, S) with a root in R and a root in S. Moreover, if t is not a root then $\deg_T t = 1 + \#f^{-1}(t)$, while if t is a root then $\deg_T t = \#f^{-1}(t)$. It follows that

$$\sum_{\substack{a \in R \\ b' \in S}} (x_a y_b) \left(x_1^{-1} \cdots x_r^{-1}\right) \left(y_1^{-1} \cdots y_s^{-1}\right) \sum_T \left(\prod_{i \in R} x_i^{\deg i}\right) \left(\prod_{j' \in S} y_j^{\deg j'}\right)$$
$$= (x_1 + \cdots + x_r)^s (y_1 + \cdots + y_s)^r, \tag{5.130}$$

where T ranges over all free bipartite trees on (R, S). Then (5.100) follows immediately from (5.130).

Example. Let T be as in Figure 5.25. Suppose we choose 4 and $1'$ as the roots. The corresponding path P is $43'31'$, so the cyclic part of f written in two-line notation is $\begin{pmatrix} 1' & 3 & 3' & 4 \\ 4 & 3' & 3 & 1' \end{pmatrix}$, and in cycle notation is $(1', 4)(3', 3)$. The digraph D_f is shown in Figure 5.26.

The number $c(K_{rs})$ of spanning trees of K_{rs} was first obtained (by different methods than here) by M. Fiedler and J. Sedláček, *Časopis pro pěstováni matematiky* **83** (1958), 214–225; T. L. Austin, *Canad. J. Math.* **12** (1960),

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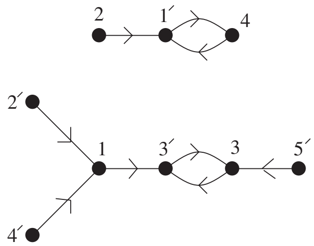

Figure 5.26 The digraph D_f of a function $f : R \cup S \to R \cup S$

535–545 (a special case of Theorem II); and H. I. Scoins, *Proc. Camb. Phil. Soc.* **58** (1962), 12–16.

31. (a) Easy!

(b) Given a function $f : S \to T$, let D_f be the digraph with vertex set $S \cup T$ and edges $s \to f(s)$ for $s \in S$. Now fix $A \subseteq [n]$, and consider the sum

$$F_A = \sum_g \prod_{i \in A} x_{g(i)}, \tag{5.131}$$

where g ranges over all *acyclic* (i.e., D_g has no directed cycles) functions $g : A \to A \cup \{n+1\}$. Then D_g is an oriented tree with root $n+1$, and the exponent of x_j in the product in (5.131) is equal to $(\deg j) - 1$ if $j \neq n+1$, and to $\deg(n+1)$ if $j = n+1$, where $\deg k$ denotes the total number of vertices adjacent to k (ignoring the direction of the edges). Since the root and orientation of D_g can be determined from the underlying free tree on $A \cup \{n+1\}$, it follows from Theorem 5.3.4 that

$$F_A = x_{n+1}\left(x_{n+1} + \sum_{i \in A} x_i\right)^{\#A-1}.$$

Next consider

$$G_A = \sum_h \prod_{i \in A'} x_{h(i)},$$

where h ranges over *all* functions $h : A' \to A' \cup [n+2]$. By (a), we have

$$G_A = \left(x_{n+2} + \sum_{i \in A'} x_i\right)^{n-\#A}.$$

If now $f : [n] \to [n+2]$, then the component of D_f containing $n+1$ will be equal to D_g for a unique $A \subseteq [n]$ and acyclic $g : A \to A \cup \{n+1\}$.

The remainder of D_f is equal to D_h for a unique $h : A' \to A' \cup \{n+2\}$. Thus

$$\begin{aligned}(x_1 + \cdots + x_{n+2})^n &= \sum_{f:[n]\to[n+2]} \prod_{i=1}^{n} x_{f(i)} \\ &= \sum_{A\subseteq[n]} F_A G_A,\end{aligned}$$

and the proof follows.

This result is equivalent to one of A. Hurwitz, *Acta Math.* **26** (1902), 199–203. See also [2.3, Exer. 20, p. 163] and [171, Exer. 2.3.44–30]. The proof given here is a minor variation of one of J. Françon, *Discrete Math.* **8** (1974), 331–343 (repeated in [2.3, pp. 129–130]). Françon uses an elegant "coding" of functions $[n] \to [n]$ due to D. Foata and A. Fuchs, *J. Combinatorial Theory* **8** (1970), 361–375, and obtains many related results in a systematic way. For a generalization, see A. J. Stam, *J. Math. Anal. Appl.* **122** (1987), 439–443.

(c) Put $x_{n+1} = x$, $x_{n+2} = y + nz$, $x_1 = x_2 = \cdots = x_n = -z$ and collect the A such that $\#A = k$ in (b). This famous identity, one of several equivalent ones called "Abel's identity" (see the fourth entry of Exercise 37(b)), is to due N. Abel, *J. reine angew. Math.* (= *Crelle's J.*) **1** (1826), 159–160, or *Oeuvres Complètes*, vol. 1, p. 102. For some other proofs, see [2.3, pp. 128–129] and [171, Exer. 1.2.6–51]. For additional references, see H. W. Gould, *Amer. Math. Monthly* **69** (1962), 572. For a combinatorial treatment of many identities related to Abel's identity, see V. Strehl, *Discrete Math.* **99** (1992), 321–340.

(d) This is equivalent to the case $x = 1$, $y = n$, $z = -1$ of (c). (It can also be proved directly by considering functions $[n] \to [n+1]$.)

32. (a) Fix $j \in \mathbb{P}$. Given a rooted tree τ, let $w(\tau) = \prod t_{jk}^{a_k}$, where τ has a_k vertices at distance k from the root. By a simple refinement of (5.41), we have

$$\sum_{n\geq1} \left[\sum_{\tau} w(\tau)\right] \frac{x^n}{n!} = t_{j0}xe^{t_{j1}xe^{t_{j2}xe^{\cdot^{\cdot^{\cdot}}}}} = E_j, \quad \text{say,}$$

where τ ranges over all rooted trees on $[n]$.

Now let C be a collection of j such trees $\tau_1, \ldots, \tau_j$ arranged in a j-cycle, and define $w(C) = \prod w(\tau_i)$. Then

$$\sum_{n\geq1} \left[\sum_{C} w(C)\right] \frac{x^n}{n!} = \sum_{j\geq1} \frac{1}{j} E_j^j,$$

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where C ranges over all "cycles of rooted trees"on the vertex set $[n]$, since by Proposition 5.1.3, E_j^j enumerates j-tuples $(\tau_1, \ldots, \tau_j)$ of rooted trees, and each j-cycle corresponds to j distinct j-tuples.

Finally by Corollary 5.1.6 the exponential generating function for disjoint unions of cycles of rooted trees on $[n]$ (or digraphs of functions $f : [n] \to [n]$) is given by

$$\exp \sum_{j \geq 1} \frac{1}{j} E_j^j,$$

as desired.

(b) $\tilde{Z}_n(t_{jk} = 1)$ is just the number n^n of functions $f : [n] \to [n]$, so the first equality follows. The second equality is a consequence of (5.41) and Proposition 5.3.2.

(c) A necessary and sufficient condition that $f^a = f^{a+b}$ is that (i) every cycle of D_f has length dividing b, and (ii) every vertex of D_f is at distance at most a from a cycle. Hence (c) follows by substituting in (a)

$$t_{jk} = \begin{cases} 1, & \text{if } j|b \text{ and } k \leq a \\ 0, & \text{otherwise} \end{cases}$$

(d) Since $f = f^{1+b}$ for some $b \in \mathbb{P}$ if and only if every vertex of D_f is at distance at most one from a cycle, we obtain from (a) by setting $t_{j0} = t_{j1} = 1$, $t_{jk} = 0$ if $k > 1$ (or from (c) by letting $b = m!$ and $m \to \infty$) that

$$\begin{aligned}
\sum_{n \geq 0} h(n) \frac{x^n}{n!} &= \exp \sum_{j \geq 1} \frac{1}{j} (xe^x)^j \\
&= \exp \log(1 - xe^x)^{-1} \\
&= (1 - xe^x)^{-1} \\
&= \sum_{m \geq 0} x^m e^{mx} \\
&= \sum_{m \geq 0} \sum_{r \geq 0} m^r \frac{x^{m+r}}{r!} \\
&= \sum_{n \geq 0} \left(\sum_{k=1}^{n} k^{n-k} (n)_k \right) \frac{x^n}{n!},
\end{aligned}$$

so (5.104) follows.

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Putting $b = 1$ in (5.103) yields

$$\sum_{n\geq 0} g(n)\frac{x^n}{n!} = \exp xe^x$$
$$= \sum_{m\geq 0} \frac{x^m e^{mx}}{m!}, \tag{5.132}$$

and (5.105) follows in a similar manner to (5.104).

(e) Note that f satisfies $f^a = f^{a+1}$ for some $a \in \mathbb{P}$ if and only if every cycle of the digraph D_f has length one. Hence we want the number of planted forests on $[n]$, which by Proposition 5.3.2 is $(n+1)^{n-1}$.

(f) While a proof using generating functions is certainly possible, there is a very simple direct argument. Namely, for each $i \in [n]$, we have $n-1$ choices for $f(i)$. Hence there are $(n-1)^n$ such functions. Note that the proportion $P(n)$ of functions $f : [n] \to [n]$ without fixed points is $(n-1)^n/n^n$, so $\lim_{n\to\infty} P(n) = 1/e$. From equation (2.12) this is also the limiting value of the proportion of *permutations* $f : [n] \to [n]$ without fixed points.

Equation (5.101) can be deduced from the general "composition theorem" of B. Harris and L. Schoenfeld, in *Graph Theory and Its Applications* (B. Harris, ed.), Academic Press, New York/London, 1970, pp. 215–252. In that paper equation (5.102) is essentially derived, though it is not explicitly written down. Special cases of (5.102) had appeared in earlier papers; in particular, equations (5.105) and (5.132) are obtained by B. Harris and L. Schoenfeld, *J. Combinatorial Theory* **3** (1967), 122–135, along with considerable additional information concerning the number $g(n)$ of idempotents in the symmetric semigroup Λ_n. The first explicit statement of (5.102) seems to be [3.32, §3.3.15, Ex. 3.3.31], and a refinement appears in [73, §3.2].For asymptotic properties of Λ_n, see B. Harris, *J. Combinatorial Theory (A)* **15** (1973), 66–74, and B. Harris, *Studies in Pure Mathematics*, Birkhäuser, Basel/Boston/Stuttgart, 1983, pp. 285–290.

33. The functions c and $2-\zeta$ are *not* multiplicative, so Theorem 5.1.11 does not apply. Since

$$(2-\zeta)^{-1} = \sum_{k\geq 0} (\zeta - 1)^k,$$

the correct generating function is the unappealing

$$E_c(x) = \sum_{k\geq 0} f^{\langle k\rangle}(x),$$

where $f(x) = f^{\langle 1\rangle}(x) = e^x - x - 1$ (set $f^{\langle 0\rangle}(x) = x$).

34. (a) Straightforward generalization of Theorem 5.1.11.
(b) Let $\zeta : \mathbb{P} \to K$ be given by $\zeta(n) = 1$ for all n. Thus $\zeta^2(n) = q_n$ and $\zeta^{-1}(n) = \mu_n$. Since $\varphi(\zeta) = e_k(x)$, the result follows from (a).
(c) Define $\zeta_t : \mathbb{P} \to K$ by $\zeta_t(n) = t^n$. Then $\chi_n(t) = \mu\zeta_t(n)$. Now

$$\begin{aligned} \varphi(\zeta_t) &= \sum_{n\geq 0} t^n \frac{x^{kn+1}}{(kn+1)!} \\ &= t^{-1/k} e_k(t^{1/k} x), \end{aligned}$$

while $\varphi(\mu) = e_k^{\langle -1\rangle}(x)$. Thus (5.106) follows from (a). When $k = 2$, (5.106) becomes

$$t \sum_{n\geq 0} \chi_n(t^2) \frac{x^{2n+1}}{(2n+1)!} = \sinh(t \sinh^{-1} x).$$

To get (5.107), use Exericse 1.173(c).
For further results on Ψ_n and related posets, see A. R. Calderbank, P. Hanlon, and R. W. Robinson, *Proc. London Math. Soc. (3)* **53** (1986), 288–320; S. Sundaram, *Contemporary Math.* **178** (1994), 277–309; and the references given in this latter paper.

35. (a) Let T be a plane tree with $n + 1$ vertices for which s_i internal vertices have i successors. Label the vertices of T in preorder with the numbers $0, 1, \ldots, n$. Let $\pi(T)$ be the partition of $[n]$ whose blocks are the sets of vertices with a common parent. This sets up a bijection with noncrossing partitions of $[n]$ of type $s_1, \ldots, s_n$, and the proof follows from Theorem 5.3.10. This result was first proved (by other means) by G. Kreweras, *Discrete Math.* **1** (1972), 333–350 (Theorem 4). The bijective proof just sketched was first found by P. H. Edelman (unpublished). Later independently N. Dershowitz and S. Zaks, *Discrete Math.* **62** (1986), 215–218, gave the same bijection between plane trees amd noncrossing partitions, though they don't explicitly mention enumerating noncrossing partitions by type.
(b) Assume $\operatorname{char} K = 0$. By (a) we have for $n > 0$ that

$$\begin{aligned} h(n) &= \sum_{s_1+2s_2+\cdots=n} f(1)^{s_1} f(2)^{s_2} \cdots \frac{(n)_{k-1}}{s_1! s_2! \cdots} \\ &= \sum_{k\geq 1} (n)_{k-1} [x^n] \frac{(F(x)-1)^k}{k!} \end{aligned}$$

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$$= [x^n] \int_0^{F(x)-1} (1+t)^n dt$$
$$= [x^n] \frac{F(x)^{n+1} - 1}{n+1}$$
$$= [x^n] \frac{F(x)^{n+1}}{n+1}.$$

Hence by Lagrange inversion (Theorem 5.4.2, with $k = 1$ and n replaced by $n + 1$) we get

$$h(n) = [x^{n+1}] \left(\frac{x}{F(x)} \right)^{\langle -1 \rangle},$$

and the proof follows when $\operatorname{char} K = 0$. The case $\operatorname{char} K = p$ is an easy consequence of the characteristic zero case.

This result is due to R. Speicher, *Math. Ann.* **298** (1994), 611–628 (p. 616). Speicher's proof avoids the use of (a), so he in fact deduces (a) from (5.108) (see his Corollary 1).

(c) This can be proved by an argument similar to (a), though the details are more complicated. The result is due to A. Nica and R. Speicher, *J. Algebraic Combinatorics*, **6** (1997), 141–160 (Theorem 1.6), and is related to the "free probabability theory" developed by D.-V. Voiculescu. See also R. Speicher, *Mem. Amer. Math. Soc.*, vol. 132, no. 627, 1998, 88 pages, and R. Speicher, *Sém. Lotharingien de Combinatoire* (electronic) **39** (1997), B39c.

NOTE. If one defines $\zeta(n) = 1$ for all n, then the function $h = f\zeta$ is as in (b). Since $\Gamma_\zeta = 1/(1+x)$, there results

$$\left(\sum_{n \geq 1} h(n) x^n \right)^{\langle -1 \rangle} = \frac{1}{1+x} \left(\sum_{n \geq 1} f(n) x^n \right)^{\langle -1 \rangle}.$$

It follows from the case $C(x) = 1/(1+x)$ of Exercise 51 that this formula is equivalent to (5.108).

36. (a) Let $u = \left(\frac{1}{2}(1 + 2x - e^x) \right)^{\langle -1 \rangle}$ and $v = (\log(1+2x) - x)^{\langle -1 \rangle}$. Thus $1+2u-2x = e^u$. If we replace u by $x+w$, then we obtain $1+2w = e^{x+w}$, whence $w^{\langle -1 \rangle} = \log(1+2x) - x$. Therefore $w = v$, so $y = u - v = x$.

(b) It follows from equation (5.27), equation (5.99), and part (a) of this exercise that $E_t(2x) - E_g(x) = x$, from which the proof is immediate.

(c) Let us call a subset of the boolean algebra B_n of the type enumerated by $g(n)$ a *power tree*. Represent a total partition π of $[n]$ (where

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$n > 1$) as a tree T, as in Figure 5.3. Remove any subset of the endpoints of T, in 2^n ways. The labels of the remaining vertices form a power tree. This correspondence associates each total partition of $[n]$ with 2^n power trees, such that each power tree appears exactly once, yielding (b). This elegant argument is due to C. H. Yan.

37. (a) First note that (ii) and (iii) are obviously equivalent, since $f(u) = \log \sum_{n\geq 0} p_n(1)\frac{u^n}{n!}$. Given (ii), then (i) follows by expanding in powers of u both sides of the identity

$$(\exp xf(u))(\exp yf(u)) = \exp(x+y)f(u).$$

Conversely, given (i) write

$$L(x,u) = \log \sum_{n\geq 0} p_n(x)\frac{u^n}{n!}.$$

It follows from (i) that $L(x,u) + L(y,u) = L(x+y,u)$, from which it is easy to deduce that $L(x,u) = xf(u)$ for some $f(u) = a_1u + a_2u^2 + \cdots$ (with $a_1 \neq 0$.)

For the equivalence of (i) and (iv), see G.-C. Rota and R. C. Mullin, in *Graph Theory and Its Applications* (B. Harris, ed.), Academic Press, New York, 1970, pp. 167–213 (Thm. 1) or G.-C. Rota, D. Kahaner, and A. M. Odlyzko, *J. Math. Anal. Appl.* **42** (1973), 684–760 (Thm. 1). These two papers develop a beautiful theory of "finite operator calculus" with many applications to analysis and combinatorics. For additional information and references, see S. Roman, *The Umbral Calculus*, Academic Press, Orlando, 1984. For asymptotic properties of polynomials of binomial type, see E. R. Canfield, *J. Combinatorial Theory (A)* **23** (1977), 275–290.

(b)

$$\sum_n x^n\frac{u^n}{n!} = \exp xu$$

$$\sum_n (x)_n\frac{u^n}{n!} = (1+u)^x \quad = \quad \exp[x\log(1+u)]$$

$$\sum_n x^{(n)}\frac{u^n}{n!} = (1-u)^{-x} \quad = \quad \exp[x\log(1-u)^{-1}]$$

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$$\sum_n x(x-an)^{n-1}\frac{u^n}{n!} = \exp x\sum_{n\geq 1}(-an)^{n-1}\frac{u^n}{n!}$$

$$\sum_n\sum_k S(n,k)x^k\frac{u^n}{n!} = \exp x(e^u-1)$$

$$\sum_n\sum_k \frac{n!}{k!}\binom{n+(a-1)k-1}{n-k}x^k\frac{u^n}{n!} = \exp\frac{xu}{(1-u)^a}$$

$$\sum_n\sum_k \binom{n}{k}k^{n-k}x^k\frac{u^n}{n!} = \exp xue^u.$$

A further interesting example, for which an explicit formula is not available, consists of the polynomials $n!Q_n(x)$ of Exercise 4.82. For a vast generalization see J. Schneider, *Electron. J. Combin.* **21** (2014), Paper 1.43. For two additional examples, see Exercise 38.

(c) Rota and Mullin, *loc. cit.*, Thm. 2, and Rota, Kahaner, and Odlyzko, *loc. cit.*, Thm. 3.2.

(d) Rota and Mullin, *loc. cit.*, Cor. 2, and Rota, Kahaner, and Odlyzko, *loc. cit.*, Cor. 3.3.

(e) Let $g(u) = \sum_{n\geq 0} p_n(1)\frac{u^n}{n!}$, so by (a)(iii) we have $\sum_{n\geq 0} p_n(x)\frac{u^n}{n!} = g(u)^x$. By Exercise 58 there is a power series $f(u)$ satisfying

$$\begin{aligned} f(u)^x &= \sum_{n\geq 0}\frac{x}{x+\alpha n}[u^n]g(u)^{x+\alpha n} \\ &= \sum_{n\geq 0}\frac{x}{x+\alpha n}\frac{p_n(x+\alpha n)}{n!}, \end{aligned}$$

and the proof follows from (a)(iii). This result appears as part of Proposition 7.4 (p. 711) of Rota, Kahaner, and Odlyzko, *ibid*. The version of the proof given here was suggested by E. Rains.

38. (a) Follows from Example 3.15.8 and condition (iii) of Exercise 37(a).
    (b) Instead of Example 3.15.8 use equation (5.77).

39. Let $g(n)$ (respectively, $h(n)$) be the number of series-parallel posets on $[n]$ that cannot be written as a nontrivial disjoint union (respectively, ordinal sum). Let $G(x) = \sum_{n\geq 1} g(n)\frac{x^n}{n!}$ and $H(x) = \sum_{n\geq 1} h(n)\frac{x^n}{n!}$. It is easy to see that every series-parallel poset with more than one element is either a disjoint union or ordinal sum, but not both. Hence

$$F(x) = G(x) + H(x) - x. \tag{5.133}$$

Every series-parallel poset P is a unique disjoint union $P_1 + \cdots + P_k$, where each P_i is not a nontrivial disjoint union (i.e., is connected). Hence by Corollary 5.1.6,

$$1 + F(x) = e^{G(x)}. \tag{5.134}$$

Similarly P is a unique ordinal sum $P_1 \oplus \cdots \oplus P_k$, where each P_i is not a nontrivial ordinal sum. If there are exactly k summands, then by Proposition 5.1.3 the exponential generating function is $H(x)^k$. Hence

$$F(x) = \sum_{k\geq 1} H(x)^k = H(x)/(1 - H(x)). \tag{5.135}$$

It is a simple matter to eliminate $G(x)$ and $H(x)$ from (5.133), (5.134), and (5.135), thereby obtaining (5.112).

This result first appeared in R. Stanley, *Proc. Amer. Math. Soc.* **45** (1974), 295–299.

40. (a) The "unlabelled" version of this problem is due to P. A. MacMahon, *The Electrician* **28** (1892), 601–602, and is further developed by J. Riordan and C. E. Shannon, *J. Math. and Physics* **21** (1942), 83–93. The labelled version given here turns out to be equivalent to the fourth problem of Schröder [291] discussed in the Notes. The numbers $s(n)$ satisfy $s(n) = 2t(n)$ for $n \geq 2$, where $t(n)$ is the number of total partitions of an n-set, as defined in Example 5.2.5. Note also that if $f(n)$ is as in Exercise 26, then $f(n) = 2^n s(n)$, $n \geq 1$. (See Exercise 36 for related results.)

    The first published appearance of the formula (5.113) appears in L. Carlitz and J. Riordan, *Duke Math. J.* **23** (1955), 435–445 (eqn. (2.13)). As discussed in this reference, earlier (essentially equivalent) results were obtained by R. M. Foster (unpublished) and W. Knödel, *Monatshefte Math.* **55** (1951), 20–27. Additional aspects appear in J. Riordan, *Acta math.* **137** (1976), 1–16. See also [2.17, Sect. 6.10].

    (b) For this result and a number of related ones, see P. J. Cameron, *Electron. J. Combin.* **2** (1995), paper R4 (1995).

    (c) See [3.13, Thm. 4 and Cor. 1 on p. 351] The table of values given in Exercise 5, p. 353, of this reference is incorrect.

41. (a) Let F be a forest on the vertex set $[n]$ such that every component of F is an alternating tree rooted at some vertex i all of whose neighbors are less than i. We obtain an alternating tree T on $\{0, 1, \ldots, n\}$ by adding a vertex 0 and connecting it to the roots of the components of F. Hence if $g(n)$ denotes the number of alternating trees on the vertex set $[n]$ rooted at some vertex i all of whose neighbors are less than i, then the

exponential formula (Corollary 5.1.6) yields

$$F(x) = \exp \sum_{n \geq 1} g(n) \frac{x^n}{n!}. \tag{5.136}$$

It is also easy to see that $g(n) = nf(n-1)/2$ for $n > 1$ (consider the involution on alternating trees with vertex set $[n]$ that sends vertex i to $n+1-i$), from which the stated functional equation is immediate.

Alternating trees first arose in the theory of general hypergeometric systems, as developed by I. M. Gelfand and his collaborators. In the paper I. M. Gelfand, M. I. Graev, and A. Postnikov, in *The Arnold-Gelfand Mathematical Seminars*, Birkhäuser, Boston, 1997, pp. 205–221 (§6), it is shown that $f(n)$ is the number of "admissible bases" of the space of solutions to a certain system of linear partial differential equations whose solutions are called *hypergeometric functions on the group of unipotent matrices*. The basic combinatorial properties of alternating trees were subsequently determined by A. Postnikov, *J. Combinatorial Theory (A)* **79** (1997), 360–366. See also A. Postnikov, Ph.D. thesis, M.I.T., 1997 (Section 1.4). In particular, Postnikov established parts (a), (b), and (g) of the present exercise. Further discussion of alternating trees appears in R. Stanley, *Proc. Nat. Acad. Sci.*, **93** (1996), 2620–2625, and A. Postnikov and R. Stanley, *J. Combin. Theory Ser. A* **91** (2000), 544-597. See also Exercise 6.19(p, q).

(b) Let $H(x) = x(F(x)+1)$. Then $H = x(1+e^{H/2})$, so $H(x) = (x/(1+e^{x/2}))^{\langle -1 \rangle}$. The proof follows from an application of Lagrange inversion. (See A. Postnikov, *J. Combinatorial Theory (A)* **79** (1997), 260–366, and Ph.D. thesis, M.I.T., 1997 (Theorem 1.4.1), for the details.) It is an open problem to find a bijective proof.

(c) This follows from equation (5.136) by reasoning as in Example 5.2.2.

(d) Let $T(x) = \log F(x)^q = q \log F(x)$. It follows from (a) that

$$T(x) = \frac{qx}{2}\left(1 + e^{T/q}\right).$$

Now apply equation (5.64) to the case $F(x) = 2x/q(1+e^{x/q})$ and $H(x) = e^x$. (Here we are using $F(x)$ and $H(x)$ in the generic sense of (5.64), and not with the specific meaning of this exercise.) This argument is due to A. Postnikov.

(e) Let E be the operator on polynomials $P(q)$ defined by $EP(q) = P(q+1)$. Then (d) can be restated as

$$P_n(q) = \frac{q}{2^n}(E+1)^n q^{n-1}.$$

The proof now follows by iterating the case $\alpha = 1$ of the following lemma.

Lemma. *Let $P(q) \in \mathbb{C}[q]$ such that every zero of $P(q)$ has real part m. Let $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Then every zero of the polynomial $P(q+1)+\alpha P(q)$ has real part $m - \frac{1}{2}$.*

For the history of this lemma and an elementary proof, see A. Postnikov and R. Stanley, *ibid.* (§9.3).

(f) Let $R_n(q) = Q_n\left(q - \frac{n}{2}\right)$. Then $R_n(q)$ has real coefficients, is monic of degree $n-1$, and by (e) has only purely imaginary zeros (allowing 0 to be purely imaginary). Hence $R_n(q)$ has the form $q^j \prod_k \left(q^2 + a_k\right)$, $a_k \in \mathbb{R}$. Thus $R_n(-q) = (-1)^{n-1} R_n(q)$, which is equivalent to $Q_n(q) = (-1)^{n-1} Q_n(-q-n)$.

(g) See A. Postnikov, *J. Combinatorial Theory (A)* **79** (1997), 360–366 (§4.1), and Ph.D. thesis, M.I.T., 1997 (Section 1.4.2).

(h,i) The question of counting the number of regions of $\mathcal{L}_n$ was raised by N. Linial (private communication, March 27, 1995), so $\mathcal{L}_n$ is now known as the *Linial arrangement*. It was conjectured by R. Stanley that $\chi(L_n, q) = (-1)^n P_n(-q)$. This conjecture was proved by A. Postnikov, Ph.D. thesis, M.I.T., 1997 (a special case of Theorem 1.5.7), and later (using Theorem 3.11.10) by C. A. Athanasiadis, *Advances in Math.* **122** (1996), 193–233 (Theorem 4.2). See also R. Stanley, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625 (Corollary 4.2) and A. Postnikov and R. Stanley, *ibid.* (§9.2).

(j) An alternating graph G cannot contain an odd cycle and hence is bipartite. We can partition the vertices into two sets A and B (possibly empty, and unique except for the isolated vertices of G) such that (α) every edge goes from A to B, and (β) if $i \in A, j \in B$, and there is an edge between i and j, then $i < j$. Call a pair (i,j) *admissible* (with respect to A and B) if $i \in A, j \in B$, and $i < j$. Let $h_k(n)$ be the number of ways to choose two disjoint sets A and B whose union is $\{1, 2, \ldots, n\}$, and then choose a k-element set of admissible pairs (i,j). Suppose that the elements of B are $a_1 < a_2 < \cdots < a_k$. Then the number of admissible pairs is $\nu(a_1, \ldots, a_k) = (a_1 - 1) + (a_2 - 2) + \cdots + (a_k - k)$. Hence the generating function for the subsets of such pairs according to the number of edges is $(q+1)^{\nu(a_1,\ldots,a_k)}$, so

$$\begin{aligned}\sum_k h_k(n) q^k &= \sum_{1 \le a_1 < \cdots < a_k \le n} (q+1)^{\nu(a_1,\ldots,a_k)} \\ &= \sum_{0 \le b_1 \le \cdots \le b_k \le n-k} (q+1)^{b_1 + \cdots + b_k}.\end{aligned}$$

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By Proposition 1.3.19 we have that for *fixed* k,

$$\sum_{0\le b_1\le\cdots\le b_k\le n-k} q^{b_1+\cdots+b_k} = \binom{n}{k}.$$

It follows that

$$\sum_k h_k(n)q^k = \sum_{k=0}^{n} \binom{n}{k}_{q+1}.$$

Now an alternating graph with r isolated vertices and k edges gets counted exactly 2^r times by $g_k(n)$ (since each isolated vertex can belong to either A or B, but there is no choice for the other vertices). Hence if $u_k(n)$ denotes the number of alternating graphs on the vertices $1, 2\ldots, n$ with no isolated vertices and with k edges, then

$$\sum_{r=0}^{n} \binom{n}{r} 2^r \sum_k u_k(n-r)q^k = \sum_k h_k(n)q^k$$
$$\sum_{r=0}^{n} \binom{n}{r} \sum_k u_k(n-r)q^k = \sum_k g_k(n)q^k.$$

From this it is routine to deduce the stated result. The case $q = 1$ appeared in R. Stanley, Problem 10572, *Amer. Math. Monthly* **104** (1997), 168; solution by S. C. Locke, **106** (1999), 168. Locke's solution is different from the one given here.

(k) Let $w_1w_2\cdots w_n \in \mathfrak{S}_n$. Define a tree T_w with edges labelled $1, 2, \ldots, n$ as follows: If $i < j$ then the edges labelled w_i and w_j have a common vertex if and only if the sequence $w_iw_{i+1}\cdots w_j$ is either increasing or decreasing. Then T_w is an edge labelled alternating tree, and every such tree occurs exactly twice in this way (when $n > 1$), namely, from $w_1w_2\cdots w_n$ and its reverse $w_n\cdots w_2w_1$. Hence when $n > 1$ there are $n!/2$ edge labelled alternating trees with $n + 1$ vertices. This exercise is due to A. Postnikov (private communication, December, 1997).

42. (a) From $y = xe^y$ we have $y' = e^y + xy'e^y$, so $xy' = xe^y/(1 - xe^y) = y/(1-y) = -1+(1-y)^{-1}$. Thus $(1-y)^{-1} = 1+xy' = 1+\sum_{n\ge 1} n^n \frac{x^n}{n!}$.

(b) Since $(1 - R(x))^{-1} = 1 + R(x) + R(x)^2 + \cdots$, by Proposition 5.1.3 we seek a bijection $\varphi : \mathcal{R}_n^1 \cup \mathcal{R}_n^2 \cup \cdots \to \mathcal{T}_n^*$, where for $n \ge 1$ $\mathcal{R}_n^j$ is the set of j–tuples $(\tau_1, \ldots, \tau_j)$ of (nonempty) rooted trees whose total vertex set is $[n]$, and where $\mathcal{T}_n^*$ is the number of double rooted trees on $[n]$. Given $(\tau_1, \ldots, \tau_j) \in \mathcal{R}_n^j$, let v_i be the root of τ_i. Let P be a path with successive vertices $v_1, v_2, \ldots, v_j$. Label v_1 by s and v_j by e, and

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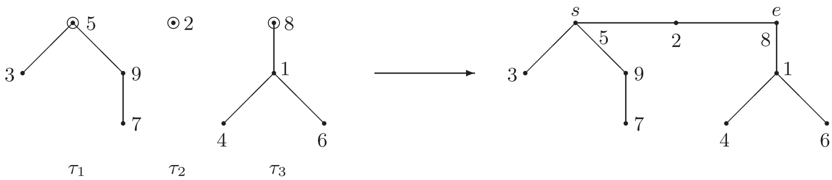

Figure 5.27 A bijection from j-tuples of rooted trees to double rooted trees

attach to each v_i the remainder of the tree τ_i. This yields the desired double rooted tree on $[n]$. This bijection is illustrated in Figure 5.27.

43. Let T be a leaf-labelled tree as in the problem. Iterate the following procedure until all vertices are labelled except the root. At the start, the leaves are labelled $1, \dots, k$. Assume now that labels $1, 2, \dots, m$ have been used. Label by $m+1$ the vertex v satisfying: (a) v is unlabelled and all successors of v are labelled, and (b) among all unlabelled vertices with all successors labelled, the vertex having the successor with the *least* label is v. Now let the blocks of the partition π consist of the labels of the successors of each nonleaf vertex v. It can be checked that this procedure yields the desired bijection.

    Similar bijections appear in Erdős-Székely [82] and W. Y. C. Chen, *Proc. Natl. Acad. Sci. USA* **87** (1990), 9635–9639. See also W. Y. C. Chen, *European J. Combinatorics* **15** (1994), 337–343. (A further bijection was discovered independently by M. Haiman.) The Erdős–Székely bijection has the minor defect of not preserving the leaf labels when the nonroot vertices are labelled. Erdős and Székely go on to deduce from their bijection many standard results on the enumeration of trees, including our Theorem 5.3.4 (or Corollary 5.3.5) and Theorem 5.3.10.

44. Let $r_j = \#\{i \,:\, a_i = j\}$. Given the permutation $w = w_1 \cdots w_n$, define a word $\varphi(w) = x_{m_1} \cdots x_{m_n} x_0$ as follows: If w_i is the first occurrence of a letter k, then $m_i = a_k$. Otherwise $m_i = 0$. One checks that φ is a map between the set S of permutations we wish to count and the set T of elements of the monoid $\mathcal{B}^*$ defined by equation (5.50) containing r_j copies of x_j and $1 + \sum(a_i - 1)$ copies of x_0, and that every element of T is the image of $\prod r_j!$ elements of S. The proof follows from Theorem 5.3.10. Is there a simpler proof?

    An easy bijection shows that the result of this exercise is equivalent to the statement that the number of nonnesting partitions of $[n]$ (as defined in Exercice 6.19(uu)) with r_j blocks of size j is given by $n!/((n-k+1)!\,r_1!\,r_2! \cdots)$. Note the curious fact that by Exercise 35(a) this number is also the number of noncrossing partitions of $[n]$ with r_j blocks of size j. It is not difficult to give a bijective proof of this fact.

45. Let $y = \sum_{n\geq 1} t_n x^n$ and $z = \sum_{n\geq 0} f_n x^n$. It is easy to see that $(k+1)xy^k$ is the generating function for recursively labelled trees for which the root has exactly k subtrees. Hence

$$y = x + 2xy + 3xy^2 + \cdots = \frac{x}{(1-y)^2}.$$

It is then routine to use the Lagrange inversion formula to obtain the stated formula for t_n. Similarly $z = 1/(1-y)$, so $y = x/(1-y)^2 = xz^2$ and $z = 1/(1-xz^2)$. Again it is routine to use Lagrange inversion to find f_n, or to observe from $z = 1/(1-xz^2)$ that $z = 1+xz^3$, the generating function for ternary trees. With a little more work these arguments can be "bijectivized," yielding a bijection from recursively labelled forests to ternary trees (and similarly from recursively labelled trees to pairs of ternary trees). Recursively labelled trees and forests were first defined by A. Björner and M. L. Wachs, *J. Combinatorial Theory (A)* **52** (1989), 165–187.

46. Define a ternary tree $\gamma(T)$ whose vertices are the edges of T as follows. Let j be the smallest vertex of T (in this case, $j = 1$), and let k be the largest vertex for which jk is an edge e. Define three subtrees of T as follows. T_1 is the connected component containing vertex 1 of the graph $T - e$. T_2 is the connected component containing vertex k of the graph obtained from T by removing edge e and vertices $k+1, k+2, \ldots, n$. T_3 is the graph obtained from T by removing vertices $1, 2, \ldots, k-1$. Define e to be the root of $\gamma(T)$, and recursively define $\gamma(T_i)$ to be the ith subtree of the root. It is easy to see that γ is a bijection from noncrossing trees on $[n]$ to ternary trees with $n-1$ vertices. See Figure 5.28 for an example. In this figure the vertices of $\gamma(T)$ are shown as open circles and the edges as dotted lines. Three edge directions with empty subtrees have been drawn to make the ternary structure clear. Essentially the bijection just described was suggested independently by R. Simion and A. Postnikov. Noncrossing trees were first enumerated by S. Dulucq and J.-G. Penaud, *Discrete Math.* **117** (1993), 89–105 (Lemme 3.11). For further information and references, see M. Noy, *Discrete Math.* **180** (1998), 301–313. Dulucq and Penaud, *ibid.* (Proposition 2.1), also give a bijection between plane ternary trees with $n-1$ vertices and ways of drawing n chords with no common endpoints between $2n$ points on a circle such that the intersection graph G of the set of chords is a tree. (The chords are the vertices of G, with an edge connecting two vertices u and v if and only if u and v intersect (as chords).)

47. (a) Let $w \in \mathfrak{S}_n$, and let (i,j) be a transposition in $\mathfrak{S}_n$. It is easy to see that if i and j are in different cycles of w then these two cycles are merged into a single cycle in the product $(i,j)w$. From this it follows that a

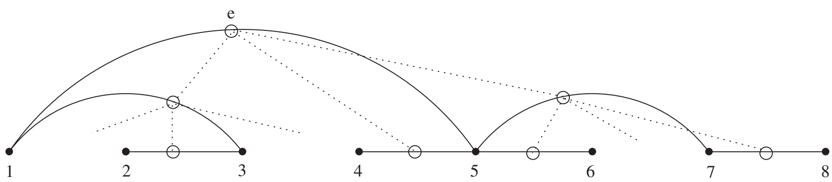

Figure 5.28 A ternary tree constructed from a noncrossing tree

product $\tau_1 \cdots \tau_{n-1}$ of $n-1$ transpositions is an n-cycle if and only if the graph on the vertex set $[n]$ whose edges are the pairs transposed by the τ_k's is a tree. There are n^{n-2} trees on $[n]$ (Proposition 5.3.2) and $(n-1)!$ ways to linearly order their edges. Hence there are $(n-1)!n^{n-2}$ ways to write *some* n-cycle as a product of $n-1$ transpositions. By "symmetry" all $(n-1)!$ n-cycles have the same number of representations as a product of $n-1$ transpositions. Hence any particular n-cycle, such as $(1, 2, \ldots, n)$, has n^{n-2} such representations. This result is usually attributed to J. Dénes, *Publ. Math. Institute Hungar. Acad. Sci.* (= *Magyar Tud. Akad. Mat. Kutato Int. Kozl.*) **4** (1959), 63–71, and has spawned a large literature. However, a much more general theorem was announced (with a sketch of the proof) by A. Hurwitz, *Math. Ann.* **39** (1891), 1–66 (see part (c) of this exercise). Bijective proofs of this exercise were given by P. Moszkowski, *European J. Combin.* **10** (1989), 13–16; I. P. Goulden and S. Pepper, *Discrete Math.* **113** (1993), 263–268; and C. M. Springer, in *Eighth International Conference on Formal Power Series and Algebraic Combinatorics, University of Minnesota, June 25–29, 1996*, pp. 427–438.

(b) The formula $g(n) = \frac{1}{2n-1}\binom{3(n-1)}{n-1}$ was first proved by J. A. Eidswick, *Discrete Math.* **73** (1989), 239–243, and J. Q. Longyear, *Discrete Math.* **78** (1989), 115–118. A number of proofs were given subsequently, including I. P. Goulden and D. M. Jackson, *J. Algebra* **16** (1994), 364–378, and C. M. Springer, *ibid.* (Both these papers prove much more general results.) We sketch a bijective proof based on a suggestion of A. Postnikov. Given a noncrossing tree on $[n]$, label the edges with the labels $1, 2, \ldots, n-1$ such that the following condition holds. For every vertex i, if the vertices adjacent to i are $j_1 < \cdots < j_r < k_1 < \cdots < k_s$ with $j_r < i < k_1$, and if $\lambda(m)$ denotes the label of the edge im, then

$$\lambda(j_r) < \lambda(j_{r-1}) < \cdots < \lambda(j_1) < \lambda(k_s) < \lambda(k_{s-1}) < \cdots < \lambda(k_1).$$

Let τ_i be the transposition (a, b), where ab is the edge of T labelled i. Then it is not hard to show that $\tau_1\tau_2 \cdots \tau_{n-1} = (1, 2, \ldots, n-1)$ and

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that each equivalence class is obtained exactly once in this way, thus giving the desired bijection.

(c) This result was stated with a sketch of a proof by A. Hurwitz in 1891 (reference in (a)). The first complete proof was given by I. P. Goulden and D. M. Jackson, *Proc. Amer. Math. Soc.* **125** (1997), 51–60, based on the theory of symmetric functions. A reconstruction of the proof of Hurwitz, together with much interesting further information, was given by V. Strehl, *Sém. Lothar. Combin.* **37** (1996), Art. S37c. A direct combinatorial proof was given by M. Bousquet-Mélou and G. Schaeffer, *Adv. in Appl. Math.* **24** (2000), 337–368. Some further aspects of "transitive factorizations" are discussed in I. P. Goulden and D. M. Jackson, *European J. Combin.* **21** (2000), 1001–1016, and in *The Mathematical Legacy of Richard Stanley*, Amer. Math. Soc., Providence, RI, 2016, pp. 189–201.

48. (a) Let G be a connected graph on $[n]$. Define a certain spanning tree τ_G of G as follows. Start at vertex 1, and always move to the *greatest* adjacent unvisited vertex if there is one; otherwise backtrack. Stop when every vertex has been visited, and let τ_G consist of the vertices and edges visited. We leave to the reader the proof of the following crucial lemma.
Lemma. *Let τ be a tree on $[n]$. A connected graph G satisfies $\tau_G = \tau$ if and only if τ is a spanning tree of G, and every other edge of G has the form $\{i,k\}$, where (i,j) is an inversion of τ and k is the unique predecessor of j in the rooted tree (with root 1) τ.*
Thus the $n-1$ edges of τ must be edges of G, while any subset of the $i(\tau)$ "inversion edges" defined by the previous lemma may constitute the remaining edges of G. Hence

$$\sum_G t^{e(G)} = t^{n-1}(1+t)^{\mathrm{inv}(\tau)}, \tag{5.137}$$

where G ranges over all connected graphs on $[n]$ satisfying $\tau_G = \tau$. Summing (5.137) over all τ completes the proof.
Equation (5.115) was first proved using an indirect generating function method by C. L. Mallows and J. Riordan, *Bull. Amer. Math. Soc.* **74** (1968), 92–94. (See also [231, Sect. 4.5].) The elegant proof given here is due to I. M. Gessel and D.-L. Wang, *J. Combinatorial Theory (A)* **26** (1979), 308–313. Gessel and Wang also give a similar result related to the enumeration of acyclic digraphs and tournaments. For some further results related to inversions in trees, see G. Kreweras, *Period. Math. Hungar.* **11**(4) (1980), 309–320; J. S. Beissinger, *J. Combinatorial Theory (B)* **33** (1982), 87–92, as well as Exercises 49(c) and 50(c). A remarkable conjectured connection between tree inversions

and invariant theory appears in M. Haiman, *J. Algebraic Combinatorics* **3** (1994), 17–76 (§2.3), subsequently proved by Haiman using the geometry of Hilbert schemes. For an overview see M. Haiman, in *Current Developments in Mathematics, 2002*, Int. Press, Somerville, MA, 2003, pp. 39–111.

(b) Substitute $t-1$ for t in equation (5.115), take the logarithm of both sides, and differentiate with respect to x. An explicit statement of the formula appears in [115, (14.7)]. For a generalization, see R. Stanley, in *Mathematical Essays in Honor of Gian-Carlo Rota* (B. Sagan and R. Stanley, eds.), Birkhäuser, Boston/Basel/Berlin, 1998, pp. 359–375 (Theorem 3.3).

49. (a) If some $b_i > i$, then at least $n-i+1$ cars prefer the $n-i$ spaces $i+1, i+2, \ldots, n$ and hence are unable to park. Thus the stated condition is necessary. The sufficiency can be proved by induction on n. Namely, suppose that $a_1 = k$. Define for $1 \le i \le n-1$,

$$a'_i = \begin{cases} a_{i+1}, & \text{if } a_{i+1} \le k \\ a_{i+1}-1, & \text{if } a_{i+1} > k. \end{cases}$$

Then the sequence $(a'_1, \ldots, a'_{n-1})$ satisfies the condition so by induction is a parking function. But this means that for the original sequence $\alpha = (a_1, \ldots, a_n)$, the cars $C_2, \ldots, C_n$ can park after car C_1 occupies space k. Hence α is a parking function, and the proof follows by induction (the base case $n = 1$ being trivial).

(b) Add an additional parking space 0 after space n, and allow 0 also to be a preferred parking space. Consider the situation where the cars $C_1, \ldots, C_n$ enter the street as before (beginning with space 1), but if a car is unable to park it may start over again at 1 and take the first available space. Of course now every car can park, and there will be exactly one empty space. If the preferences $(a_1, \ldots, a_n)$ lead to the empty space i, then the preferences $(a_1+k, \ldots, a_n+k)$ will lead to the empty space $i+k$ (addition in G). Moreover, α is a parking function if and only if the space 0 is left empty. From this the proof follows.

Parking functions *per se* were first considered by A. G. Konheim and B. Weiss, *SIAM J. Applied Math.* **14** (1966), 1266–1274, in connection with a hashing problem. They proved the formula $P(n) = (n+1)^{n-1}$ using recurrence relations. (The characterization (a) of parking functions seems to be part of the folklore of the subject.) However, a result equivalent to $P(n) = (n+1)^{n-1}$ was earlier given by R. Pyke, *Ann. Math. Statist.* **30** (1959), 568–576 (special case of Lemma 1). The elegant proof given here is due to H. Pollak, described in J. Riordan, *J.*

Combinatorial Theory **6** (1969), 408–411, and D. Foata and J. Riordan, *aequationes math.* **10** (1974), 10–22 (p. 13). Some bijections between parking functions and trees on the vertex set $[n+1]$ appear in the previous reference, as well as in J. Françon, *J. Combinatorial Theory (A)* **18** (1975), 27–35; P. Moszkowski, *Period. Math. Hungar.* **20** (1989), 147–154 (§3); and J. S. Beissinger and U. N. Peled, *Electronic J. Combinatorics* **4** (1997), paper R4. For a survey of parking functions see C. H. Yan, in *Handbook of Enumerative Combinatorics*, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2015, pp. 835–893.

(c) This result is due to G. Kreweras, *Period. Math. Hungar.* **11**(4) (1980), 309–320. Kreweras deals with *suites majeures* (major sequences), which are obtained from parking functions $(a_1, \ldots, a_n)$ by replacing a_i with $n+1-a_i$.

(d) Suppose that cars $C_1, \ldots, C_{i-1}$ have already parked at spaces $u_1, \ldots, u_{i-1}$. Then C_i parks at u_i if and only if spaces $a_i, a_i+1, \ldots, u_i-1$ are already occupied. Thus a_i can be any of the numbers $u_i, u_i-1, \ldots, u_i-\tau(u,u_i)+1$. There are therefore $\tau(u,u_i)$ choices for a_i, so

$$\nu(u) = \tau(u,u_1)\cdots\tau(u,u_n) = \tau(u,1)\cdots\tau(u,n).$$

This result is implicit in Konheim and Weiss, *ibid.*

(e) Given σ, define a poset $(P_\sigma, \overset{\sigma}{<})$ on $[n]$ by the condition that $j \overset{\sigma}{<} i$ if either $0 < i-j \leq s_i$ or $0 < j-i \leq t_i$. It is easy to see that P_σ is a tree, and that T_σ consists of the linear extensions of P_σ (where we regard a linear extension of P_σ as a permutation of its elements). By definition of P_σ we have $\#\Lambda_i = s_i + t_i$, where $\Lambda_i = \{j \in P_\sigma \ : \ j \overset{\sigma}{\leq} i\}$, and the proof follows from Exercise 3.57.

NOTE. The trees T_σ by definition have the property that the elements of Λ_i form a set of consecutive integers. Hence T_σ is a *recursively labelled tree* in the sense of Exercise 45. There follows from Theorem 2.2 of the paper of Björner and Wachs cited there the curious result

$$\sum_{u \in T_\sigma} q^{\mathrm{inv}(u)} = \sum_{u \in T_\sigma} q^{\mathrm{maj}(u)}.$$

Note that this formula is a refinement of the fact that maj and inv are equidistributed over $\mathfrak{S}_n$ (Corollary 1.3.13 and Proposition 1.4.6).

(f) **First solution.** Suppose that $\alpha_1, \ldots, \alpha_k$ is a sequence of prime parking functions, where the length of α_i is d_i. Let β_i denote α_i with $d_1 + d_2 + \cdots + d_{i-1}$ added to each term. Then any permutation of all the terms of all the β_i's is a parking function, and conversely given any parking

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function one can uniquely reconstruct $\alpha_1, \ldots, \alpha_k$. From this it follows (using equation (5.116)) that

$$\sum_{n\geq 0}(n+1)^{n-1}\frac{x^n}{n!} = \frac{1}{1-\sum_{n\geq 1} Q(n)\frac{x^n}{n!}}.$$

The proof now follows from equation (5.67). The definition of prime parking functions and the above proof of their enumeration is due to I. Gessel (private communication, 1997).

Second solution. Let r_i be the number of entries of α equal to i. One checks that the parking function α is prime if and only if every partial sum of the sequence $r_1 - 1, r_2 - 1, \ldots, r_{n-1} - 1$ is positive (in which case $r_n = 0$ and $\sum_i (r_i - 1) = 1$). A version of Lemma 5.3.7 shows any sequence of integers with sum 1 has exactly one cyclic permutation all of whose partial sums are positive. From this it follows that if we regard the elements of the group $L = \mathbb{Z}/(n-1)\mathbb{Z}$ as being the integers $1, 2, \ldots, n-1$, then every coset of the subgroup M of L^n generated by $(1, 1, \ldots, 1)$ contains exactly one prime parking function. Hence $Q(n) = [L^n : M] = (n-1)^{n-1}$. This argument is due to L. Kalikow.

50\. (a) The number of regions was first computed by J.-Y. Shi, *Lecture Notes in Mathematics*, no. 1179, Springer, Berlin/Heidelberg/New York, 1986 (Chapter 7), and *J. London Math. Soc.* **35** (1987), 56–74. For this reason the arrangement $\mathcal{S}_n$ is called the *Shi arrangement*. A more elementary (though nonbijective) proof was subsequently given by P. Headley, Ph.D. thesis, University of Michigan, Ann Arbor, 1994 (Chapter VI), and *J. Algebraic Combin.* **6** (1996), 331–338. For a simple nonbijective proof, see the solution to (b). A bijection between the regions of $\mathcal{S}_n$ and parking functions of length n (as defined in Exercise 49) is due to I. Pak and R. Stanley, stated in R. Stanley, *Proc. Nat. Acad. Sci.*, **93** (1996), 2620–2625 (Theorem 5.1) and proved in R. Stanley, in *Mathematical Essays in Honor of Gian-Carlo Rota (Cambridge, MA, 1996)*, Progr. Math. **161**, Birkhäuser Boston, Boston, MA, 1998, pp. 359–375 (Theorem 2.1). This bijection has the virtue of allowing an easy proof of (c). Simpler bijections lacking this property were given by J. Lewis, Parking functions and regions of the Shi arrangement, preprint dated August 1, 1996, and C. A. Athanasiadis and S. Linusson, *Discrete Math.* **204** (1999), 27–39.

(b) We want to compute the number of n-tuples $(x_1, \ldots, x_n) \in \mathbb{F}_p^n$ such that if $i < j$, then $x_i \neq x_j$ and $x_i \neq x_j + 1$. There are $(p-n)^{n-1}$ ways to choose a weak ordered partition $\pi = (B_1, \ldots, B_{p-n})$ of $[n]$ into $p-n$

blocks such that $1 \in B_1$. Choose x_1 in p ways. Think of the elements of $\mathbb{F}_p$ as being arranged in a circle, in the clockwise order $0, 1, \dots, p-1$. We will place the numbers $1, 2, \dots, n$ on some of the p points of this circular depiction of $\mathbb{F}_p$. Place the elements of B_1 consecutively in increasing order when read clockwise, with 1 placed at x_1. Then skip one space (in clockwise order) and place the elements of B_2 consecutively in increasing order. Then skip one space and place the elements of B_3 consecutively in increasing order, etc. Let x_i be the point at which i is placed. It is easy to see that this gives a bijection between the $p(p-n)^{n-1}$ choices of (π, x_1) and the allowed values of $(x_1, \dots, x_n)$, so the proof follows from Theorem 3.11.10. This argument is due to C. A. Athanasiadis, Ph.D. thesis, M.I.T, 1996 (Theorem 6.2.1), and *Advances in Math.* **122** (1996), 193–233 (Theorem 3.3). The characteristic polynomial of the Shi arrangement was first computed by P. Headley, Ph.D. thesis, University of Michigan, Ann Arbor, 1994 (Chapter VI); *Formal Power Series and Algebraic Combinatorics, FPSAC '94, May 23–27, 1994*, DIMACS preprint, pp. 225–232 (§5); and *J. Algebraic Combinatorics* **6** (1997), 331–338 (Theorem 2.4 in the case $\Phi = A_n$), by a different method. A further proof appears in A. Postnikov, Ph.D. thesis, M.I.T., 1997 (Example 1, p. 39), and A. Postnikov and R. Stanley, *J. Combin. Theory Ser. A* **91** (2000) 544-597 (Corollary 9.3).

(c) This result is equivalent to a theorem of I. Pak and R. Stanley that is stated in R. Stanley, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625 (Theorem 5.1), and proved in R. Stanley, in *Mathematical Essays in Honor of Gian-Carlo Rota (Cambridge, MA, 1996)*, Progr. Math. **161**, Birkhäuser Boston, Boston, MA, 1998, pp. 359–375 (the case $k = 1$ of Corollary 2.20).

(d) Let $x = (x_1, \dots, x_n)$ belong to some region R of $\mathcal{S}_n$. Define $\pi_x \in \mathfrak{S}_n$ by the condition

$$x_{\pi_x(1)} > x_{\pi_x(2)} > \cdots > x_{\pi_x(n)}.$$

Let $I_x = \{(i,j) \,:\, 1 \le i < j \le n,\ x_j + 1 > x_i > x_j\}$. It is not difficult to show that the map $R \mapsto (\pi_x, I_x)$ is a bijection between the regions of $\mathcal{S}_n$ and the pairs (π_x, I_x) where $\pi_x \in \mathfrak{S}_n$ and $I_x \in J(P_{\pi_x})$. Moreover, $d(R) = \binom{n}{2} - |I_x|$, and the proof follows. This result was stated without proof in R. Stanley, *ibid.* (after Theorem 5.1).

(e) Let $\pi = \{B_1, \dots, B_{n-k}\}$ be a partition of $[n]$, and let w_i be a permutation of B_i. Let X consist of all points $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ such that $x_a - x_b = m$ if a and b appear in the same block B_i of π, a appears to the left of b in w_i, and there are exactly m ascents appearing in w_i

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between a and b. For instance, if $w_1 = 495361$ and $w_2 = 728$, then X is defined by the conditions

$$x_4 = x_9 + 1 = x_5 + 1 = x_3 + 1 = x_6 + 2 = x_1 + 2, \quad x_7 = x_2 = x_8 + 1.$$

This defines a bijection between the partitions of $[n]$ into $n-k$ linearly ordered blocks and the elements X of $L_{\mathcal{S}_n}$ of rank k.

51. Assume (i). Substituting $A(x)$ for x yields

$$\frac{x}{C(A(x))} = B^{\langle -1\rangle}(A(x)).$$

Substituting $B(x)$ for x in (i) yields

$$A^{\langle -1\rangle}(B(x)) = xC(B(x)).$$

But $\left(A^{\langle -1\rangle}(B(x))\right)^{\langle -1\rangle} = B^{\langle -1\rangle}(A(x))$, so (ii) follows. (Note that we did not need $[x]C(x) \neq 0$.)

Since $[x]C(x) \neq 0$, the compositional inverse $(C(x)-c)^{\langle -1\rangle}$ exists. We can now argue as follows. Substituting $xC(B(x))$ for x in (ii) yields

$$xC(B(x))/C(A(xC(B(x)))) = x,$$

so $C(B(x)) = C(A(xC(B(x))))$. Substituting $B(x)^{\langle -1\rangle}$ for x yields

$$C(x) = C(A(B(x)^{\langle -1\rangle}C(x))).$$

Subtract c from both sides and apply $(C-c)^{\langle -1\rangle}$ to get $x = A(B(x)^{\langle -1\rangle}C(x))$. Applying $A^{\langle -1\rangle}$ to both sides gives (i). This argument is due to Daniel Giaimo and Amit Khetan and (independently) to Yumi Odama.

52. (a) See [2.3, Section 3.7], where also the polynomials $\varphi_n(k)$ are given for $n \leq 7$.

    (b) First check that for fixed n, the quantities $[x^n]F^{\langle j+k\rangle}(x)$ and $[x^n]F^{\langle j\rangle}(F^{\langle k\rangle}(x))$ are polynomials in j and k. Since these two polynomials agree for all $j,k \in \mathbb{P}$, they must be the same polynomials. A similar argument works for the second identity. See [2.3, Thm. B, p. 148].

53. We need to compute

$$f(n) := [x^{n-1}]\left(1 - \frac{1}{2}x\right)^{-n}(1-x)^{-1} \text{ [why?]}.$$

In equation (5.64), let $x/F(x) = (1 - \frac{1}{2}x)^{-1}$ and $H'(x) = (1-x)^{-1}$. Then

$$F^{\langle -1\rangle}(x) = 1 - \sqrt{1-2x},\ H(x) = -\log(1-x),$$

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so

$$\begin{aligned} f(n) &= n[x^n](-\log\sqrt{1-2x}) \\ &= -\frac{n}{2}[x^n]\log(1-2x) \\ &= 2^{n-1}, \end{aligned}$$

exactly half the sum of the entire series.

This result is equivalent to the identity

$$2^{n-1} = \sum_{j=0}^{n-1} 2^{-j}\binom{n+j-1}{j},$$

or equivalently (putting $n+1$ for n)

$$4^n = \sum_{j=0}^{n} 2^{n-j}\binom{n+j}{j}. \tag{5.138}$$

Bromwich [30, Example 20, p. 199] attributes the result of this exercise to *Math. Trip.* 1903. Equation (5.138) also follows immediately from "Banach's match box problem," and account of which appears for instance in W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, second ed., Wiley, New York, 1957 (§5.8). This yields a simple bijective proof of (5.138).

54. By equation (5.53) we have

$$[x^{-1}]F(x)^{-n} = n[x^n]F^{\langle -1\rangle}(x).$$

The compositional inverses of the four functions are given by

$$\begin{aligned} \sin^{-1} x &= \sum_{m\geq 0} 4^{-m}\binom{2m}{m}\frac{x^{2m+1}}{2m+1} \\ \tan^{-1} x &= \sum_{m\geq 0}(-1)^m\frac{x^{2m+1}}{2m+1} \\ e^x - 1 &= \sum_{n\geq 1}\frac{x^n}{n!} \\ x + \frac{x^2}{2(1-x)} &= x + \frac{1}{2}\sum_{n\geq 2} x^n, \end{aligned}$$

yielding the four answers

$$\begin{cases} 0, & n = 2m \\ 4^{-m}\dbinom{2m}{m}, & n = 2m+1 \end{cases}$$

$$\begin{cases} 0, & n = 2m \\ (-1)^m, & n = 2m+1 \end{cases}$$

$$1/(n-1)!$$

$$\begin{cases} 1, & n = 1 \\ n/2, & n \geq 2 \end{cases}$$

Bromwich [30, Example 19, p. 199] attributes these formulas to Wolstenholme.

55. (a) Let $y \in \mathbb{Q}[[x]]$ satisfy $y = xF_1(y)$. By (5.55) with $k = 1$ we have

$$n[x^n]y = [x^{n-1}]F_1(y)^n = 1,$$

so $y = \sum_{n\geq 1} \frac{x^n}{n} = -\log(1-x)$. Hence $y^{\langle -1\rangle} = 1 - e^{-x}$, so $F_1(x) = x/y^{\langle -1\rangle} = x/(1-e^{-x})$.

NOTE. The *Bernoulli numbers* B_n are defined by

$$\frac{x}{e^x - 1} = \sum_{n\geq 0} B_n \frac{x^n}{n!}.$$

Hence

$$F_1(x) = \sum_{n\geq 0} (-1)^n B_n \frac{x^n}{n!}.$$

Essentially the same result is attributed to Wolstenholme and *Math. Trip.* 1904 by Bromwich [30, Example 18, p. 199]. Somewhat surprisingly, this result has applications to algebraic geometry. See Lemma 1.7.1 of F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, New York, 1966.

A more general result is the following: Given $f(x) = \sum_{n\geq 1} \alpha_{n-1}\frac{x^n}{n} \in \mathbb{C}[[x]]$, it follows from (5.57) that the unique power series $F(x)$ satisfying $[x^n]F(x)^{n+1} = \alpha_n$ for all $n \in \mathbb{N}$ is given by $F(x) = x/f^{\langle -1\rangle}(x)$.

(b)–(c) Note that $F_k(0) = 1$ (the case $n = 0$). Let $G_k(x) = x/F_k(x^k)$. The condition on $F_k(x)$ becomes

$$[x^n]\left(\frac{x}{G_k(x)}\right)^{n+1} = \begin{cases} 1, & \text{if } n \equiv 0 \pmod{k} \\ 0, & \text{otherwise.} \end{cases}$$

By Lagrange inversion (Theorem 5.4.2) we have

$$[x^n]\left(\frac{x}{G_k(x)}\right)^{n+1} = (n+1)[x^{n+1}]G_k^{\langle -1\rangle}(x).$$

Hence

$$G_k^{\langle -1\rangle}(x) = \sum_{m\geq 0}\frac{x^{km+1}}{km+1}.$$

When $k = 2$ we have $G_2^{\langle -1\rangle} = \frac{1}{2}\log\frac{1+x}{1-x}$, whence $G_2(x) = \frac{e^{2x}-1}{e^{2x}+1}$ and $F_2(x) = \frac{\sqrt{x}}{\tanh\sqrt{x}}$. This result appears as Lemma 1.5.1 of Hirzebruch, *ibid.*

When $k > 2$ there is no longer a simple way to invert the series $G_k^{\langle -1\rangle}(x) = \sum_{m\geq 0}\frac{x^{km+1}}{km+1}$.

56. (a) **First solution.** More generally, let $G(x) = \sum_{n\geq 0} b_n x^n$ be any power series with $b_0 = 1$. Define

$$H(x) = x\,\exp\sum_{n\geq 1} b_n\frac{x^n}{n},$$

so

$$\frac{H'(x)}{H(x)} = \frac{G(x)}{x}.$$

Set $y = F(x) = a_1x + a_2x^2 + \cdots$, $a_1 \neq 0$. Consider the formal power series

$$\log\frac{H(y^{\langle -1\rangle})}{x} := \sum_{i\geq 1} p_i x^i.$$

Then

$$\begin{aligned}
\log\frac{H(x)}{y} &= \sum_{i\geq 1} p_i y^i\\
\Longrightarrow \qquad \frac{H'(x)}{H(x)} - \frac{y'}{y} &= \sum_{i\geq 1} i p_i y^{i-1} y'.\\
\Longrightarrow \qquad y^{-n}\left(\frac{G(x)}{x} - \frac{y'}{y}\right) &= \sum_{i\geq 1} i p_i y^{i-n-1} y'.
\end{aligned}$$

Take the coefficient of $1/x$ on both sides. As in the first proof of Theorem 5.4.2 we obtain

$$[x^{-1}]\frac{G(x)}{xy^n} = np_n. \tag{5.139}$$

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Now take $G(x) = 1$ in (5.139), so $H(x) = x$. We get

$$n[x^n]\log\frac{y^{\langle -1\rangle}}{x} = np_n = [x^{-1}]\frac{1}{xy^n} = [x^n]\left(\frac{x}{y}\right)^n,$$

as desired.

Second solution. Define $H(x) = \log\frac{x}{F(x)}$. Then (5.64) becomes

$$\begin{aligned}
n[x^n]\log\frac{F^{\langle -1\rangle}(x)}{x} &= [x^{n-1}]\left(\frac{1}{x}-\frac{F'(x)}{F(x)}\right)\left(\frac{x}{F(x)}\right)^n \\
&= [x^n]\left(\frac{x}{F(x)}\right)^n - [x^{-1}]\frac{F'(x)}{F(x)^{n+1}} \\
&= [x^n]\left(\frac{x}{F(x)}\right)^n + \frac{1}{n}[x^{-1}]\frac{d}{dx}F(x)^{-n} \\
&= [x^n]\left(\frac{x}{F(x)}\right)^n,
\end{aligned}$$

as desired.

Third solution. Equation (5.53) can be rewritten (after substituting $n+k$ for n)

$$(n+k)[x^n]\frac{1}{k}\left(\frac{F^{\langle -1\rangle}(x)}{x}\right)^k = [x^n]\left(\frac{x}{F(x)}\right)^{n+k}. \tag{5.140}$$

The first proof of Theorem 5.4.2 is actually valid for any $k \in \mathbb{R}$, so we can let $k \to 0$ in (5.140) to get (after some justification) equation (5.118).

The result of this exercise goes back to J.-L. Lagrange, *Mém. Acad. Roy. Sci. Belles-Lettres Berlin* **24** (1770); *Oeuvres*, vol. 3 Gauthier-Villars, Paris, 1869, pp. 3–73. It was rediscovered by I. Schur, *Amer. J. Math.* **69** (1947), 14–26.

(b) Let $G(x) = x/F(x)$. By (a),

$$\delta_{0n} = [x^n]G(x)^n = n[x^n]\log\frac{F^{\langle -1\rangle}(x)}{x}.$$

Thus $x = \log\frac{F^{\langle -1\rangle}(x)}{x}$, so $F^{\langle -1\rangle}(x) = xe^x$. Hence

$$\begin{aligned}
G(x) &= x/(xe^x)^{\langle -1\rangle} \\
&= 1 + \sum_{n\geq 1}(-1)^{n-1}(n-1)^{n-1}\frac{x^n}{n!} \ \text{(with } 0^0 = 1\text{)},
\end{aligned}$$

by a simple application of (5.53) (or by substituting $-x$ for x in (5.67)).

57. In Corollary 5.4.3 take $H(x) = \log(1+x)$ and $x/F(x) = (1+x)^2/(2+x)$. Then $F(x) = 1-(1+x)^{-2}$, so $F^{\langle -1\rangle}(x) = (1-x)^{-1/2} - 1$. Equation (5.64) becomes

$$n[x^n]\frac{1}{2}\log(1-x)^{-1} = [x^{n-1}](1+x)^{2n-1}(2+x)^{-n}.$$

But $[x^n]\log(1-x)^{-1} = 1/n$, and the result follows.
By expanding $(1+x)^{2n-1}$ and $(2+x)^{-n}$ and taking the coefficient of x^{n-1} in their product, we see that an equivalent result is the identity (replacing n by $n+1$)

$$4^n = \sum_{j=0}^{n}(-1)^{n-j}2^j\binom{2n+1}{j}\binom{2n-j}{n}.$$

Bromwich [30, Example 18, p. 199] attributes this result to *Math. Trip.* 1906.

58. Let $F(x) = xf(x)^\alpha$ and $G(x) = g(x)^\alpha$. Then the functional equation (5.119) becomes $F(x) = xG(F(x))$, so by ordinary Lagrange inversion (Theorem 5.4.2) we get

$$m[x^m]F(x)^k = k[x^{m-k}]G(x)^m$$

for any nonnegative integers m and k. In terms of f and g this is

$$m[x^{m-k}]f(x)^{\alpha k} = k[x^{m-k}]g(x)^{\alpha m}.$$

Now set $k = t/\alpha$ and $m = \frac{t}{\alpha} + n$, so that $t = \alpha k$ and $n = m-k$. We get the desired formula with the restriction that t/α must be a nonnegative integer. However, since both sides are polynomials in t, the formula holds for all t. This result is due to E. Rains (private commmunication), and the above proof was provided by I. Gessel. For an application, see Exercise 37(e).

59. Define $g(x,y) \in K[[x,y]]$ to be the (unique) power series satisfying the functional equation $g = yF(x,g)$. Thus $g(x,1) = f(x)$. By Lagrange inversion (Theorem 5.4.2) we have $n[y^n]g(x,y)^k = k[y^{n-k}]F(x,y)^n$. Hence

$$g(x,t)^k = \sum_{n\geq 1}\left([y^n]g(x,y)^k\right)t^n = \sum_{n\geq 1}\frac{k}{n}\left([u^{n-k}]F(x,u)^n\right)t^n.$$

Setting $t=1$ yields the desired result. This argument is due to I. Gessel.

60. (a) One method of proof is to let $B(x) = A(x) - 1$ and write

$$A(x)^n = (1+B(x))^n = \sum_{j\geq 0}\binom{n}{j}B(x)^j.$$

Thus (since $\deg B(x)^j \geq j$),

$$[x^k]A(x)^n = \sum_{j=0}^{k} \binom{n}{j} [x^k]B(x)^j,$$

which is clearly a polynomial in n of degree $\leq k$.
Alternatively, let Δ be the difference operator with respect to the variable n. Then by equation (1.26) we have

$$\begin{aligned}
\Delta^{k+1}[x^k]A(x)^n &= [x^k]\sum_{i=0}^{k+1}(-1)^{k+1-i}\binom{k+1}{i}A(x)^{n+i} \\
&= [x^k]A(x)^n(A(x)-1)^{k+1} \\
&= 0.
\end{aligned}$$

Now use Proposition 1.4.2(a).

(b) Since $e^{tF(x)} = \sum_{n\geq 0} t^n F(x)^n/n!$, we have

$$\begin{aligned}
p_k(n) &= \left[t^n \frac{x^{n+k}}{(n+k)!} \right] e^{tF(x)} \\
&= \frac{(n+k)!}{n!} [x^{n+k}]F(x)^n \\
&= (n+k)_k \, [x^k] \left(\frac{F(x)}{x} \right)^n. \qquad (5.141)
\end{aligned}$$

Now use (a).

(c) We have, as in (b),

$$\begin{aligned}
\left[t^n \frac{x^{n+k}}{(n+k)!} \right] e^{tF^{\langle -1\rangle}(x)} &= (n+k)_k[x^{n+k}]F^{\langle -1\rangle}(x)^n \\
&= (n+k)_k \frac{n}{n+k} [x^k] \left(\frac{F(x)}{x} \right)^{-n-k} \quad \text{(by (5.53))} \\
&= \frac{(n+k-1)_k}{(-n)_k} p_k(-n-k) \quad \text{(by (5.141))} \\
&= (-1)^k p_k(-n-k),
\end{aligned}$$

as desired.

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(d) *Answer:* We have $p_k(n) = S(n+k, n)$ and $(-1)^k p_k(-n-k) = s(n+k, n)$. The "Stirling number reciprocity" $S(-n, -n-k) = (-1)^k s(n+k, n)$ is further discussed in I. Gessel and R. Stanley, *J. Combinatorial Theory (A)* **24** (1978), 24–33, and D. E. Knuth, *Amer. Math. Monthly* **99** (1992), 403–422.

(e) It follows from Exercise 17(b) that

$$\begin{aligned} p_k(n) &= \frac{(n+k)!}{n!}\binom{n+k-1}{n-1} \\ &= (n+k)(n+k-1)^2(n+k-2)^2\cdots(n+1)^2 n/(k-1)! \ \ (k \geq 1). \end{aligned}$$

Since $F^{\langle -1\rangle}(x) = x/(1+x) = -F(-x)$, it follows from (c) that $p_k(-n-k) = p_k(n)$.

For further information on power series $F(x)$ satisfying $F^{\langle -1\rangle}(x) = -F(-x)$, see Exercise 1.168.

(f) *Answer:* $p_k(n) = (-1)^k\binom{n+k}{k}n^k$. Thus $(-1)^k p_k(-n-k) = \binom{n+k-1}{k}(n+k)^k$, so

$$\exp t(xe^{-x})^{\langle -1\rangle} = \sum_{n\geq 0}\sum_{k\geq 0}\binom{n+k-1}{k}(n+k)^k t^n \frac{x^{n+k}}{(n+k)!}.$$

This formula is also immediate from Propositions 5.3.1 and 5.3.2.

61. (a) Clearly $\mu(\overline{P}\times\overline{Q}) = \mu(\overline{P})\mu(\overline{Q})$ by Proposition 3.8.2. Now we have the disjoint union

$$\overline{P}\times\overline{Q} = (P\times Q)\cup\{(x,\hat{0}_{\overline{Q}}) : x\in P\}\cup\{(\hat{0}_{\overline{P}}, y) : y \in Q\}\cup\{(\hat{0}_{\overline{P}},\hat{0}_{\overline{Q}})\}.$$

Write $[u, v]_R$ for the interval $[u, v]$ of the poset R. If $t \in P\times Q$ then the intervals $[t, \hat{1}]_{P\times Q}$ and $[t, \hat{1}]_{\overline{P}\times\overline{Q}}$ are isomorphic. If $x \in P$, then the interval $[(x, \hat{0}_{\overline{Q}}), \hat{1}_{\overline{P}\times\overline{Q}}]$ is isomorphic to $[x, \hat{1}]_P \times \overline{Q}$. Similarly if $y \in Q$, then $[(\hat{0}_{\overline{P}}, y), \hat{1}]_{\overline{P}\times\overline{Q}} \cong \overline{P}\times[y, \hat{1}]_Q$. Hence

$$\begin{aligned} 0 &= \sum_{t\in\overline{P}\times\overline{Q}} \mu_{\overline{P}\times\overline{Q}}(t,\hat{1}) \\ &= \sum_{t\in P\times Q}\mu_{P\times Q}(t,\hat{1}) + \left(\sum_{x\in P}\mu_P(x,\hat{1})\right)\mu(\overline{Q}) + \mu(\overline{P})\left(\sum_{y\in Q}\mu_Q(y,\hat{1})\right) \\ &\quad + \mu(\overline{P})\mu(\overline{Q}) \\ &= -\mu(\overline{P\times Q}) - \mu(\overline{P})\mu(\overline{Q}) - \mu(\overline{P})\mu(\overline{Q}) + \mu(\overline{P})\mu(\overline{Q}), \end{aligned}$$

and the proof follows. Note that Exercise 3.69(d) is a special case.

(b) In Corollary 5.5.5 put $f(i) = -\mu_i = -\mu(\overline{Q}_i)$. If type $\pi = (a_1, \ldots, a_n)$ then by property (E3) of exponential structures and (a), we have $f(1)^{a_1} \cdots f(n)^{a_n} = -\mu(\overline{Q_1^{a_1} \times \cdots \times Q_n^{a_n}}) = -\mu(\hat{0}, \pi)$. Hence

$$h(n) = -\sum_{\pi \in Q_n} \mu(\hat{0}, \pi) = \mu(\hat{0}, \hat{0}) = 1,$$

and the proof follows.

This proof was suggested by D. Grieser.

62. (a) The case $r = 0$ is trivial, so assume $r > 0$. Let Γ_A be the corresponding bipartite graph, as defined in Section 5.5. Suppose $(\Gamma_A)_i$ is a connected component of Γ_A with vertex bipartition (X_i, Y_i), where $\#X_i = \#Y_i = j$. Suppose $j \geq 2$. The edges of $(\Gamma_A)_i$ may be chosen as follows. Place a bipartite cycle on the vertices (X_i, Y_i) in $\frac{1}{2}(j-1)!j!$ ways (as in the proof of Proposition 5.5.10). Replace some edge e of this cycle with m edges, where $1 \leq m \leq r-1$. Replace each edge at even distance from e also with m edges, while each edge at odd distance is replaced with $r-m$ edges. Thus given (X_i, Y_i), there are $\frac{1}{2}(r-1)(j-1)!j!$ choices for $(\Gamma_A)_i$ when $j \geq 2$. When $j = 1$ there is only one choice. Hence by the exponential formula for 2-partitions,

$$\begin{aligned}
\sum_{n\geq 0} f_r(n)\frac{x^n}{n!^2} &= \exp\left[x + \frac{1}{2}(r-1)\sum_{j\geq 2}(j-1)!j!\frac{x^j}{j!^2}\right] \\
&= \exp\left[x + \frac{1}{2}(r-1)(-x + \log(1-x)^{-1})\right] \\
&= (1-x)^{-\frac{1}{2}(r-1)}\, e^{\frac{1}{2}(3-r)x}.
\end{aligned}$$

(b) When $r = 3$ we obtain

$$\sum_{n\geq 0} f_3(n)\frac{x^n}{n!^2} = \frac{1}{1-x} = \sum_{n\geq 0} x^n,$$

so $f_3(n) = n!^2$. D. Callan observed (private communication) that there is a very simple combinatorial proof. Any matrix of the type being enumerated can be written *uniquely* in the form $P + 2Q$, where P and Q are permutation matrices. Conversely, $P + 2Q$ is always of the type being enumerated, whence $f_3(n) = n!^2$.

63. Let $A = P_1 + P_2 + \cdots + P_{2k}$, and let Γ_A be the bipartite graph corresponding to A, as defined in Section 5.5. Write $\Gamma_m = \Gamma_{P_m}$, so Γ_A is the edge-union of $\Gamma_1, \Gamma_2, \ldots, \Gamma_{2k}$. Supose $(\Gamma_A)_i$ is a connected component of Γ_A with vertex bipartition (X_i, Y_i), where $\#X_i = \#Y_i = j \geq 1$. If $j \geq 2$ then $(\Gamma_A)_i$ is

obtained by placing a bipartite cycle on (X_i, Y_i) and then replacing each edge with k edges. This can be done in $\frac{1}{2}(j-1)!\,j!$ ways. Write $E(\Gamma)$ for the multiset of edges of the graph Γ. Then $E(\Gamma_m)\cap E((\Gamma_A)_i)$ consists of j vertex-disjoint edges of $(\Gamma_A)_i$. There are precisely two distinct ways to choose j vertex-disjoint edges of $(\Gamma_A)_i$, and each must occur k times among the sets $E(\Gamma_m)\cap E((\Gamma_A)_i)$, for fixed i and for $1 \le m \le 2k$. Hence there are $\binom{2k}{k}$ ways to choose the sets $E(\Gamma_m)\cap E((\Gamma_A)_i)$, $1 \le m \le 2k$. Thus for $j \ge 2$ there are $\frac{1}{2}(j-1)!\,j!\binom{2k}{k} = (j-1)!\,j!\binom{2k-1}{k}$ choices for each bipartition (X_i, Y_i) with $\#X_i = \#Y_i = j$. When $j = 1$ it is clear that there is only one choice. Hence by the exponential formula for 2-partitions,

$$\begin{aligned}\sum_{n\ge 0} N_k(n)\frac{x^n}{n!^2} &= \exp\left[x + \binom{2k-1}{k}\sum_{j\ge 2}(j-1)!\,j!\frac{x^j}{j!^2}\right]\\ &= \exp\left[x + \binom{2k-1}{k}(-x+\log(1-x)^{-1})\right]\\ &= (1-x)^{-\binom{2k-1}{k}}\exp\left[x\left(1-\binom{2k-1}{k}\right)\right].\end{aligned}$$

64. (a) Let M' be M with its first row multiplied by -1. If k is odd then $(\det M)^k + (\det M')^k = (\operatorname{per} M)^k + (\operatorname{per} M')^k = 0$, from which it follows that $f_k(n) = g_k(n) = 0$. Now

$$\begin{aligned}2^{n^2} f_2(n) &= \sum_M\left(\sum_{\pi\in\mathfrak{S}_n} \pm m_{1,\pi(1)}\cdots m_{n,\pi(n)}\right)^2\\ &= \sum_{\pi,\sigma\in\mathfrak{S}_n}(\operatorname{sgn}\pi)(\operatorname{sgn}\sigma)\sum_{i,j}\sum_{m_{ij}=\pm 1}\\ &\qquad m_{1,\pi(1)}\cdots m_{n,\pi(n)}m_{1,\sigma(1)}\cdots m_{n,\sigma(n)}.\end{aligned}$$

If $\pi \neq \sigma$, say $j = \pi(i) \neq \sigma(i)$, then the inner two sums have a factor $\sum_{m_{ij}=\pm 1} m_{ij} = 0$. Hence

$$\begin{aligned}2^{n^2} f_2(n) &= \sum_{\pi\in\mathfrak{S}_n}(\operatorname{sgn}\pi)^2\sum_{i,j}\sum_{m_{ij}=\pm 1}(m_{1,\pi(1)}\cdots m_{n,\pi(n)})^2\\ &= \sum_{\pi\in\mathfrak{S}_n}\sum_{i,j}\sum_{m_{ij}} 1\\ &= 2^{n^2} n!,\end{aligned}$$

so $f_2(n) = n!$. The same argument gives $g_2(n) = n!$, since the factors $(\operatorname{sgn}\pi)(\operatorname{sgn}\sigma)$ above turned out to be irrelevant.

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Nyquist, Rice, and Riordan (see reference below) attribute this result (in a somewhat more general form) to R. Fortet, *J. Research Nat. Bur. Standards* **47** (1951), 465–470, though it may have been known earlier. For a connection with Hadamard matrices, see C. R. Johnson and M. Newman, *J. Research Nat. Bur. Standards* **78B** (1974), 167–169, and M. Kac, *Probability and Related Topics in Physical Sciences*, vol. I, Interscience, London/New York, 1959 (p. 23).

(b) Now we get

$$2^{n^2} f_4(n) = \sum_{\rho,\pi,\sigma,\tau\in\mathfrak{S}_n} (\operatorname{sgn}\rho)(\operatorname{sgn}\pi)(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)$$
$$\sum_{i,j}\sum_{m_{ij}=\pm 1}\prod_{k=1}^{n} m_{k,\rho(k)}m_{k,\pi(k)}m_{k,\sigma(k)}m_{k,\tau(k)}. \tag{5.142}$$

We get a nonzero contribution only when the product P in (5.142) is a perfect square (regarded as a monomial in the variables m_{rs}). Equivalently, if we identify a permutation with its corresponding permutation matrix then $\rho+\pi+\sigma+\tau$ has entries $0,\ 2$ or 4. We claim that in this case the product $\rho\pi\sigma\tau$ is an even permutation. One way to see this is to verify that a fixed cycle C occurs an even number of times ($0,\ 2,$ or 4, with 4 possible only for singletons) among the four permutations $\rho\rho^{-1},\ \pi\rho^{-1},\ \sigma\rho^{-1},\ \tau\rho^{-1}$. Hence $\rho\rho^{-1}\pi\rho^{-1}\sigma\rho^{-1}\tau\rho^{-1}$ is even, so also $\rho\pi\sigma\tau$. It follows that the factor $(\operatorname{sgn}\rho)(\operatorname{sgn}\pi)(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)$ in (5.142) is equal to 1 for all nonzero terms. Hence the right-hand side of (5.142) is equal to $2^{n^2}N_2(n)$, where $N_2(n)$ is the number of 4-tuples $(\rho,\pi,\sigma,\tau)\in\mathfrak{S}_n^4$ with every entry of $\rho+\pi+\sigma+\tau$ equal to $0,\ 2,$ or 4. Hence (5.120) follows from Exercise 63.

The computation is identical for $g_4(n)$ since the factor $(\operatorname{sgn}\rho)(\operatorname{sgn}\pi)(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)$ turned out to be irrelevant.

Equation (5.120) is due to Nyquist, Rice, and Riordan *Quart. J. Appl. Math.* **12** (1954), 97–104, using a different technique. They prove a more general result wherein the matrix entries are identically distributed independent random variables symmetric about 0. The method used here applies equally well to this more general result. For a result on the sixth moment of the determinant of a random matrix, see Z. Lv and A. Potechin, The sixth moment of random determinants, `arxiv:2206.11356`.

(c) It is clear from the above proof technique that $f_{2k}(n) < g_{2k}(n)$ provided there are permutations $\pi_1,\dots,\pi_{2k}\in\mathfrak{S}_n$ such that $\pi_1+\cdots+\pi_{2k}$ has even entries and $\pi_1\cdots\pi_{2k}$ is an odd permutation. For $n=3$ and $k=3$

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we can take $\{\pi_1, \pi_2, \ldots, \pi_6\} = \mathfrak{S}_3$. For larger values of n and k we can easily construct examples from the example for $n = 3$ and $k = 3$.

(d) Let $M \in \mathcal{D}_{n+1}$. Multiply each column of M by ± 1 so that the first row consists of 1's. Multiply each row except the first by ± 1 so that the first column contains -1's in all positions except the first. Now add the first row to all the other rows. The submatrix obtained by deleting the first row and column will be an $n \times n$ matrix $2M'$, where M' is a 0-1 matrix. Expanding by the first column yields $\det M = \pm 2^n(\det M')$. This map $M \longmapsto M'$ produces each $n \times n$ 0-1 matrix the same number (viz., 2^{2n+1}) of times. From this it follows easily that $f_k'(n) = 2^{-kn} f_k(n+1)$ when k is even. When k is odd one can see easily that $f_k'(n) = 0$.
We leave the easy case of $g_1'(n)$ to the reader and consider $g_2'(n)$. As in (a) or (b) we have

$$2^{n^2} g_2'(n) = \sum_{\pi,\sigma \in \mathfrak{S}_n} \sum_{i,j} \sum_{m_{ij}=0,1} \prod_{k=1}^{n} m_{k,\pi(k)} m_{k,\sigma(k)}.$$

Suppose that the matrix $\pi + \sigma$ has r 2's, and hence $2n - r$ 1's. Equivalently, $\pi\sigma^{-1}$ has r fixed points. Then

$$\sum_{i,j} \sum_{m_{ij}=0,1} \prod_{k=1}^{n} m_{k,\pi(k)} m_{k,\sigma(k)} = 2^{n^2-2n+r}.$$

Since we can choose any $\pi \in \mathfrak{S}_n$ and then choose σ so that $\pi\sigma^{-1}$ has r fixed points, it follows that

$$2^{n^2} g_2'(n) = n! 2^{n^2-2n} \sum_{\pi \in \mathfrak{S}_n} 2^{c_1(\pi)} = n! 2^{n^2-2n} h(n),$$

say, where π has $c_1(\pi)$ fixed points. Setting $t_1 = 2$ and $t_2 = t_3 = \cdots = 1$ in (5.30) yields

$$\begin{aligned}
\sum_{n\geq 0} h(n) \frac{x^n}{n!} &= \exp\left(2x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \\
&= \frac{e^x}{1-x} \\
&= \sum_{n\geq 0} n! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) \frac{x^n}{n!},
\end{aligned}$$

so $h(n) = n!(1 + \frac{1}{1!} + \cdots + \frac{1}{n!})$, and $g_2'(n)$ is as claimed. (One could also give a proof using Proposition 5.5.8 instead of (5.30).)

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65. (a) Given a function $g : \mathbb{N} \times \mathbb{N} - \{(0,0)\} \to K$, define a new function $h : \mathbb{N} \times \mathbb{N} \to K$ by

$$h(m,n) = \sum g(\#A_1, \#B_1) \cdots g(\#A_k, \#B_k),$$

where the sum is over all sets $\{(A_1, B_1), \ldots, (A_k, B_k)\}$, where $A_j \subseteq [m]$ and $B_j \subseteq [n]$, satisfying:

(i) For no j do we have $A_j = B_j = \emptyset$,

(ii) The nonempty A_j's form a partition of the set $[m]$,

(iii) The nonempty B_j's form a partition of the set $[n]$.

(Set $h(0,0) = 1$.) In the same way that Corollary 5.1.6 is proved we obtain

$$\sum_{m,n\geq 0} h(m,n)\frac{x^m y^n}{m!\,n!} = \exp \sum_{\substack{i,j\geq 0 \\ (i,j)\neq(0,0)}} g(i,j)\frac{x^i y^j}{i!\,j!}.$$

Now let $A = (a_{ij})$ be an $m \times n$ matrix of the type being counted. Let Γ_A be the bipartite graph with vertex bipartition $(\{x_1, \ldots, x_m\}, \{y_1, \ldots, y_n\})$, with a_{ij} edges between x_i and y_j. The connected components $\Gamma_1, \ldots, \Gamma_k$ of Γ_A define a set $\{(A_1, B_1), \ldots, (A_k, B_k)\}$ satisfying (i)–(iii) above, namely, $i \in A_j$ if x_i is a vertex of Γ_j, and $i \in B_j$ if y_i is a vertex of Γ_j. Every connected component of Γ_A must be a path (of length ≥ 0) or a cycle (of even length ≥ 2). We have the following number of possibilities for a component with i vertices among the x_k's and j among the y_k's:

$$\begin{aligned}
(i,j) = (0,1) \text{ or } (1,0):&\quad 1\\
(1,1):&\quad 2\\
(i,i+1) \text{ or } (i+1,i):&\quad \frac{1}{2}i!(i+1)!,\quad i\geq 1\\
(i,i):&\quad i!^2 + \frac{1}{2}(i-1)!i!,\quad i\geq 2\\
\text{all others}:&\quad 0.
\end{aligned}$$

Hence

$$F(x,y) = \exp\left[x + y + 2xy + \frac{1}{2}\sum_{i\geq 1}(x^i y^{i+1} + x^{i+1}y^i)\right.$$
$$\left. + \sum_{i\geq 2}\left(1 + \frac{1}{2i}\right)x^i y^i\right],$$

which simplifies to the right-hand side of (5.121).

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(b) For any power series $G(x,y) = \sum c_{mn}x^m y^n$, let $\mathcal{D}G(x,y) = \sum c_{nn}t^n$. The operator $\mathcal{D}$ preserves infinite linear combinations; and if $G(x,y) = H(x,y,xy)$ for some function H, then $\mathcal{D}G(x,y) = \mathcal{D}H(x,y,t)$. Hence

$$\begin{aligned}\sum_{n\geq 0} f(n,n)\frac{t^n}{n!^2} &= \mathcal{D}F(x,y)\\ &= (1-t)^{-\frac{1}{2}} e^{\frac{t(3-t)}{2(1-t)}}\mathcal{D}\exp\left[\frac{(x+y)(1-\frac{1}{2}t)}{1-t}\right].\end{aligned}$$

But

$$\begin{aligned}\mathcal{D}\exp\left[\frac{(x+y)(1-\frac{1}{2}t)}{1-t}\right] &= \mathcal{D}\sum_{n\geq 0}\frac{(x+y)^n}{n!}\left(\frac{1-\frac{1}{2}t}{1-t}\right)^n\\ &= \sum_{n\geq 0}\binom{2n}{n}\frac{t^n}{(2n)!}\left(\frac{1-\frac{1}{2}t}{1-t}\right)^{2n},\end{aligned}$$

and the proof follows.

66. (a) If $r \neq s$ then the matrix $\boldsymbol{L} - r\boldsymbol{I}$ has s equal rows and hence has rank at most $r+1$. Thus $\boldsymbol{L}$ has at least $s-1$ eigenvalues equal to r. If $r = s$ then another r rows of $\boldsymbol{L} - r\boldsymbol{I}$ are equal, so $\boldsymbol{L}$ has at least $r+s-1$ eigenvalues equal to r.

(b) By symmetry, $\boldsymbol{L}$ has at least $r-1$ eigenvalues equal to s.

(c) Since the rows of $\boldsymbol{L}$ sum to 0, there is at least one 0 eigenvalue. The trace of $\boldsymbol{L}$ is $2rs$. Since this is the sum of the eigenvalues, the remaining eigenvalue must be $2rs - (s-1)r - (r-1)s = r+s$.

(d) By the Matrix-Tree Theorem (Theorem 5.6.8) we have

$$c(K_{rs}) = \frac{1}{r+s}(r+s)r^{s-1}s^{r-1} = r^{s-1}s^{r-1},$$

agreeing with Exercise 30.

67. (a) For each edge $e = \{i,j\}$ associate an indeterminate $x_{ij} = x_{ji}$. Let $\boldsymbol{L} = \left(L_{ij}\right)$ be the $n \times n$ matrix

$$L_{ij} = \begin{cases} -x_{ij}, & \text{if } i \neq j\\ \sum_{\substack{1\leq k\leq n\\ k\neq i}} x_{ik}, & \text{if } i = j,\end{cases}$$

Let $\boldsymbol{L_0}$ denote $\boldsymbol{L}$ with the last row and column removed. By the Matrix-Tree Theorem (Theorem 5.5.6.8), we have

$$\sum_T f(T) = \det \boldsymbol{L_0}(f),$$

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where $\boldsymbol{L_0}(f)$ is obtained from $\boldsymbol{L_0}$ by substituting $f(e)$ for x_e. Since the (i,i)-entry of $\boldsymbol{L_0}$ has the form $x_{in}+$ other terms, and since x_{in} appears nowhere else in $\boldsymbol{L_0}$, it follows that we can replace the (i,i)-entry of $\boldsymbol{L_0}$ with a new indeterminate y_i without affecting the distribution of values of $\det \boldsymbol{L_0}$. Hence $P_n(q)$ is just the number of invertible $(n-1)\times(n-1)$ symmetric matrices over $\mathbb{F}_q$, whose number is given by Exercise 1.198.

(b) It is easy to see that as G ranges over all simple graphs on the vertex set $[n]$, the reduced Laplacian matrices $\boldsymbol{L_0}(G)$ range over all symmetric $(n-1)\times(n-1)$ $(0,1)$-matrices. Since the determinant of a square matrix A over $\mathbb{F}_2$ is equal to 1 if and only if A is invertible, it follows from (a) that the desired number of graphs is $P_n(2)$. The present exercise appears also in Chapter 9, Exercise 13(b), of R. Stanley, *Algebraic Combinatorics*, second ed., Springer, New York, 2018.

This exercise is related to an unpublished question raised by M. Kontsevich. For further information see R. Stanley, *Ann. Comb.* **2** (1998), 351–363; J. R. Stembridge, *Ann. Comb.* **2** (1998), 365–385; and P. Belkale and P. Brosnan, *Duke Math. J.* **116** (2003), 147–188.

68. The argument parallels that of Example 5.6.10. Let V be the vector space of all functions $f:\Gamma\to\mathbb{C}$. Define a linear transformation $\Phi: V\to V$ by

$$(\Phi f)(u)=\sum_{v\in\Gamma}\sigma(v)f(u+v).$$

It is easy to check that the characters $\chi\in\hat{\Gamma}$ are the eigenvectors of Φ, with eigenvalue $\sum_{v\in\Gamma}\sigma(v)\chi(v)$. Moreover, the matrix of Φ with respect to the basis Γ of V is just

$$[\Phi]=\left(\sum_{v\in\Gamma}\sigma(v)\right)\cdot\boldsymbol{I}-\boldsymbol{L}(D).$$

Hence the eigenvalues of $\boldsymbol{L}(D)$ are given by $\sum_{v\in\Gamma}\sigma(v)(1-\chi(v))$ for $\chi\in\hat{\Gamma}$, and the proof follows from Corollary 5.6.6. Note that Example 5.6.10 corresponds to the case $\Gamma=(\mathbb{Z}/2\mathbb{Z})^n$ and σ given by $\sigma(v)=1$ if v is a unit vector, while $\sigma(v)=0$ otherwise.

69. (a) Easily seen that $\tau(D,v)=a_1a_2\cdots a_{p-1}$.

(b) D is connected and balanced, and the oudegree of vertex v_i is $a_{i-1}+a_i$ (with $a_0=a_p=0$). Hence by Theorem 5.6.2,

$$\epsilon(D,e)=a_1a_2\cdots a_{p-1}\prod_{i=1}^{p-1}(a_{i-1}+a_i-1)!.$$

70. The argument is completely parallel to that used to prove Corollary 5.6.15. The digraph D_n becomes the graph with vertex set $[0, d-1]^{n-1}$ and edges $(a_1a_2\cdots a_{n-1}, a_2a_3\cdots a_n)$, yielding the answer $d!^{d^{n-1}}d^{-n}$. This result seems to have been first obtained in [333].
71. Let d be the degree of the vertices of G. First note that the number q of edges of G is given by $W(2) = 2q$ (since G has no loops or multiple edges). Now $2q = dp$ where p is the number of vertices of G, so p is determined as well. It is easy to see that the numbers λ_j satisfying (5.122) for all $\ell \geq 1$ are unique (consider e.g. the generating function $\sum_{\ell\geq 1} W(\ell)x^\ell$), and hence by the proof of Corollary 4.7.3 are the nonzero eigenvalues of the adjacency matrix $\boldsymbol{A}$ of G. Since $\boldsymbol{A}$ has p eigenvalues in all, it follows that $p-m$ of them are equal to 0. A number of arguments are available to show that the largest eigenvalue λ_1 is equal to d. Since G is regular, the eigenvalues of the Laplacian matrix $\boldsymbol{L}$ of G are the numbers $d - \lambda_j$, together with the eigenvalue d of multiplicity $p-m$. Hence by the Matrix-Tree Theorem (Theorem 5.6.8),
$$c(G) = \frac{\lambda_1^{p-m}}{p}\prod_{j=2}^{m}(\lambda_1 - \lambda_j).$$
72. There is a standard bijection $T \mapsto T^*$ between the spanning trees T of G and those of G^*, namely, if T has edge set $\{e_1, \ldots, e_r\}$, then T^* has edge set $E^* - \{e_1^*, \ldots, e_r^*\}$, where E^* denotes the edge set of G^*. Hence $c(G) = c(G^*)$. Let $\boldsymbol{L}_0(G^*)$ denote $\boldsymbol{L}(G^*)$ with the row and column indexed by the outside vertex deleted. It is easy to see that $\boldsymbol{L}_0(G^*) = 4\boldsymbol{I} - \boldsymbol{A}(G')$, and the proof follows from Theorem 5.6.8.

    This result is due to D. Cvetković and I. Gutman, *Publ. Inst. Math. (Beograd)* **29** (1981), 49–52. They give an obvious generalization to planar graphs all of whose bounded regions have the same number of boundary edges. See also D. Cvetković, M. Doob, I. Gutman, and A. Torgašev, *Recent Results in the Theory of Graph Spectra*, Annals of Discrete Mathematics **36**, North-Holland, Amsterdam, 1988 (Theorem 3.34). For some related work, see T. Chow, *Proc. Amer. Math. Soc.* **125** (1997), 3155–3161; M. Ciucu, *J. Combinatorial Theory (A)* **81** (1998), 34–68; and D. E. Knuth, *J. Alg. Combinatorics* **6** (1997), 253–257.
74. (a) Let $\boldsymbol{J}$ be the $p \times p$ matrix of all 1's. As in the proof of Lemma 5.6.14 we have that $\boldsymbol{A}^\ell = \boldsymbol{J}$ and that the eigenvalues of $\boldsymbol{A}$ are $p^{1/\ell}$ (once) and 0 ($p-1$ times). (Note that since $\operatorname{tr}\boldsymbol{A}$ is an integer, it follows that $p = r^\ell$ for some $r \in \mathbb{P}$. Part (d) of this exercise gives a more precise result.)

    (b) The number of loops is $\operatorname{tr}\boldsymbol{A} = r$, where $p = r^\ell$ as above.

(c) Since by hypothesis there is a walk between any two vertices of D, it follows that D is connected. Since $\boldsymbol{A}$ has a unique eigenvalue equal to r, there is a unique corresponding eigenvector E (up to multiplication by a nonzero scalar). Since E is also an eigenvector of $\boldsymbol{A}^\ell = \boldsymbol{J}$ with eigenvalue $r^\ell = p$, it follows that E is the (column) vector of all 1's. The equation $\boldsymbol{A}E = rE$ shows that every vertex of D has outdegree r. If we take the transpose of both sides of the equation $\boldsymbol{A}^\ell = \boldsymbol{J}$, then we get $\left(\boldsymbol{A}^t\right)^\ell = \boldsymbol{J}$. Thus the same reasoning shows that $\boldsymbol{A}^t E = rE$, so every vertex of D has indegree r.

(d) The above argument shows that $r = d$ (or $p = d^\ell$).

(e) Since every vertex of D has outdegree r, we have $\boldsymbol{L} = r\boldsymbol{I} - \boldsymbol{A}$. Hence by (a) the eigenvalues of $\boldsymbol{L}$ are r ($p-1$ times) and 0 (once). It follows from Corollary 5.6.7 that

$$\begin{aligned}\epsilon(D,e) &= \frac{1}{p} r^{p-1}(r-1)!^p \\ &= r^{-(\ell+1)} r!^{r^\ell}.\end{aligned}$$

The *total* number of Eulerian tours is just

$$\epsilon(D) = rp \cdot \epsilon(D,e) = r!^{r^\ell}.$$

(f) We want to find all $p \times p$ matrices $\boldsymbol{A}$ of nonnegative integers such that $\boldsymbol{A}^\ell = \boldsymbol{J}$. If we ignore the hypothesis that the entries of $\boldsymbol{A}$ are nonnegative integers, then a simple linear algebra argument shows that $\boldsymbol{A} = r^{-\ell+1}\boldsymbol{J} + \boldsymbol{N}$ where $\boldsymbol{N}^\ell = \boldsymbol{0}$ and $\boldsymbol{NJ} = \boldsymbol{JN} = \boldsymbol{0}$. Equivalently, if $\boldsymbol{e_i}$ denotes the ith unit coordinate vector, then $\boldsymbol{N}^\ell = \boldsymbol{0}$, $\boldsymbol{N}(\boldsymbol{e_1} + \cdots + \boldsymbol{e_p}) = 0$, and the space of all vectors $a_1\boldsymbol{e_1} + \cdots + a_p\boldsymbol{e_p}$ with $\sum a_i = 0$ is $\boldsymbol{N}$-invariant. Conversely, for any such $\boldsymbol{N}$ the matrix $\boldsymbol{A} = r^{-\ell+1}\boldsymbol{J} + \boldsymbol{N}$ satisfies $\boldsymbol{A}^\ell = \boldsymbol{J}$. If we choose $\boldsymbol{N}$ to have integer entries and let c be a large enough integer so that the matrix $\boldsymbol{B} = c\boldsymbol{J} + \boldsymbol{N}$ has nonnegative entries, then $\boldsymbol{B}$ will be the adjacency matrix of a digraph with the *same* number of paths (not necessarily just one path) of length ℓ between any two vertices. For instance, let $p = 3$ and (writing column vectors as row vectors for simplicity) define $\boldsymbol{N}$ by $\boldsymbol{N}[1,1,1] = [0,0,0]$, $\boldsymbol{N}[1,-1,0] = [2,-1,-1]$, and $\boldsymbol{N}[2,-1,-1] = [0,0,0]$. Then

$$2\boldsymbol{J} + \boldsymbol{N} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 4 & 1 & 1 \end{bmatrix},$$

and $(2\boldsymbol{J} + \boldsymbol{N})^2 = 12\boldsymbol{J}$. Hence $2\boldsymbol{J} + \boldsymbol{N}$ is the adjacency matrix of a digraph with 12 paths of length two between any two vertices. It is more difficult to obtain a digraph, other than the de Bruijn graphs, with a *unique* path of length ℓ between two vertices, but such examples were given by M. Capalbo and H. Fredricksen (independently). The adjacency matrix of Capalbo's example (with a unique path of length two between any two vertices) is the following:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

For further information, see F. Curtis, J. Drew, C.-K. Li, and D. Pragel, *J. Combin. Theory Ser. A* **105** (2004), 35–50, and the references therein.