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GLOBAL POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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1. Introduction. The semilinear elliptic boundary value problem

$$(1.1)\qquad \begin{cases} Lu + f(x, u) = 0, & x \in \Omega \\ u(x) = g(x), & x \in \partial\Omega \end{cases}$$

will be considered in an exterior domain $\Omega \subset \mathbf{R}^n$, $n \geqq 2$, with boundary $\partial\Omega \in C^{2+\alpha}$, $0 < \alpha < 1$, where

$$(1.2)\qquad Lu = \sum_{i,j=1}^{n} a_{ij}(x)D_iD_ju + \sum_{i=1}^{n} b_i(x)D_iu, \quad x \in \Omega,$$

$D_i = \partial/\partial x_i$, $i = 1, \ldots, n$. The coefficients a_{ij}, b_i in (1.2) are assumed to be real-valued functions defined in $\Omega \cup \partial\Omega$ such that each $a_{ij} \in C^{\alpha}_{\mathrm{loc}}(\Omega)$, $b_i \in C^{\alpha}_{\mathrm{loc}}(\Omega)$, and $(a_{ij}(x))$ is uniformly positive definite in every bounded domain in Ω. The Hölder exponent α is understood to be fixed throughout, $0 < \alpha < 1$. The regularity hypotheses on f and g are stated as H1 near the beginning of Section 2.

Special attention will be directed toward the case of null boundary data in (1.1). This boundary value problem is entered here separately for convenience:

$$(1.3)\qquad \begin{cases} Lu + f(x, u) = 0, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Our main objective is to prove, under suitable hypotheses, the existence of infinitely many classical solutions of (1.3) which are positive throughout the entire domain Ω. In the case that the differential equation in (1.3) is the Schrödinger equation

$$(1.4)\qquad \Delta u + f(x, u) = 0, \quad x \in \Omega,$$

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specific conditions on f will be given which guarantee the existence of global positive solutions. The main results of this type are Theorems 4.1 and 4.3 (superlinear case) and Theorem 5.2 (sublinear case). The methods in the two cases are completely different, and require separate preliminary theorems in Sections 2 and 3. Such results are not the same as nonoscillation theorems for (1.4), i.e., criteria ensuring that (1.4) has a positive solution in $\Omega \cap \{x \in \mathbf{R}^n: \ |x| > t\}$ for some $t > 0$. (See [**6**], [**7**].)

A related problem is to prove the existence of a positive solution $u(x)$ of (1.4) throughout an arbitrary exterior domain Ω such that $u(x)$ is bounded, and in dimension $n \geqq 3$, $\lim_{|x|\to\infty} u(x) = 0$. Theorem 4.4 is a result of this type, under weaker requirements on $f(x, u)$ than before.

The preparatory existence theorems based on subsolutions and supersolutions of (1.1) are stated in Section 2 under varying hypotheses, as appropriate for the applications in Sections 4 and 5. Some preliminary results for nonlinear ordinary differential equations in semi-infinite intervals are developed in Section 3. These results are then used in Sections 4 and 5 to generate functions v and w satisfying the hypotheses of Theorems 2.3, 2.7, and 2.9. In the case of a superlinear ordinary differential equation, Theorem 3.1 gives sufficient conditions for a solution of the initial value problem (3.2) to exist and be positive in the entire half-axis of its definition, and also gives estimates for this solution. In the sublinear case, Theorem 3.5 states criteria for equation (3.1) to have a bounded positive solution in $[t_0, \infty)$ for any t_0 for which the differential equation is defined.

Our basic plan is to construct a solution $u(x)$ of (1.1) or (1.3) which is squeezed between a subsolution $w(x)$ and a supersolution $v(x)$. We require such functions v and w which satisfy (2.1), or similar inequalities, and $0 \leqq w(x) \leqq v(x)$ throughout Ω. In general the choice $w(x) = 0$ identically in Ω is ineffective since the conclusion $w(x) \leqq u(x) \leqq v(x)$ of our theorems would not preclude the trivial solution $u(x)$ of (1.3); and even if that were ruled out by limiting the class of functions f, a positive solution $u(x)$ throughout Ω would not be guaranteed. Accordingly we must construct subsolutions $w(x)$ and supersolutions $v(x)$ which are nontrivial. The three main problems that we solve are listed below.

(1) The superlinear case with null boundary condition. For a spherical boundary, we can choose both v and w to be radial functions vanishing identically on the boundary. This is accomplished via Theorem 3.1. For an arbitrary boundary $\partial\Omega \in C^{2+\alpha}$, this approach fails, but the modification presented in Theorems 2.7 and 4.3 solves the problem, again by an appeal to the ODE Theorem 3.1.

(2) The sublinear case with null boundary condition. Now the superlinear Theorem 3.1 is not applicable and we need a sublinear analogue to construct a supersolution v of (1.3); our sublinear Theorem 3.5 is such a result, proved by a fixed point argument. However, the function v constructed in this way is bounded whereas the harmonic function w used in problem (1) is unbounded, and so the requirement that $w(x) \leqq v(x)$ throughout Ω would necessarily fail for such a $w(x)$. Instead a function w is defined in a fixed nonempty bounded subdomain $R \subset \Omega$ as a positive eigenfunction of a linear eigenvalue problem in R, as guaranteed by the Krein-Rutman theorem [**2**]. The sublinear conclusions in Theorem 5.2 and Corollary 5.3 then follow from Theorem 2.9.

(3) The superlinear case without boundary condition. Now we can use $w(x) = 0$ identically in Ω to prove in Theorem 4.4 the existence of a bounded positive solution $u(x)$ in Ω, and $u(x) \to 0$ as $|x| \to \infty$ in dimensions $n \geqq 3$, under weaker hypotheses on $f(x, u)$ than in problems (1) and (2). This is accomplished by application of a non-linear ODE theorem of Nehari to construct $v(x)$, and use of the maximum principle to guarantee the positivity of the solution $u(x)$ throughout Ω.

The approach to problem (3) fails in problem (1), and so the conclusions in (1) could not be thereby strengthened, since $v(x) = 0$ identically on $\partial\Omega$ would be needed and the conclusion $0 = w(x) \leqq u(x) \leqq v(x)$ would not ensure that $u(x)$ is positive in Ω. Accordingly it cannot be proved that there exists a positive solution of the boundary value problem (4.13) which is bounded in $\Omega \subset \mathbf{R}^2$, or has limit zero at ∞ in $\Omega \subset \mathbf{R}^n$, $n \geqq 3$: These are still open questions.

2. Existence of solutions of (1.1). Let $|x|$ denote the Euclidean norm of a point x in real n-space $\mathbf{R}^n$. Define

$$\begin{aligned} S_t &= \{x \in \mathbf{R}^n: |x| = t\}, && t > 0\,; \\ G_t &= \{x \in \mathbf{R}^n: |x| > t\}, && t > 0\,; \\ G_{s,t} &= \{x \in \mathbf{R}^n: s < |x| < t\}, && 0 < s < t\,; \\ \Omega_t &= \{x \in \Omega: |x| < t\}, && t > 0\,, \end{aligned}$$

where Ω denotes an exterior domain in $\mathbf{R}^n$, i.e., $G_a \subset \Omega$ for some positive number a, fixed throughout. For $t > a$, evidently Ω_t is a bounded subdomain of Ω containing in particular all $x \in G_{a,\,t}$, and $\partial\Omega_t = S_t \cup \partial\Omega$.

For any bounded domain $M \subset \mathbf{R}^n$, the norm of a function $u:\bar{M} \to \mathbf{R}^1$ in the Hölder space $C^{m+\alpha}(\bar{M})$ will be denoted by $\|u\|_{m+\alpha,\bar{M}}$, $0 < \alpha < 1$, $m = 0, 1, 2, \ldots$. Also the abbreviation below will be used:

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$$||u||_{m+\alpha,j} = ||u||_{m+\alpha,\bar{\Omega}_{a+j}}, \quad j = 1, 2, \dots .$$

The notation $C_{\text{loc}}^{m+\alpha}(\Omega)$ will denote the set of all functions $u: \Omega \to \mathbf{R}^1$ such that $u \in C^{m+\alpha}(\bar{M})$ for every bounded domain $M \subset \Omega$. The notation $C_{\text{loc}}^{m+\alpha}(\Omega \times \mathbf{R}^1)$ is defined similarly.

The functions f and g in (1.1) are required to satisfy the hypothesis H1 below throughout the sequel, and also H2 for some of our results:

H1. $f \in C_{\text{loc}}^{\alpha}(\Omega \times \mathbf{R}^1)$; $g \in C^{2+\alpha}(\overline{\partial\Omega})$.

H2. For every bounded domain $M \subset \Omega$ and for every $T > 0$, there exists a positive number K, depending on M and T, such that $f(x, t) + Kt$ is nondecreasing on $0 \leqq t \leqq T$ for all $x \in \bar{M}$.

H2 holds, for example, if (i) $f(x, t)$ is nondecreasing with respect to t in $0 \leqq t < \infty$, for each fixed $x \in \Omega$, or if (ii) $f(x, t)$ is continuously differentiable with respect to t at every (x, t), as is easily verified. Other hypotheses will be added in Sections 3-5 as required.

A *solution* of (1.1) is understood to be a function $u \in C_{\text{loc}}^{2+\alpha}(\Omega)$ such that u satisfies (1.1) identically.

LEMMA 2.1. *Let L, Ω, α, and a be fixed as described above and suppose that $f(x, u)$ and $g(x)$ satisfy* H1. *If nonnegative functions v and w exist in $\Omega \cup \partial\Omega$ such that $v, w \in C_{\text{loc}}^{2+\alpha}(\Omega)$, $w(x) \leqq v(x)$ throughout Ω, and*

$$\begin{aligned} &Lv + f(x, v) \leqq 0 \ \text{in } \Omega, \quad v \geqq g \ \text{on } \partial\Omega \\ &Lw + f(x, w) \geqq 0 \ \text{in } \Omega, \quad w \leqq g \ \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

then there exists a sequence of functions u_j in $\Omega \cup \partial\Omega$ with the following properties:

(A) $u_j \in C^{2+\alpha}(\bar{\Omega}_{a+j})$;

(B) $u_j(x) = g(x)$ *if* $x \in \partial\Omega$;

(C) $u_j(x) = v(x)$ *if* $|x| \geqq a + j$;

(D) $Lu_j + f(x, u_j(x)) = 0$ *for all* $x \in \Omega_{a+j}$;

(E) $w(x) \leqq u_j(x) \leqq v(x)$ *for all* $x \in \Omega \cup \partial\Omega$;

(F) $u_{j+1}(x) \leqq u_j(x)$ *for all* $x \in \Omega \cup \partial\Omega$, $j = 1, 2, \dots .$

The proof is based on a theorem of Amann [**1**, p. 283] applied to the bounded domains Ω_{a+j}, and is similar to the proof given by Noussair in [**5**].

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LEMMA 2.2. *Let* $\{u_j\}$ *be the sequence in Lemma* 2.1. *For every positive integer i there exists a positive constant* $K_0 = K_0(i)$, *independent of j, such that*

$$\|u_j\|_{2+\alpha,i} \leqq K_0 \quad \text{for all } j \geqq i.$$

This key lemma is proved from Schauder *a priori* estimates, L^p estimates, and Sobolev embedding. The technique is indicated in the authors' articles [**5**, **6**].

THEOREM 2.3. *Let L be the elliptic operator* (1.2), *let* Ω *and* α *be as described in Section* 1, *and suppose that* $f(x, u)$ *and* $g(x)$ satisfy H1. *If nonnegative functions v and w in* $C_{\text{loc}}^{2+\alpha}(\Omega)$ *exist satisfying* (2.1) *in* $\Omega \cup \partial\Omega$ *and* $w(x) \leqq v(x)$ *throughout* Ω, *then the boundary value problem* (1.1) *has a solution* $u^*(x)$ *such that* $w(x) \leqq u^*(x) \leqq v(x)$ *throughout* $\Omega \cup \partial\Omega$. *Furthermore,* $u^*(x)$ *is maximal in the sense that* $u^*(x) \geqq u(x)$ *in* $\Omega \cup \partial\Omega$ *for any solution of* (1.1) *satisfying* $w(x) \leqq u(x) \leqq v(x)$ *in* $\Omega \cup \partial\Omega$.

Proof. Since the sequence $\{u_j : j \geqq 1\}$ constructed in Lemma 2.1 is bounded in the norm $\| \;\|_{2+\alpha,1}$ by Lemma 2.2, the compactness of the injection $C^{2+\alpha}(\bar{\Omega}_{a+1}) \to C^2(\bar{\Omega}_{a+1})$ implies that $\{u_j : j \geqq 1\}$ has a subsequence $\{u^1_j\}$ which converges in the norm $\| \;\|_{2,1}$ to a function u^1 on $\bar{\Omega}_{a+1}$. From property (B) of Lemma 2.1, u^1 satisfies the boundary condition $u^1(x) = g(x)$ on $\partial\Omega$. Since $\{u_j : j \geqq i\}$ is bounded in the norm $\| \;\|_{2+\alpha,i}$ by Lemma 2.2 and the injection $C^{2+\alpha}(\bar{\Omega}_{a+i}) \to C^2(\bar{\Omega}_{a+i})$ is compact, a subsequence $\{u^i_j : j \geqq i\}$ exists for each $i = 1, 2, \ldots$ which converges in the norm $\| \;\|_{2,i}$ to a function $u^i \in C^2(\bar{\Omega}_{a+i})$. This is accomplished inductively by choosing $\{u^i_j\}$ as a convergent subsequence of $\{u^{i-1}_j\}$ in the $C^2(\bar{\Omega}_{a+i})$ norm, $i = 2, 3, \ldots$. Evidently $\bar{\Omega}_{a+i} \subset \bar{\Omega}_{a+i+1}$ and $u^{i+1} = u^i$ on $\bar{\Omega}_{a+i}$ for each $i = 1, 2, \ldots$, and hence a function u^* in the entire domain $\bar{\Omega}$ can be defined by $u^*(x) = u^i(x)$ if $x \in \bar{\Omega}_{a+i}$, $i = 1, 2, \ldots$.

It will now be proved that $u^*(x)$ is the required solution of (1.1). For any bounded domain $M \subset \Omega$, $\bar{M} \subset \bar{\Omega}_{a+i}$ for some integer i, and hence the diagonal sequence $\{u^j_j\}$ converges in the $C^2(\bar{M})$ norm to $u^i = u^*$ on $\bar{M}$. In particular u^j_j and Lu^j_j converge uniformly on $\bar{M}$ to u^* and Lu^*, respectively. Since $Lu^j_j = -f(x, u^j_j)$ for $x \in \bar{M}$ by property (D) of Lemma 2.1 it follows that u^* is a solution of (1.1) in $C^2(\bar{M})$, and hence $u^* \in C^{2+\alpha}(\bar{M})$ by a standard regularity argument based on Schauder estimates. By Lemma 2.1, $w(x) \leqq u^j_j(x) \leqq v(x)$ for each $j = 1, 2, \ldots$, and hence the solution $u^*(x)$ also satisfies $w(x) \leqq u^*(x) \leqq v(x)$ throughout $\Omega \cup \partial\Omega$.

The maximality of $u^*(x)$ can be proved from Amann's theorem [**1**] by an argument similar to that in [**5**].

We now consider the boundary value problem (1.3), with null boundary data, in the case that a nonnegative function w satisfying $Lw + f(x, w(x)) \geqq 0$ is known to exist only in the subdomain G_a of Ω, where $a > 0$ is fixed as before. Lemma 2.1 cannot be applied since the second condition (2.1) does not hold. Let $w_0(x)$ denote the extension of $w(x)$ to all of Ω defined by

$$(2.2) \qquad w_0(x) = \begin{cases} w(x) & \text{if } x \in G_a \\ 0 & \text{if } x \in \bar{\Omega}_a . \end{cases}$$

LEMMA 2.4. *Let L, Ω, α, and a be as above and suppose that $f(x, u)$ satisfies* H1 *and* H2. *If there exist nonnegative functions $v \in C_{\text{loc}}^{2+\alpha}(\Omega)$, $w \in C_{\text{loc}}^{2+\alpha}(G_a)$ such that*

$$(2.3) \qquad \begin{aligned} &Lv + f(x, v(x)) \leqq 0 \textit{ in } \Omega, && v \geqq 0 \textit{ on } \partial\Omega \\ &Lw + f(x, w(x)) \geqq 0 \textit{ in } G_a, && w = 0 \textit{ on } S_a \end{aligned}$$

and $w(x) \leqq v(x)$ throughout $G_a \cup S_a$, then for each $j = 1, 2, \ldots$ there exists a function $w_j \in C^{2+\alpha}(\bar{\Omega}_{a+j})$ satisfying

$$Lw_j + f(x, w_j(x)) \geqq 0 \quad \textit{in } \Omega_{a+j}, \quad w_j = 0 \quad \textit{on } \partial\Omega$$

and $w_0(x) \leqq w_j(x) \leqq v(x)$ throughout $\bar{\Omega}_{a+j}$.

Sketch of proof. By H2 there exists a positive number $K = K_j$ such that $f(x, t) + K_j t$ is nondecreasing on $0 \leqq t \leqq T_j$ for all $x \in \bar{\Omega}_{a+j}$, where

$$T_j = \sup_{x \in \bar{\Omega}_{a+j}} v(x).$$

Let $w_j(x)$ be the unique solution of the linear boundary value problem

$$\begin{cases} (L - K_j)w_j + f(x, w_0(x)) + K_j w_0(x) = 0, & x \in \Omega_{a+j} \\ w_j(x) = 0, & x \in \partial\Omega \\ w_j(x) = w(x), & x \in S_{a+j} \end{cases}$$

where $w_0(x)$ is given by (2.2). It can then be verified by use of Schauder estimates and the maximum principle that $w_j(x)$ satisfies all the conditions of Lemma 2.4.

LEMMA 2.5. *Under the hypotheses of Lemma* 2.4, *there exists a sequence of functions u_j in $\Omega \cup \partial\Omega$ with properties* (A), (B), (C), (D) *of Lemma* 2.1, *with $g(x) = 0$ in* (B), *and*

(E′) $\quad w_0(x) \leqq w_j(x) \leqq u_j(x) \leqq v(x) \quad$ *for all $x \in \bar{\Omega}_{a+j}$*, $j = 1, 2, \ldots .$

LEMMA 2.6. *Let $\{u_j\}$ be the sequence in Lemma 2.5 under the same hypotheses. Then, for every positive integer i there corresponds a positive*

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constant $K_0 = K_0(i)$, *independent of* j, *such that*

$$\|u_j\|_{2+\alpha,i} \leqq K_0 \quad \text{for all } j \geqq i.$$

These lemmas are proved in the same way as Lemmas 2.1 and 2.2.

THEOREM 2.7. *Let* L, Ω, α, *and* a *be as above and suppose that* $f(x, u)$ *satisfies* H1 *and* H2. *Suppose there exist functions* $v \in C^{2+\alpha}_{\mathrm{loc}}(\Omega)$, $w \in C^{2+\alpha}_{\mathrm{loc}}(G_a)$ *satisfying* (2.3) *such that* $0 \leqq w(x) \leqq v(x)$ *throughout* $G_a \cup S_a$. *Then the boundary value problem* (1.3) *has a solution* $u(x)$ *satisfying* $w_0(x) \leqq u(x) \leqq v(x)$ *throughout* $\Omega \cup \partial\Omega$.

Proof. The proof of Theorem 2.3 needs to be altered only by replacing Lemmas 2.1 and 2.2 by Lemmas 2.5 and 2.6, respectively.

In a third case, to be applied to sublinear boundary value problems in the sequel, a nonnegative function $w(x)$ satisfying $Lw + f(x, w) \geqq 0$ is known to exist only in the closure of a fixed nonempty bounded domain $R \subset \Omega$, where $\partial R \in C^{2+\alpha}$. Also, we assume that $w(x) = 0$ identically on ∂R, and suppose that a positive number $A \geqq a$ has been selected so that $R \subset \Omega_A$.

Let w_0 denote the extension of $w(x)$ to $\mathbf{R}^n$ defined by $w_0(x) = 0$ for $x \notin R$. Since restrictions of w_0 to bounded domains Ω_{A+j} will not serve as subsolutions, because the required regularity fails on ∂R, we need the following lemma to obtain an analogue of Theorem 2.3. The proof is similar to that of Lemma 2.4 and will be omitted.

LEMMA 2.8. *Let* L, Ω, R, α, *and* A *be as above, and suppose that* $f(x, u)$ *satisfies* H1 *and* H2. *If there exist functions* $v \in C^{2+\alpha}_{\mathrm{loc}}(\Omega)$ *and* $w \in C^{2+\alpha}(\bar{R})$ *such that* $0 \leqq w(x) \leqq v(x)$ *throughout* $\bar{R}$, *and*

$$\text{(2.4)} \qquad \begin{aligned} &Lv + f(x, v) \leqq 0 \quad \text{in } \Omega, \qquad v \geqq 0 \text{ on } \partial\Omega \\ &Lw + f(x, w) \geqq 0 \quad \text{in } R, \quad\ w = 0 \text{ on } \partial R \end{aligned}$$

then there exists a sequence of functions $w_j \in C^{2+\alpha}(\Omega_{A+j})$ *satisfying*

$$Lw_j + f(x, w_j) \geqq 0 \quad \text{in } \Omega_{A+j}, \quad w_j = 0 \quad \text{on } \partial\Omega$$

such that $w_0(x) \leqq w_j(x) \leqq v(x)$ *throughout* $\bar{\Omega}_{A+j}$.

THEOREM 2.9. *Under the hypotheses of Lemma* 2.8, *the boundary value problem* (1.3) *has solution* $u(x)$ *satisfying* $0 \leqq w_0(x) \leqq u(x) \leqq v(x)$ *in* $\Omega \cup \partial\Omega$.

We now consider positive solutions of the differential equation

$$\text{(2.5)} \qquad Lu + f(x, u) = 0, \quad x \in G_a$$

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in the case $\Omega = G_a$, without any boundary condition.

LEMMA 2.10. *Let L and α be as in Section 1 and suppose that $f(x, u) \geqq 0$ and satisfies H1. If there exists a positive function v in $G_a \cup S_a$ for some $a > 0$, with $v \in C_{\text{loc}}^{2+\alpha}(G_a)$, satisfying $Lv + f(x, v) \leqq 0$ in G_a, then there exists a sequence of functions u_j in $G_a \cup S_a$ with the following properties*:

(A) $u_j \in C^{2+\alpha}(\bar{G}_{a,a+j})$;

(B) $u_j(x) = v(x)$ *if* $x \in S_a \cup S_{a+j} \cup G_{a+j}$;

(C) $Lu_j + f(x, u_j) = 0$ *in* $G_{a,a+j}$; *and*

(D) $0 \leqq u_j(x) \leqq v(x)$ *in* $G_a \cup S_a$,
$j = 1, 2, \ldots .$

This follows from Lemma 2.1 in the case that $\Omega = G_a$ and $w(x) = 0$ identically in Ω, where the boundary function g on $\partial\Omega = S_a$ is defined as $v|_{\partial\Omega}$.

THEOREM 2.11. *Under the hypotheses of Lemma 2.10, equation (2.5) has a solution $u(x)$ such that $0 < u(x) \leqq v(x)$ throughout $G_a \cup S_a$.*

Proof. The procedure used for Theorem 2.3 shows only, by use of property (D) of Lemma 2.10, that $0 \leqq u(x) \leqq v(x)$. However, $Lu \leqq 0$ in $G_{a,b}$ for arbitrary $b > a$ by (2.5) and the nonnegativity of $f(x, u)$, $u = v > 0$ on S_a, and $u \geqq 0$ on S_b. Hence $u > 0$ throughout $G_{a,b}$ by the maximum principle. Since b is arbitrary, u is a positive solution of (2.5) in G_a.

3. Some properties of ordinary differential equations. In Sections 4 and 5 we want to obtain specific criteria for the existence of positive solutions of (1.3) in the case that $L = \Delta$. In order to be able to construct suitable functions v and w satisfying conditions (2.1), such that $w(x) \leqq v(x)$ throughout Ω, we need information about semilinear ordinary differential equations of the type

$$\text{(3.1)} \qquad \frac{d^2y}{dt^2} + yg(t, y) = 0, \quad 0 < t < \infty$$

under the following hypotheses.

Hypotheses.

H3. $g(t, y)$ is continuous and positive for $0 < t < \infty$, $0 < y < \infty$.

H4. $g(t, y)$ is nondecreasing in y for all $t \in (0, \infty)$.

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H5. $$\lim_{s \to 0+} \int^{\infty} tg(t, st)\, dt = 0.$$

For example, H3-H5 are satisfied in the case $g(t, y) = p(t)y^{\gamma-1}$, $\gamma > 1$, if $p(t)$ is positive and continuous in $(0, \infty)$ and

$$\int^{\infty} t^{\gamma} p(t) dt < \infty.$$

For example, H5 holds if both H5′ and H5″ below are satisfied.

H5′. There exists a positive number ϵ such that $s^{-\epsilon}g(t, s)$ is a strictly increasing function of s in $0 \leqq s < \infty$ for each fixed t in $(0, \infty)$.

H5″. There exists a positive constant C such that

$$\int^{\infty} tg(t, Ct) dt < \infty.$$

In fact, if H5′ and H5″ both hold, then for all $s < C$ and for all $t > 0$ we have

$$(st)^{-\epsilon}g(t, st) < (Ct)^{-\epsilon}g(t, Ct),$$

so

$$0 < tg(t, st) < C^{-\epsilon}s^{\epsilon}tg(t, Ct)$$

and H5 follows from H5″.

The strengthened form H5′ is needed for Nehari's Theorem 3.4 below, and indeed for much of Nehari's paper [**4**].

For arbitrary $t_0 > 0$ and arbitrary $b > 0$, the initial value problem

$$\frac{d^2y}{dt^2} + yg(t, y) = 0, \quad y(t_0) = 0, \quad y'(t_0) = b \tag{3.2}$$

has a unique solution $y(t)$ near $t = t_0$. The theorem below shows that, in fact, this solution is positive globally provided b is sufficiently small.

THEOREM 3.1. *If* H3, H4, *and* H5 *hold, then for arbitrary* $t_0 > 0$ *there exists a number* $b_0 > 0$ *such that the solution* $y(t)$ *of the initial value problem* (3.2) *exists and is positive throughout* (t_0, ∞) *for* $0 < b \leqq b_0$. *Furthermore, for arbitrary* ϵ, $0 < \epsilon < 1$, *there exists* $\delta > 0$ *such that* $y'(t) \geqq (1 - \epsilon)b$ *for all* b *in* $0 < b < \delta$ *and for all* $t \geqq t_0$.

Proof. Because of H3, $yy'' < 0$ if $y \neq 0$, and so the local solution of (3.2) for $t > t_0$ must lie between the t-axis and the tangent line to the graph at any point. It follows that a solution $y(t)$ of (3.2) can be continued to ∞. If

$y(t)$ is nonpositive at any t in (t_0, ∞), there must exist $t_1 > t_0$ such that $y(t_1) = 0$ and $y(t) > 0$ for all t in (t_0, t_1). Since $y(t)$ is concave in (t_0, t_1),

$$y(t) \leqq y'(t_0)(t - t_0) \leqq bt \quad \text{in } [t_0, t_1].$$

Then from (3.2), H4, and H5 in turn, for $t_0 < t < t_1$ and $0 < \epsilon < 1$,

$$\begin{aligned} y'(t_0) &= y'(t) + \int_{t_0}^{t} y(s)g(s, y(s))ds \\ &\leqq y'(t) + b\int_{t_0}^{t} sg(s, bs)ds \qquad (3.3) \\ &\leqq y'(t) + b\,\epsilon \end{aligned}$$

if $0 < b < \delta$, for some $\delta > 0$. Therefore for such b,

$$b \leqq y'(t) + b\,\epsilon \quad \text{or} \quad y'(t) \geqq (1 - \epsilon)\, b > 0.$$

This shows that $y'(t) > 0$ throughout (t_0, t_1), contradicting $y(t_1) = 0$. Therefore $y(t)$ is defined and positive for all $t > t_0$. The last statement in Theorem 3.1 is proved by repeating the argument in (3.3) for any $t > t_0$.

COROLLARY 3.2. *There exists* $\delta > 0$ *such that the solution* $y(t)$ *of* (3.2) *satisfies*

$$0 \leqq \frac{b}{2}\left(t - t_0\right) \leqq y(t) \qquad (3.4)$$

for all $t \geqq t_0$ *and for all* b *in* $0 < b < \delta$.

In fact, corresponding to $\epsilon = 1/2$ in Theorem 3.1, there exists $\delta > 0$ such that $y'(t) \geqq b/2$ for $0 < b < \delta$ and for all $t > t_0$, implying (3.4).

More generally, we have the comparison theorem below comparing the solution $y(t)$ of (3.2) with the solution $z(t)$ of another initial value problem of the same type:

$$\frac{d^2z}{dt^2} + zG(t, z) = 0, \quad z(t_0) = 0, \quad z'(t_0) = b/2. \qquad (3.5)$$

COMPARISON THEOREM 3.3. *Let* y, z *be the solutions of* (3.2), (3.5), *respectively, where* g *and* G *satisfy all the hypotheses* H3, H4, *and* H5. *Then there exists* $\delta > 0$ *such that* $0 < b < \delta$ *implies that*

$$0 \leqq z(t) \leqq y(t) \quad \textit{for all } t \geqq t_0.$$

Proof. Since $z''(t) < 0$, $z'(t) \leqq z'(t_0) = b/2$ for $t \geqq t_0$. Then for $\epsilon = 1/2$ in Theorem 3.1, there exists $\delta > 0$ such that

$$y'(t) - z'(t) \geqq (1 - \epsilon)\, b - \frac{b}{2} = 0$$

for $t \geqq t_0$ and $0 < b < \delta$, from which the conclusion follows.

The theorem below was proved by Nehari [**4**, Theorem IX, p. 120].

THEOREM 3.4. (Nehari). *For arbitrary* $t_1 > 0$, *equation* (3.1) *has a bounded solution* $y(t)$ *in* (t_1, ∞) *with* $y(t_1) = 0$ *and* $y(t) > 0$ *for all* $t > t_1$ *if* H3 *and* H5′ *hold and*

$$\text{(3.6)} \qquad \int^{\infty} tg(t, Ct^{1/2})\, dt < \infty$$

for every positive constant C. In particular, for arbitrary $t_0 > 0$, (3.1) *has a bounded positive solution in* $[t_0, \infty)$ *under these hypotheses.*

We next consider the differential equation (3.1) under the following sublinear hypotheses, in addition to H3:

H6. $g(t, y)$ is nonincreasing in y for all $t \in (0, \infty)$.

H7. For arbitrary $t_0 > 0$, there exist constants B and C, allowed to depend on t_0, such that $B > C > 0$ and

$$\text{(3.7)} \qquad B \int_{t_0}^{\infty} tg(t, C)\, dt < B - C\,.$$

In particular H7 implies that the integral converges. For example, H7 is implied by the following hypothesis:

H7′. There exists a positive number ϵ such that $s^{\epsilon} g(t, s)$ is a nonincreasing function of s in $(0, \infty)$ for each $t \in (0, \infty)$, and $\int^{\infty} tg(t, 1)\, dt < \infty$.

In fact, if H7′ holds, $C > 1$ can be chosen so that

$$\int_{t_0}^{\infty} tg(t, 1)\, dt < \frac{1}{2}\, C^{\epsilon} \quad \text{and} \quad B = 2C.$$

Then H7 holds because

$$B \int_{t_0}^{\infty} tg(t, C)\, dt = \frac{B}{C^{\epsilon}} \int_{t_0}^{\infty} tC^{\epsilon} g(t, C)\, dt$$

$$\leqq \frac{B}{C^{\epsilon}} \int_{t_0}^{\infty} tg(t, 1)\, dt < \frac{B}{2} = B - C\,.$$

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Clearly H7′ holds, hence H7 also holds, in the classical sublinear case

$$g(t, y) = p(t)\, y^{\gamma - 1}, \quad 0 < \gamma < 1,$$

where $p(t)$ is positive and continuous in $(0, \infty)$ and $\int^{\infty} tp(t)\, dt < \infty$. All that is needed is to choose an ϵ in $0 < \epsilon < 1 - \gamma$. Then

$$s^{\epsilon} g(t, s) = s^{\gamma - 1 + \epsilon} p(t)$$

is decreasing in s, and H7′ becomes obvious.

THEOREM 3.5. *If* H3, H6, *and* H7 *hold, then for arbitrary* $t_0 > 0$ *equation* (3.1) *has a bounded positive solution* $y(t)$ *in* $[t_0, \infty)$ *such that* $C \leqq y(t) \leqq B$ *for all* $t \geqq t_0$.

Proof. Let $\mathscr{B}$ denote the Banach space of all real-valued continuous functions y in $[t_0, \infty)$ such that $\|y\| < \infty$, where

$$\|y\| = \sup_{t \geqq t_0} |y(t)|.$$

Consider the closed, bounded, convex subset $\mathscr{S}$ of $\mathscr{B}$ defined by

$$\mathscr{S} = \{ y \in \mathscr{B} : C \leqq y(t) \leqq B \text{ for } t \geqq t_0 \}.$$

Let Φ be the mapping on $\mathscr{S}$ defined by

$$(3.8) \qquad (\Phi y)(t) = B - \int_t^{\infty} (s - t) y(s) g(s, y(s))\, ds, \quad t \geqq t_0.$$

Then Φ maps $\mathscr{S}$ into $\mathscr{S}$ by H6 and H7, for if $y \in \mathscr{S}$,

$$B \geqq (\Phi y)(t) \geqq B - B \int_t^{\infty} s g(s, C)\, ds \geqq C, \quad t \geqq t_0.$$

Let $\{y_n\}$ be a convergent sequence in $\mathscr{S}$; $\lim_{n \to \infty} \|y_n - y\| = 0$, $y \in \mathscr{S}$ since $\mathscr{S}$ is closed. Also

$$|(\Phi y_n - \Phi y)(t)| \leqq \int_t^{\infty} (s - t) | y_n(s) g(s, y_n(s)) - y(s)\, g(s, y(s)) |\, ds.$$

Since the integrand has uniform limit 0 in $[t_0, \infty)$ and is bounded above by $Bsg(s, C)$ for $s \geqq t_0$, Lebesgue's dominated convergence theorem shows, on account of H7, that

$$\lim_{n \to \infty} \|\Phi y_n - \Phi y\| = 0.$$

Therefore $\Phi: \mathscr{S} \to \mathscr{S}$ is continuous. The set of functions $\{\Phi y : y \in \mathscr{S}\}$ is easily seen to be uniformly bounded and equicontinuous by standard arguments, and hence the Ascoli-Arzela theorem shows that Φ maps $\mathscr{S}$

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into a compact subset of $\mathcal{S}$. Then Φ has a fixed point $y(t)$ in $\mathcal{S}$ by the Schauder-Tychonoff fixed point theorem. According to (3.8), $y(t)$ is a solution of the integral equation $y(t) = (\Phi y)(t)$ for $t \geqq t_0$, and standard procedure shows that $y(t)$ has two continuous derivatives and satisfies (3.1).

4. Positive solutions of superlinear Schrödinger equations. We now specialize the boundary value problem (1.3) to the Schrödinger equation (1.4), i.e., $L = \Delta$ in (1.3). In addition to H1 our hypotheses are:

H8. $f(x, u)$ is positive for all $x \in \Omega \cup \partial\Omega$ and for all $u > 0$.

H9. There exists a function $g \in C^{\alpha}_{\text{loc}}(\mathbf{R}_+ \times \mathbf{R}_+)$, where $\mathbf{R}_+ = (0, \infty)$, such that $g(r, u)$ is nondecreasing in u for each $r > 0$, and

$$f(x, u) \leqq ug(|x|, u) \tag{4.1}$$

for all $x \in \Omega \cup \partial\Omega$ and for all $u > 0$.

For example, we can take $g(r, u)$ to be

$$g(r, u) = \sup_{|x| \leqq r, x \in \Omega} f(x, u)/u.$$

In the classical superlinear case $f(x, u) = p(x)u^{\gamma}$, $\gamma > 1$, an appropriate function g is given by

$$g(r, u) = \left[\max_{|x| = r} p(x)\right] u^{\gamma - 1}.$$

For simplicity we specialize Ω to the domain

$$\Omega = \{x \in \mathbf{R}^n : |x| > a\}, \quad a > 0.$$

The boundary value problem (1.3) then becomes

$$\begin{cases} \Delta u + f(x, u) = 0, & |x| > a \\ u(x) = 0, & |x| = a. \end{cases} \tag{4.2}$$

Positive subsolutions $w(x)$ are immediately available in the form of harmonic functions

$$\begin{aligned} w(x) &= A \log\frac{|x|}{a} && \text{if } n = 2 \\ w(x) &= A(a^{2-n} - |x|^{2-n}) && \text{if } n \geqq 3 \end{aligned} \tag{4.3}$$

for $|x| \geqq a$, where A is an arbitrary positive constant to be chosen later. Clearly the second condition (2.1) is satisfied in the present case:

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$$(4.4) \qquad \Delta w + f(x, w) \geqq 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega .$$

Positive supersolutions will be constructed in the form $v(x) = \zeta(r)$ for $r = |x| \geqq a$, where $\zeta \in C^{2+\alpha}[a, b]$ for all $b > a$ and $\zeta(r)$ satisfies the ordinary differential equation

$$(4.5) \qquad r^{1-n} \frac{d}{dr}\left(r^{n-1} \frac{d\zeta}{dr}\right) + \zeta(r) g(r, \zeta(r)) = 0 .$$

If $\zeta(r)$ is a solution of (4.5) and $\zeta(a) = 0$, then (4.1) implies that $v(x) = \zeta(|x|)$ satisfies the first condition (2.1):

$$(4.6) \qquad \Delta v + f(x, v) \leqq 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega .$$

In the case $n = 2$, the change of variables $r = \rho e^t$, $y(t) = \zeta(\rho e^t)$, for a fixed constant $\rho \in (0, a)$, transforms (4.5) to the form (3.1):

$$(4.7) \qquad y''(t) + \rho^2 e^{2t} y(t) g(\rho e^t, y(t)) = 0.$$

Similarly if $n \geqq 3$, the change of variables

$$r = \beta(t) = (\nu t)^\nu, \quad y(t) = t\,\zeta(\beta(t)), \quad \nu = \frac{1}{n-2}$$

transforms (4.5) into

$$(4.8) \qquad y''(t) + t^{-4}[\beta(t)]^{2n-2} y(t) g\left(\beta(t), \frac{y(t)}{t}\right) = 0 .$$

Hypothesis H5 in the case (4.7) is

$$\lim_{s \to 0+} \int^\infty t e^{2t} g(\rho e^t, st) dt = 0, \quad n = 2,$$

which is equivalent to

$$\lim_{s \to 0+} \int^\infty r \log \frac{r}{\rho}\, g\left(r, s\log \frac{r}{\rho}\right) dr = 0, \quad n = 2 .$$

Since $g(r, u)$ is nondecreasing in u by H9, this is implied by

$$(4.9) \qquad \lim_{s \to 0+} \int^\infty r \log r\; g(r, 2s \log r) dr = 0, \quad n = 2 .$$

Similarly H5 in the case (4.8) becomes

$$\lim_{s \to 0+} \int^\infty t^{-3}[\beta(t)]^{2n-2} g(\beta(t), s) dt = 0, \quad n \geqq 3,$$

which is equivalent to

$$(4.10)\quad \lim_{s \to 0+} \int^{\infty} r\, g(r, s)dr = 0, \quad n \geqq 3 .$$

For example, these are satisfied in the case $g(r, u) = p(r)u^{\gamma - 1}$, $\gamma > 1$, $p(r) > 0$, if respectively

$$\int^{\infty} r(\log r)^{\gamma}\, p(r)dr < \infty, \quad n = 2 ;$$

$$\int^{\infty} r\, p(r)dr < \infty, \quad n \geqq 3 .$$

THEOREM 4.1 *Under hypotheses* H1, H8, H9, *and* (4.9) *or* (4.10), *the boundary value problem* (4.2) *has infinitely many solutions which are positive for* $|x| > a$.

Proof. If $n = 2$, let $t_0 = \log(a/\rho)$. Since the present hypotheses imply the hypotheses H3, H4, and H5 of Section 3, Theorem 3.1 and Corollary 3.2 show that equation (4.7) has a solution $y(t)$ in $[t_0, \infty)$ satisfying $y(t_0) = 0$, $y'(t_0) = b$, and

$$0 \leqq \frac{b}{2}(t - t_0) \leqq y(t)$$

throughout (t_0, ∞) if $0 < b < \delta$, for some $\delta > 0$. The corresponding statement for equation (4.5) is that (4.5) has a solution $\zeta(r)$ satisfying $\zeta(a) = 0$, $\zeta'(a) = b/a$, and

$$(4.11)\quad 0 \leqq \frac{b}{2} \log \frac{r}{a} \leqq \zeta(r), \quad r \geqq a ,$$

provided $0 < b < \delta$.

As already noted, the functions v and w defined by

$$v(x) = \zeta(\,|x|\,) \quad \text{and} \quad w(x) = \frac{b}{a} \log \frac{|x|}{a} \quad \text{for } |x| = r \geqq a$$

satisfy (4.4) and (4.6), respectively, and also by (4.11), $0 \leqq w(x) \leqq v(x)$ throughout Ω. Since the hypotheses of Theorem 2.3 are all satisfied, the boundary value problem (4.2) has a solution $u(x)$ satisfying $w(x) \leqq u(x) \leqq v(x)$, i.e.,

$$(4.12)\quad \frac{b}{2} \log \frac{|x|}{a} \leqq u(x) \leqq \zeta(\,|x|\,)$$

for all $|x| \geqq a > 0$. Obviously $u(x) > 0$ for $|x| > a$.

For any $b_1 < \delta$ for which (4.11) holds, we can choose b_2 in $0 < b_2 < b_1/2$ and a corresponding $\zeta_2(r)$ satisfying $\zeta_2(a) = 0$, $\zeta_2'(a) = b_2/a$ such that

$$\zeta_2(r) < \frac{b_1}{2} \log \frac{r}{a}, \quad a < r < a + \epsilon$$

for some $\epsilon > 0$. Then in this neighbourhood, by (4.11),

$$\frac{b_2}{2} \log \frac{r}{a} \leqq \zeta_2(r) < \frac{b_1}{2} \log \frac{r}{a} \leqq \zeta_1(r)\,.$$

It follows that the positive solutions $u_1(x)$ and $u_2(x)$ of (4.2) corresponding to b_1 and b_2, respectively, as constructed above, satisfy

$$0 < u_2(x) \leqq \zeta_2(\,|x|\,) < \frac{b_1}{2} \log \frac{|x|}{a} \leqq u_1(x)$$

in $a < |x| < a + \epsilon$. Therefore $u_1(x)$ and $u_2(x)$ are distinct positive solutions of the boundary value problem (4.2), and consequently (4.2) has infinitely many distinct positive solutions.

If $n \geqq 3$, we take $t_0 = (n - 2)a^{n-2}$ in (3.2) and apply Theorem 3.1 and Corollary 3.2 to (4.8) to conclude that equation (4.5) has a positive solution $\zeta(r)$ for $r > a$ satisfying $\zeta(a) = 0$, $\zeta'(a) = (n - 2)b/a$, and the analogue of (4.11):

$$0 \leqq \frac{ba^{n-2}}{2} \left[a^{2-n} - r^{2-n}\right] \leqq \zeta(r), \quad r \geqq a\,,$$

provided $0 < b < \delta$. The remainder of the proof closely parallels that for $n = 2$, with (4.12) replaced by

$$\frac{ba^{n-2}}{2} \left[a^{2-n} - |x|^{2-n}\right] \leqq u(x) \leqq \zeta(\,|x|\,)$$

for all $|x| \geqq a > 0$.

The corollary below is obtained if (4.9), (4.10) are slightly strengthened to the following:

H.10. There exist positive constants ϵ and C such that $s^{-\epsilon}g(r, s)$ is a strictly increasing function of s in $[0, \infty)$ for each fixed $r > 0$, and

$$\int^{\infty} r \log r\, g(r, C \log r)dr < \infty, \quad n = 2$$

$$\int^{\infty} rg(r, C)dr < \infty, \qquad n \geqq 3\,.$$

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COROLLARY 4.2. *If* H1, H8, H9, *and* H10 *hold, the boundary value problem* (4.2) *has infinitely many positive solutions in* G_a.

Proof. In the case of (4.7), (4.8), the hypothesis H10 is equivalent to H5′ and H5″, which imply H5. The proof is completed as in Theorem 4.1.

We next consider the analogue of (4.2) for an arbitrary exterior domain Ω:

$$\text{(4.13)} \quad \begin{cases} \Delta u + f(x, u) = 0, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega . \end{cases}$$

As before, $G_a \subset \Omega$ for a fixed positive number a. The method of solution is now more complicated since the radial subsolutions (4.3) of (4.13) cannot be chosen to satisfy the boundary conditions identically. Our procedure will be to apply Theorem 2.7 instead of Theorem 2.3.

THEOREM 4.3. *If* H1, H2, H8, H9, *and* H10 *hold, the boundary value problem* (4.13) *has infinitely many distinct nonnegative solutions, which are strictly positive in the subdomain* G_a *of* Ω.

Proof. The subsolutions $w(x)$ given by (4.3) satisfy the second condition (2.3):

$$\text{(4.14)} \quad \Delta w + f(x, w) \geqq 0 \quad \text{in } G_a, \quad w = 0 \text{ on } S_a ,$$

as required for Theorem 2.7. Define

$$a_0 = \inf_{x \in \partial\Omega} |x|.$$

If $n = 2$, take $t_0 = \log (a_0/\rho)$ in Section 3, where ρ in (4.7) is chosen to satisfy $0 < \rho < a_0$. Since $g(r, u)$ is defined for $r \geqq a_0$ by H9, the hypotheses of Theorem 4.3 imply the hypotheses H3, H4, H5′, and H5″ of Section 3, and H5′ and H5″ imply H5. Then Theorem 3.1 and Corollary 3.2 show that (4.7) has a solution $y(t)$ in $[t_0, \infty)$ satisfying $y(t_0) = 0$, $y'(t_0) = b$, and $0 \leqq (b/2)(t - t_0) \leqq y(t)$ for $t \geqq t_0$, provided $0 < b < \delta$ for some $\delta > 0$. Hence (4.5) has a solution $\zeta(r)$ for $r > a_0$ satisfying $\zeta(a_0) = 0$, $\zeta'(a_0) = b/a_0$, and

$$\text{(4.15)} \quad 0 \leqq \frac{b}{2} \log \frac{r}{a_0} \leqq \zeta(r), \quad r \geqq a_0 .$$

The function $v(x) = \zeta(\,|x|\,)$ for $|x| = r \geqq a_0$ therefore satisfies the first condition (2.3):

$$\text{(4.16)} \quad \Delta v + f(x, v) \leqq 0 \text{ in } \Omega, \quad v \geqq 0 \text{ on } \partial\Omega$$

since these inequalities hold in the larger set G_{a_0}, and also, by (4.15),

$$0 \leqq \frac{b}{2} \log \frac{|x|}{a_0} \leqq v(x) \quad \text{in } \bar{G}_{a_0}.$$

Since $0 < a_0 < a$, $w(x) = (b/2) \log (|x|/a)$ satisfies $0 \leqq w(x) \leqq v(x)$ in $\bar{G}_a$.

In view of (4.14) and (4.16), all the hypotheses of Theorem 2.7 are fulfilled, and hence the boundary value problem (4.13) has a solution $u(x)$ satisfying $w_0(x) \leqq u(x) \leqq v(x)$, where

$$w_0(x) = \begin{cases} \dfrac{b}{2} \log \dfrac{|x|}{a} & \text{if } x \in G_a \\ 0 & \text{if } x \in \bar{\Omega}_a. \end{cases}$$

Then clearly $u(x) > 0$ if $|x| > a$.

The proof for $n \geqq 3$ is virtually the same, using (4.8) instead of (4.7), and will be deleted.

We now consider the related problem of proving the existence of a positive solution of the same differential equation

$$\text{(4.17)} \qquad \Delta u + f(x, u) = 0, \quad x \in G_{a_0}$$

in an arbitrary exterior domain Ω, without any boundary condition on $\partial\Omega$. Evidently it is enough to consider the case $\Omega = G_{a_0}$ since we can choose $a_0 = \inf |x|$ over $\partial\Omega$. This problem is somewhat simpler than the boundary value problem (4.2) since Theorem 2.11 can be used, in which $w(x) = 0$ identically is used as a subsolution of (4.17). A positive supersolution $v(x)$ of (4.17) will be constructed in the form $v(x) = \zeta(r)$ for $r = |x| \geqq a_0$, where $\zeta(r)$ satisfies (4.5).

THEOREM 4.4. *For arbitrary $a_0 > 0$, equation* (4.17) *has a bounded positive solution $u(x)$ in G_{a_0} if* (1) H1, H8, *and* H9 *hold*; (2) *for some $\epsilon > 0$, $s^{-\epsilon}g(r, s)$ is strictly increasing in s, $0 \leqq s < \infty$, for each $r \geqq a_0$*; *and* (3) *one of the following is satisfied*:

$$\text{(4.18)} \qquad \int^{\infty} r \log r \, g(r, C_0 (\log r)^{1/2})dr < \infty, \quad n = 2$$

$$\text{(4.19)} \qquad \int^{\infty} rg(r, C_0 r^{(2-n)/2})dr < \infty, \qquad n \geqq 3$$

for every positive constant C_0.

Proof. If $n = 2$, let $t = \log(r/\rho)$, $t_0 = \log(a_0/\rho)$, where $0 < \rho < a_0$ in (4.7). Since the present assumptions (1) and (2) imply H3 and H5′, Nehari's Theorem 3.4 shows that equation (4.7) has a bounded positive solution $y(t)$ in $[t_0, \infty)$ if the analogue of (3.6) holds, i.e., if

$$\int^{\infty} te^{2t}\, g(\rho e^t, Ct^{1/2})dt < \infty$$

for every positive constant C, which is equivalent to (4.18). It then follows, if (4.18) is satisfied, that equation (4.5) has a bounded positive solution $\zeta(r) = y(\log r/\rho)$ in $[a_0, \infty)$. Since $g \in C^\alpha$ by H9, standard regularity theory [**3**] shows that $\zeta \in C^{2+\alpha}[a_0, b]$ for all $b > a_0$. As already noted in (4.6), the function v in $G_{a_0} \cup S_{a_0}$ defined by $v(x) = \zeta(\,|x|\,)$ satisfies

$$\Delta v + f(x, v) \leqq 0 \text{ in } G_{a_0}, \quad \text{and} \quad v \in C^{2+\alpha}(\bar{G}_{a_0,b})$$

$$\text{for all } b > a_0\,.$$

Since the hypotheses of Theorem 2.11 are satisfied, equation (4.17) has a solution $u(x)$ satisfying

$$0 < u(x) \leqq v(x) \quad \text{in } G_{a_0} \cup S_{a_0}\,.$$

If $n \geqq 3$, the proof is similar, with (4.8) used instead of (4.7), where $t = (n - 2)r^{n-2}$, $t_0 = (n - 2)a_0^{n-2}$, and $\zeta(r) = y(t)/t$. In this case, for a bounded positive solution $y(t)$ of (4.8) in $[t_0, \infty)$, we have the stronger conclusion that $v(x) = \zeta(\,|x|\,)$ has limit zero as $r = |x| \to \infty$.

5. Sublinear boundary value problems. The existence of a nontrivial nonnegative solution of the boundary value problem (4.13) will now be proved under hypotheses H1, H2, and H11-H13 below.

H11. There exists a positive function $g \in C^\alpha_{\text{loc}}(\mathbf{R}_+ \times \mathbf{R}_+)$ such that $g(r, u)$ is nonincreasing in u for each $r > 0$ and

$$f(x, u) \leqq ug(\,|x|, u)$$

for all $x \in \Omega \cup \partial\Omega$ and for all $u > 0$. H11 is a sublinear hypothesis, and replaces the superlinear hypothesis H9 of Section 4.

Theorem 2.9 will be applied in the case that $v(x) = \zeta(r)$ for $r = |x| \geqq a_0$, where $\zeta(r)$ satisfies (4.5) and $a_0 = \inf_{x\in\partial\Omega} |x|$. As in Section 4 we write (4.5) in the canonical form (4.7) or (4.8) if $n = 2$ or $n \geqq 3$, respectively. Hypothesis (3.7) of Theorem 3.5 becomes in the cases (4.7) and (4.8), respectively,

$$\text{(3.7a)} \quad B\rho^2 \int_{t_0}^{\infty} te^{2t} g(\rho e^t, C)dt < B - C, \quad n = 2$$

$$t = \log(r/\rho), \quad t_0 = \log(a_o/\rho) > 0;$$

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$$(3.7\text{b})\quad B\int_{t_0}^{\infty} t^{-3}[\beta(t)]^{2n-2}g\left(\beta(t), \frac{C}{t}\right)dt < B - C, \quad n \geqq 3$$

$$t = (n-2)r^{n-2}, \quad t_0 = (n-2)a_0^{n-2} > 0$$

for some constants B and C, allowed to depend on t_0, such that $B > C > 0$. Hypothesis H7 in the case of equation (4.5) is then equivalent to the following:

H12. For arbitrary $a_0 > 0$, there exist constants B and C, possibly depending on a_0, such that $B > C > 0$ and

$$B\,\rho^2 \int_{a_0}^{\infty} r \log\frac{r}{\rho}\, g(r, C)dr < B - C, \quad n = 2$$

$$B\int_{a_0}^{\infty} r\, g\left(r, \left(\frac{C}{n-2}\right) r^{2-n}\right)dr < B - C, \quad n \geqq 3\,.$$

In particular, H12 is implied by the following:

H12′. There exists a positive constant ϵ such that $s^{\epsilon}g(r, s)$ is a nonincreasing function of s in $(0, \infty)$ for each fixed $r > 0$, and

$$(5.1)\qquad \int^{\infty} r \log r\, g(r, 1)dr < \infty, \qquad n = 2;$$

$$(5.2)\qquad \int^{\infty} rg\left(r, \left(\frac{1}{n-2}\right) r^{2-n}\right)dr < \infty, \quad n \geqq 3\,.$$

In the sublinear prototype $g(r, u) = p(r)u^{\gamma-1}$, $0 < \gamma < 1$, equation (4.5) becomes

$$r^{1-n}\frac{d}{dr}\left(r^{n-1}\frac{d\zeta}{dr}\right) + p(r)\zeta^{\gamma} = 0\,,$$

and H12′ reduces to

$$\int^{\infty} r \log r\, p(r)dr < \infty, \quad n = 2;$$

$$\int^{\infty} r^{\sigma}\, p(r)dr < \infty, \qquad n \geqq 3\,,$$

where $\sigma = (n-1) - \gamma(n-2)$. In this case, any number ϵ in $(0, 1-\gamma)$ will do for H12′.

THEOREM 5.1. *If* H11 *and* H12 *hold, then for arbitrary* $a_0 > 0$ *equation* (4.5) *has a positive solution* $\zeta \in C_{\text{loc}}^{2+\alpha}\,[a_0, \infty)$ *such that* $r^{n-2}\zeta(r)$ *is bounded in* $[a_0, \infty)$, $n \geqq 2$.

Proof. This is a Corollary of Theorem 3.5 since H11 and H12 imply H3, H6, and H7. We merely note that the boundedness of $r^{n-2}\zeta(r)$ is equivalent to the boundedness of $y(t)$, as stated in Theorem 3.5, in view of the changes of variable in (4.7) and (4.8). Standard regularity theory [3] shows that $\zeta \in C^{2+\alpha}_{\text{loc}}\,[a_0, \infty)$ since $g \in C^{\alpha}_{\text{loc}}$ by H11.

With $\zeta(r)$ as in Theorem 5.1, the function v defined by $v(x) = \zeta(\,|x|\,)$ then satisfies (2.4), i.e.,

$$\text{(5.3)} \qquad \Delta v + f(x, v) \leqq 0 \text{ in } \Omega, \quad v > 0 \text{ on } \partial\Omega$$

as needed for Theorem 2.9. To construct a function w satisfying (2.4) we use the following hypothesis:

H13. There exists a positive number ϵ such that $s^{\epsilon-1}f(x, s)$ is a nonincreasing function of s in $(0, \infty)$ for each fixed $x \in R$, where R is a fixed nonempty bounded domain with $R \subset \Omega_A$ for some $A > 0$.

In the prototype $f(x, s) = p(x)s^{\gamma}$, $x \in \Omega$, $s > 0$, $0 < \gamma < 1$, H13 holds for any nonempty bounded domain $R \subset \Omega$ since $s^{\epsilon-1+\gamma}$ is nonincreasing for $\epsilon \leqq 1 - \gamma$.

THEOREM 5.2 *If* H1, H2, H11, H12, *and* H13 *hold, the sublinear boundary value problem* (4.13) *has a nontrivial nonnegative bounded solution* $u(x)$ *in* $\Omega \cup \partial\Omega$ *such that* $|x|^{2-n}u(x)$ *is bounded in* Ω. *In particular* $u(x) \to 0$ *as* $|x| \to \infty$ *if* $n \geqq 3$.

Proof. The linear eigenvalue problem

$$\text{(5.4)} \qquad \begin{cases} \Delta w + \lambda f(x, 1)w = 0 & \text{in } R \\ \quad w(x) = 0 & \text{on } \partial R \end{cases}$$

has a positive eigenfunction $w \in C^{2+\alpha}(\bar{R})$ in R corresponding to the smallest eigenvalue $\lambda_1 > 0$ by the Krein-Rutman theorem [4]. Since (5.4) is linear, such a positive eigenfunction $w(x)$ of (5.4) can be chosen to satisfy the following three conditions:

$$\text{(5.5)} \qquad \sup_{x \in R} w(x) \leqq 1,$$

$$\text{(5.6)} \qquad \lambda_1 \sup_{x \in R}\, [w(x)]^{\epsilon} \leqq 1, \quad \text{and}$$

$$\text{(5.7)} \qquad v(x) \geqq w(x) \quad \text{for all } x \in R$$

where $v(x) = \zeta(\,|x|\,)$ is the function constructed above from Theorem 5.1, and satisfies (5.3). Then H13, (5.4), (5.5), and (5.6) imply that

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$$
\begin{aligned}
0 &= \Delta w + \lambda_1 w(x) f(x, 1) \\
&\leqq \Delta w + \lambda_1 w(x)[w(x)]^{\epsilon - 1} f(x, w(x)) \\
&= \Delta w + \lambda_1 [w(x)]^{\epsilon} f(x, w(x)) \leqq \Delta w + f(x, w(x))
\end{aligned}
$$

for all $x \in R$, from which $w(x)$ satisfies the second condition (2.4). Since (5.3) holds and $0 \leqq w(x) \leqq v(x)$ throughout $\bar{R}$ by (5.7), Theorem 2.9 shows that the boundary value problem (4.13) has a solution $u(x)$ satisfying

$$(5.8) \qquad 0 \leqq w_0(x) \leqq u(x) \leqq v(x), \quad x \in \Omega \cup \partial\Omega$$

where $w_0(x) = w(x)$ for $x \in \bar{R}$ and $w_0(x) = 0$ otherwise.

Theorem 5.2 applies in particular to the sublinear problem

$$
(5.9) \qquad \begin{array}{ll}
\Delta u + p(x) u^{\gamma} = 0\,, & x \in \Omega \\
u(x) = 0\,, & x \in \partial\Omega \\
0 < \gamma < 1\,, &
\end{array}
$$

where $p(x) > 0$ for all $x \in \Omega \cup \partial\Omega$ and

$$p \in C^{\alpha}_{\mathrm{loc}}\,[a_0, \infty), \quad a_0 = \inf_{x \in \partial\Omega} |x|\,.$$

Define

$$p_m(r) = \sup_{|x| = r} p(x).$$

Hypotheses H1, H2, and H13 then hold automatically, H11 holds with the definition $g(r, u) = p_m(r) u^{\gamma - 1}$, $0 < \gamma < 1$, and H12 reduces to

$$
(5.10) \qquad \begin{array}{ll}
\displaystyle\int^{\infty} r \log r\, p_m(r) dr < \infty\,, & n = 2 \\
\displaystyle\int^{\infty} r^{\sigma} p_m(r) dr < \infty\,, & n \geqq 3
\end{array}
$$

where $\sigma = (n - 1) - \gamma(n - 2)$, $n \geqq 3$.

COROLLARY 5.3. *If* (5.10) *holds, the sublinear boundary value problem* (5.9) *has a nontrivial nonnegative bounded solution* $u(x)$ *in* $\Omega \cup \partial\Omega$ *with* $|x|^{2-n} u(x)$ *bounded in* Ω.

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