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UNIQUE FACTORIZATION THEOREMS FOR SUBALGEBRAS OF THE INCIDENCE ALGEBRA

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1. Introduction. H. Scheid [**4**] has found necessary and sufficient conditions on a partially ordered set $S(\leqq)$ which is a direct sum of a countable number of trees for a certain subalgebra $G(+, *)$ of the incidence algebra $F(+, *)$ to be an integral domain. In this paper we prove that under similar conditions on S, $G(+, *)$ is actually a unique factorization domain or, failing this, that there is a subalgebra $H(+, *)$ of $F(+, *)$ which is a unique factorization domain and contains G. Similar results are then obtained as corollaries in the regular convolution rings of Narkiewicz.

2. Definitions and notations. Throughout this paper $S(\leqq)$ will denote a *locally finite partially ordered* set. By this we mean $\leqq$ is a partial ordering of the nonempty set S and for all $a, b \in S$ with $a \leqq b$ the *interval* $[a, b]$ of all $x \in S$ with $a \leqq x \leqq b$ is a finite set. A one-element interval is called a *unit interval*. A *chain* is a totally ordered set. If $C(\leqq)$ is a finite chain of n elements then the *length* $\lambda(C)$ of C is $n - 1$ while if C is infinite then $\lambda(C) = \infty$. A *tree* T is a partially ordered set with a least element such that every interval of T is a finite chain. A tree consisting of a single element is called *trivial*. The *length* $\lambda(T)$ of the tree T is the supremum of the lengths of all the intervals of T. The incidence algebra [**5**] on $S(\leqq)$ will be denoted by $F(+, *)$. The set F consists of all functions f from $S \times S$ to the field K such that $f(x, y) = 0$ if $x \nleqq y$. The addition in $F(+, *)$ is pointwise addition and the product is defined for $f, g \in F$ at (x, y) in $S \times S$ by

$$(f * g)(x, y) = \sum (f(x, z)g(z, y) : x \leqq z \leqq y).$$

The characteristic function of unit intervals $\epsilon \in F$ is the unity of $F(+, *)$ and $f \in F$ is a unit if and only if $x \in S$ implies $f(x, x) \neq 0$.

We shall use $\cong$ to denote isomorphism between both algebras and partially ordered sets. The field K will be of characteristic zero. We shall denote the ring of formal power series over K in a finite number n of indeterminants by K_n and in a countably infinite number of indeterminants by K_ω. The subalgebra $G(+, *)$ of $F(+, *)$ consists of all $g \in F$ such that if $x, y, u, v \in S$ with $[x, y] \cong [u, v]$ then $g(x, y) = g(u, v)$. Thus G consists of all functions in F

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which cannot distinguish between isomorphic intervals. If $I = [x, y]$ we shall write $f(x, y) = f(I)$. Note that $g \in G$ is a unit if and only if for some unit interval I_0, $g(I_0) \neq 0$.

3. The main theorems. Scheid [**4**] proved the following.

THEOREM A. (a) *If* $S(\leqq) = T_1 \oplus \ldots \oplus T_n$ *is the direct sum of nontrivial trees* $T_1, \ldots, T_n$ *then* $G(+, *)$ *is an integral domain if and only if* $\lambda(T_i) = \infty$ *for* $1 \leqq i \leqq n$.

(b) *If* $S(\leqq) = T_1 \oplus T_2 \oplus \ldots$ *is the direct sum of a countably infinite number of trees* $T_1, T_2, \ldots$ *then* $G(+, *)$ *is an integral domain if and only if for each* T_i *with* $\lambda(T_i) < \infty$ *there are infinitely many* T_j *with* $\lambda(T_j) \geqq \lambda(T_i)$.

In proving our theorems we shall exhibit an isomorphism between the subalgebra and either K_n or K_ω. The ring K_n is known [**7**, p. 148] to be a unique factorization domain as is K_ω[1].

LEMMA. *If* $I = C_1 \oplus \ldots \oplus C_n$ *is the direct sum of finite nontrivial chains* $C_1, \ldots, C_n$ *then* I *satisfies the Jordan-Dedekind chain condition that all maximal chains* M *in* I *are of the same length, namely*

$$\lambda(M) = \lambda(I) = \lambda(C_1) + \ldots + \lambda(C_n).$$

If $I' = C_1' \oplus \ldots \oplus C_m'$ *then* $I \cong I'$ *if and only if* $m = n$ *and the number of* C_i' *of any given length equals the number of* C_i *of that length.*

Proof. The facts stated in this lemma are probably well known but for lack of a reference we prove the first part. Suppose $I = C_1 \oplus \ldots \oplus C_n$ as in the lemma and let $\lambda(C_i) = l_i$, $C_i = \{a_{i0}, a_{il}, \ldots, a_{il_i}\}$, $1 \leqq i \leqq n$. Each a in I is then of the form

$$a = (a_{1j_1}, a_{2j_2}, \ldots, a_{nj_n})$$

where $0 \leqq j_i \leqq l_i$ for $1 \leqq i \leqq n$. The elements $a_0 = (a_{10}, a_{20}, \ldots, a_{n0})$ and $a_1 = (a_{1l_1}, a_{2l_2}, \ldots, a_{nl_n})$ are the least and greatest elements of I, respectively, and must be contained in any maximal chain of I. Let

$$a_0 = x_0 < x_1 < x_2 < \ldots < x_L = a_1$$

be a maximal chain of I. For convenience of notation let us think of an element $a \in I$ as an ordered n-tuple $(j_1, \ldots, j_n)$ of nonnegative integers. Thus $a_0 = (0, 0, \ldots, 0)$ and $a_1 = (l_1, \ldots, l_n)$. Now x_1 in the maximal chain must be an n-tuple of $n - 1$ zeros and one 1 for if not then there would be a y in I with $x_0 < y < x_1$ contrary to the maximality of the chain. Similarly x_2 must be an n-tuple of $n - 1$ zeros and a 2 or $n - 2$ zeros and two 1's. In general x_{k+1} is an n-tuple which agrees with x_k except that a 1 has been added to some entry of x_k where $0 \leqq k < L$. Thus $L = l_1 + \ldots + l_n = \lambda(C_1) + \ldots + \lambda(C_n)$ and all maximal chains in I are of the same length, $\lambda(I) = \lambda(C_1) + \ldots + \lambda(C_n)$.

The second part of the lemma follows from a known theorem [**6**, p. 200].

Our first theorem concerns the case when S is the direct sum of infinitely many trees but we impose a slightly stronger condition on S than Scheid did in part (b) of Theorem A.

THEOREM 1. *If $S = T_1 \oplus T_2 \oplus \ldots$ is the direct sum of a countably infinite number of trees $T_1, T_2, \ldots$ and if for each tree T_i and each interval C of T_i there are infinitely many T_j each containing an interval isomorphic to C, then $G(+, *)$ is a unique factorization domain.*

Proof. Since each interval of S is a direct sum of a finite number of chains of finite length it follows from the lemma that two intervals of S are isomorphic if and only if their representations as direct sums of chains contain the same number of chains of each possible positive length. Thus we may associate with each isomorphism equivalence class of intervals of S the sequence

$$r(I) = (r_1(I), r_2(I), \ldots) \tag{1}$$

where $r_j(I)$ is the number of chains of length j in the representation of I as a direct sum of chains. Thus two intervals I and I' of S are isomorphic if and only if $r(I) = r(I')$.

We divide the proof into two cases.

Case 1. Suppose there is a non-negative integer m such that $\lambda(T_i) \leqq m$ for all $i \geqq 1$ and $\lambda(T_i) = m$ for some $i \geqq 1$. Then the hypothesis of the theorem implies that $\lambda(T_i) = m$ for infinitely many $i \geqq 1$. If $m = 0$ then all the trees are trivial, S is a single element set and $F(+, *) = G(+, *) \cong K$, which is trivially a unique factorization domain. Thus assume that $m \geqq 1$.

Note that $r_j(I) = 0$ for all $j > m$ and all intervals I of S. We define π_j for $1 \leqq j \leqq m$ to be the characteristic function of chains of length j. Then we shall show that every $g \in G$ has a unique representation in the form

$$g = \sum a(i_1, \ldots, i_m)\pi_1^{i_1} * \ldots * \pi_m^{i_m} \tag{2}$$

where the summation extends over all m-tuples $(i_1, \ldots, i_m)$ of nonnegative integers, $a(i_1, \ldots, i_m) \in K$, $\pi_j^{i_j} = \pi_j * \ldots * \pi_j$ with i_j factors if $i_j \geqq 1$ and $\pi_j^0 = \epsilon$. Once this representation has been established, there is an immediate isomorphism between $G(+, *)$ and the formal power series ring K_m. For if $x_1, \ldots, x_m$ are the indeterminants in K_m, simply correspond π_j to x_j for $1 \leqq j \leqq m$.

If $(i_1, \ldots, i_m)$ is a fixed m-tuple of nonnegative integers and I is an interval of S then we claim that

$$\pi_1^{i_1} * \ldots * \pi_m^{i_m}(I) = 0$$

unless

$$\lambda(I) = i_1 + 2i_2 + \ldots + mi_m. \tag{3}$$

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Recall from the proof of the lemma that

$$\lambda(I) = r_1(I) + 2r_2(I) + \ldots + mr_m(I).$$

To prove the claim suppose $I = [a, b]$ and that

$$\pi_1{}^{i_1} * \ldots * \pi_m{}^{i_m}(I) \neq 0.$$

Then there is a chain, $a = x_0 \leqq x_1 \leqq \ldots \leqq x_m = b$ in I such that

$$\pi_j{}^{i_j}\,(x_{j-1}, x_j) \neq 0$$

for $1 \leqq j \leqq m$. Thus $\lambda(x_{j-1}, x_j) = ji_j$ and the chain $x_0 \leqq x_1 \leqq \ldots \leqq x_m$ refines to a maximal chain in I of length $i_1 + 2i_2 + \ldots + mi_m$ which equals $\lambda(I)$ by the lemma. This proves the claim.

Thus if the series in (2) is evaluated at an interval I the only terms which can be nonzero are those for which the m-tuple $(i_1, \ldots, i_m)$ satisfies (3) and these are finite in number. In fact the number of solutions $(i_1, \ldots, i_m)$ of (3) is the number of partitions of the integer $\lambda(I)$ into positive integers less than or equal to m.

Now let L be a fixed nonnegative integer and consider all intervals I of S such that $\lambda(I) = L$. Since we are not distinguishing between isomorphic intervals we need only consider those m-tuples $(r_1(I), \ldots, r_m(I))$ for which $r_1(I) + 2r_2(I) + \ldots + mr_m(I) = L$. We now arrange all such m-tuples as follows:

$(L, 0, 0, 0, \ldots, 0)$, $(L - 2, 1, 0, 0, \ldots, 0)$, $\ldots$, $(0 \text{ or } 1, [L/2], 0, 0, \ldots, 0)$,
$(L - 3, 0, 1, 0, \ldots, 0)$, $(L - 5, 1, 1, 0, \ldots, 0)$, $\ldots$, $(0 \text{ or } 1, [(L - 3)/2], 1, 0, 0, \ldots, 0)$
$(L - 6, 0, 2, 0, \ldots, 0)$, $(L - 8, 1, 2, 0, \ldots, 0)$, $\ldots$, $(0 \text{ or } 1, [(L - 6)/2], 2, 0, \ldots, 0)$,

$\ldots$

$(0 \text{ or } 1 \text{ or } 2, 0, [L/3], 0, \ldots, 0)$, $(0 \text{ or } 1, 0 \text{ or } 1, [L/3], 0, \ldots, 0)$,
$(L - 4, 0, 0, 1, 0, \ldots, 0)$, $(L - 6, 1, 0, 1, 0, \ldots, 0)$, $\ldots$, $(0 \text{ or } 1, [(L - 4)/2], 0, 1, 0, \ldots, 0)$
$(L - 7, 0, 1, 1, 0, \ldots, 0)$, $(L - 9, 1, 1, 1, 0, \ldots, 0)$, $\ldots$, $(0 \text{ or } 1, [(L - 7)/2], 1, 1, 0, \ldots, 0)$

$\ldots$

where the brackets denote the greatest integer function.

As an example, if $m = 4$ and $L = 7$ the above arrangement is

$(7, 0, 0, 0)$, $(5, 1, 0, 0)$, $(3, 2, 0, 0)$, $(1, 3, 0, 0)$,
$(4, 0, 1, 0)$, $(2, 1, 1, 0)$, $(0, 2, 1, 0)$,
$(1, 0, 2, 0)$,
$(3, 0, 0, 1)$, $(1, 1, 0, 1)$,
$(0, 0, 1, 1)$.

For any given m-tuple $(i_1, \ldots, i_m)$ the hypothesis of the theorem ensures the existence of an interval I of S with $r(I) = (i_1, i_2, \ldots, i_m, 0, 0, \ldots)$. We now select intervals $I_1, I_2, \ldots$ such that $\lambda(I_j) = L$ and the m-tuples $(r_1(I_j), r_2(I_j), \ldots, r_m(I_j))$ for $j = 1, 2, \ldots$ are the m-tuples in the above arrangement, i.e.,

$$(r_1(I_1), r_2(I_1), \ldots, r_m(I_1)) = (L, 0, 0, \ldots, 0)$$
$$(r_1(I_2), r_2(I_2), \ldots, r_m(I_2)) = (L-2, 1, 0, \ldots, 0)$$
$$\ldots$$

We now proceed to evaluate (2) successively at $I_1, I_2, \ldots$ forcing equality and thereby determining the coefficients $a(i_1, \ldots, i_m)$:

$$g(I_1) = \sum a(i_1, \ldots, i_m)\, \pi_1^{i_1} * \ldots * \pi_m^{i_m}\,(I_1)$$

and since the only subintervals of I_1 which are chains are chains of length 1 we have

$$g(I_1) = a(L, 0, 0, \ldots, 0)\, \pi_1^{L}\,(I_1),$$

and

$$a(L, 0, 0, \ldots, 0) = g(I_1)/(\pi_1^{L}\,(I_1)).$$

Actually $\pi_1^L(I_1) = L!$ but we only need that it is nonzero. Continuing,

$$g(I_2) = \sum a(i_1, \ldots, i_m)\, \pi_1^{i_1} * \ldots * \pi_m^{i_m}(I_2)$$

and since the only subintervals of I_2 which are chains are chains of length 1 or 2, we have

$$g(I_2) = \sum a(i_1, i_2, 0, 0, \ldots, 0)\, \pi_1^{i_1} * \pi_2^{i_2}(I_2).$$

There is only one subinterval of I_2 which is a chain of length 2 and thus

$$g(I_2) = a(L, 0, 0, \ldots, 0)\pi_1^{L}(I_2) + a(L-2, 1, 0, \ldots, 0)\pi_1^{L-2} * \pi_2(I_2).$$

Since $\pi_1^{L-2} * \pi_2(I_2) \neq 0$, the coefficient $a(L-2, 1, 0, \ldots, 0)$ is thus determined. Continuing in this way we can evaluate all the coefficients $a(i_1, \ldots, i_m)$ for which

$$i_1 + 2i_2 + \ldots + mi_m = L. \tag{4}$$

Since every m-tuple $(i_1, \ldots, i_m)$ satisfies (4) for one and only one L this procedure uniquely determines all the coefficients in (2) and completes the proof of this case.

Case 2. Suppose that for each integer $m \geqq 1$ there is a tree T_i which contains an interval of length m. In this case we claim that each $g \in G$ has a unique representation in the form

$$g = \sum a(i_1, i_2, \ldots)\, \pi_1^{i_1} * \pi_2^{i_2} * \ldots \tag{5}$$

where the summation now extends over all sequences $(i_1, i_2, \ldots)$ of nonnegative integers all but a finite number of which are zero.

Once this representation is established the isomorphism between $G(+, *)$ and K_ω is immediate.

We throw the proof back to case 1 by considering for each nonnegative integer L all sequences $(i_1, i_2, \ldots)$ satisfying

$$i_1 + 2i_2 + \ldots = L. \tag{6}$$

The number of solutions $(i_1, i_2, \ldots)$ of (6) is the number of unrestricted partitions of L into positive integers and in any solution $i_j = 0$ for $j > L$. Thus if we take $m = L$ in case 1 the solutions of (6) coincide with the solutions of (4) and the coefficients $a(i_1, \ldots, i_L, 0, 0, \ldots)$ in (5) can be determined as in case 1. When the series in (5) is evaluated at an interval of S it reduces to a finite sum as in case 1 and equals the value of g at that interval.

In part (a) of Theorem A, Scheid found necessary and sufficient conditions for $G(+, *)$ to be an integral domain if S is the direct sum of a finite number of trees. We have not been able to prove (or disprove) that such a G is a unique factorization domain but we do have the following.

THEOREM 2. *If $S = T_1 \oplus \ldots \oplus T_n$ is the direct sum trees with $\lambda(T_i) = \infty$ for $1 \leqq i \leqq n$ then there is a subalgebra $H(+, *)$ of $F(+, *)$ which is a unique factorization domain and contains $G(+, *)$.*

Proof. Each interval I of S is of the form

$$I = C_1 \oplus \ldots \oplus C_n \tag{7}$$

where C_i is a chain from T_i for $1 \leqq i \leqq n$. Define

$$\Lambda(I) = (\lambda(C_1), \ldots, \lambda(C_n)) \tag{8}$$

where C_i is given by (7). Note that $\Lambda(I) = \Lambda(I')$ implies $I \cong I'$ but not conversely. We now define for each n-tuple $(i_1, \ldots, i_n)$ of nonnegative integers the function $e(i_1, \ldots, i_n)$ to be the characteristic function of those intervals I of S with $\Lambda(I) = (i_1, \ldots, i_n)$. Then $e(i_1, \ldots, i_n) \in G$ if and only if $i_1 = i_2 = \ldots = i_n$, for if $\Lambda(I) = (i_1, \ldots, i_n)$ and $i_j \neq i_k$ for some j, k with $1 \leqq j < k \leqq n$, say for convenience, $i_1 \neq i_2$ then by choosing I' with $\Lambda(I') = (i_2, i_1, i_3, \ldots, i_n)$ we have $I \cong I'$ but $e(i_1, \ldots, i_n)\ (I) = 1$ while $e(i_1, \ldots, i_n)\ (I') = 0$.

We now define the subset H of F to consist of all $h \in F$ representable in the form

$$h = \sum a(i_1, \ldots, i_n) e(i_1, \ldots, i_n) \tag{9}$$

where the summation extends over all n-tuples $(i_1, \ldots, i_n)$ of nonnegative integers and $a(i_1, \ldots, i_n) \in K$.

We claim that any pair of functions $e(i_1, \ldots, i_n)$, $e(k_1, \ldots, k_n)$ multiply according to the rule

$$e(i_1, \ldots, i_n) * e(k_1, \ldots, k_n) = e(i_1 + k_1, \ldots, i_n + k_n). \tag{10}$$

For suppose for some interval $I = [a, b]$ of S that

$$e(i_1, \ldots, i_n)*e(k_1, \ldots, k_n)(I) \neq 0. \tag{11}$$

If $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ then there is an $x = (x_1, \ldots, x_n) \in S$ with a $\leqq x \leqq b$ such that

$$e(i_1, \ldots, i_n)(a, x) \neq 0 \text{ and } e(k_1, \ldots, k_n)(x, b) \neq 0.$$

Thus

$$\Lambda(a, x) = (\lambda(a_1, x_1), \ldots, \lambda(a_n, x_n)) = (i_1, \ldots, i_n)$$

and

$$\Lambda(x, b) = (\lambda(x_1, b_1), \ldots, \lambda(x_n, b_n)) = (k_1, \ldots, k_n)$$

and hence

$$\Lambda(I) = (i_1 + k_1, \ldots, i_n + k_n).$$

Furthermore the above x is unique and there is exactly one nonzero term in the convolution (11). Thus (10) is established.

Because of (10) there is an isomorphism between $H(+, *)$ and the power series ring K_n if we make the correspondence

$$\sum a(i_1, \ldots, i_n)e(i_1, \ldots, i_n) \leftrightarrow \sum a(i_1, \ldots, i_n)x_1^{i_1} \ldots x_n^{i_n}.$$

To complete the proof we must show that $G \subset H$. To this end we define for each n-tuple $(i_1, \ldots, i_n)$ the function

$$\bar{e}(i_1, \ldots, i_n) = \sum e(\pi(i_1), \ldots, \pi(i_n)) \tag{12}$$

where the summation extends over all distinct permutations $(\pi(i_1), \ldots, \pi(i_n))$ of $(i_1, \ldots, i_n)$. For example

$$\bar{e}(i, i, \ldots, i) = e(i, i, \ldots, i)$$

and

$$\bar{e}(1, 0, 0, \ldots, 0) = e(1, 0, 0, \ldots, 0) + e(0, 1, 0, \ldots, 0) + \ldots + e(0, 0, 0, \ldots, 0, 1).$$

Then each $\bar{e} \in G$ and if $g \in G$ then g is expressible in the form

$$g = \sum b(i_1, \ldots, i_n)\bar{e}(i_1, \ldots, i_n). \tag{13}$$

To see this, note that $\bar{e}(i_1, \ldots, i_n)$ is the characteristic function of intervals isomorphic to the interval I with $\Lambda(I) = (i_1, \ldots, i_n)$ and the coefficient $b(i_1, \ldots, i_n)$ in (13) is simply $g(I)$. Since each $\bar{e}$ is a finite sum of the e's, (13) implies that $G \subset H$ and the proof is complete.

Equation (13) leads one to suspect that $G(+, *)$ is isomorphic to K_n also but the $\bar{e}$ functions do not have a "nice multiplication rule". Perhaps some other choice of "generating functions" in place of the $\bar{e}$'s would produce a representation of the type in (13) and have a "nice multiplication rule" but we have been unable to find such functions.

We return now to the case when S is the direct sum of infinitely many trees. In Theorem 1 we imposed a stronger condition on S than Scheid did in part (*b*) of Theorem A. Those S which satisfy Scheid's condition but not ours all satisfy

Condition B. Suppose $S = T_1 \oplus T_2 \oplus \ldots$ is the direct sum of trees $T_1, T_2, \ldots$, a finite number of which, say $T_1, \ldots, T_n$, are of infinite length while there is an integer $m \geqq 0$ such that $\lambda(T_i) \leqq m$ for all $i > n$ and $\lambda(T_i) = m$ for infinitely many $i > n$.

THEOREM 3. *If S satisfies Condition B then there is a subalgebra $H(+, *)$ of $F(+, *)$ which is a unique factorization domain isomorphic to K_{m+n} and contains G.*

Proof. The method of proof is a combination of the methods used in proving Theorems 1 and 2.

Each interval I of S is of the form

$$I = C_1 \oplus C_2 \oplus \ldots \oplus C_n \oplus C_{n+1} \oplus \ldots \tag{14}$$

where C_i is a chain from T_i for $i \geqq 1$ and all but a finite number of the C_i are trivial one element chains. Since $T_1, \ldots, T_n$ are of infinite length the first n chains in (14) may be of any arbitrary length while $\lambda(C_i) \leqq m$ for all $i > n$. For each interval I of S we define

$$\Lambda'(I) = (r_1'(I), \ldots, r_m'(I), \lambda(C_1), \ldots, \lambda(C_n)) \tag{15}$$

where $r_j'(I)$ is the number of chains C_i in (14) of length j with $i > n$. We define π_j' for $1 \leqq j \leqq m$ to be the characteristic function of intervals I of S with

$$\Lambda'(I) = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

where the 1 appears in position j. Thus π_j' is the characteristic function of a chain of length j which is an interval in a tree T_i with $i > n$.

We also define for each n-tuple $(k_1, \ldots, k_n)$ of nonnegative integers the function $e'(k_1, \ldots, k_n)$ to be the characteristic function of intervals I of S with

$$\Lambda'(I) = (0, 0, \ldots, 0, k_1, k_2, \ldots, k_n).$$

Note that none of the π_j''s or the e''s are in G except for $e'(0, 0, \ldots, 0) = \epsilon$.

We then let H be the set of all $h \in F$ representable in the form

$$h = \sum a(i_1, \ldots, i_{m+n})\pi_1'^{i_1} * \ldots \pi_m'^{i_m} e'(i_{m+1}, \ldots, i_{m+n}), \tag{16}$$

where the summation extends over all $m + n$-tuples $(i_1, \ldots, i_{m+n})$ of nonnegative integers and the coefficients $a(i_1, \ldots, i_{m+n}) \in K$.

As in the proof of Theorem 2, two e' functions multiply according to

$$e'(k_1, \ldots, k_n) * e'(k_1', \ldots, k_n') = e'(k_1 + k_1', \ldots, k_n + k_n')$$

and $H(+, *)$ is isomorphic to K_{m+n}.

The proof will be complete if we show that $G \subset H$. To see this observe for $1 \leqq j \leqq m$,

$$(17) \quad \pi_j = \pi_j' + e'(j, 0, 0, \ldots, 0) + e'(0, j, 0, \ldots, 0) + \ldots + e'(0, 0, 0, \ldots, 0, j)$$

and for $j > m$,

$$(18) \quad \pi_j = e'(j, 0, 0, \ldots, 0) + e'(0, j, 0, \ldots, 0) + \ldots + e'(0, 0, \ldots, 0, j).$$

We then claim that each $g \in G$ has a representation of the form

$$(19) \qquad g = \sum b(i_1, i_2, \ldots)\pi_1{}^{i_1} * \pi_2 *^{i_2} \ldots$$

where $b(i_1, i_2, \ldots) \in K$ and the summation extends over all sequences $(i_1, i_2, \ldots)$ of nonnegative integers all but a finite number of which are zero. Once (19) has been established we are done since (17), (18) and (19) imply $G \subset H$.

The coefficients $b(i_1, \ldots, i_m, 0, 0, \ldots)$ in (19) can be evaluated by the method used in case 1 of the proof of Theorem 1. The difficulty in extending the same method to evaluate all the $b(i_1, i_2, \ldots)$ is that not all possible direct sums of chains appear in S. For example, no interval of S is the direct sum of $n + 1$ chains each of length $m + 1$. However, this is not a serious difficulty for our present purpose. It merely allows some freedom in selecting the coefficients $b(i_1, i_2, \ldots)$ for which no interval I exists in S with $r(I) = (i_1, i_2, \ldots)$. In fact all such b's may be chosen arbitrarily. Thus each $g \in G$ has a representation of the type (19) although this representation is not unique. This completes the proof of Theorem 3.

If $S = T_1 \oplus T_2 \oplus \ldots$ is the direct sum of trees and $\lambda(T_i) = 1$ for all $i \geqq 1$ then $G(+, *) \cong K_1$ and every element $g \in G$ is uniquely expressible in the form $\sum(a_n\pi_1{}^n : n \geqq 0)$. Furthermore π_1 is the only prime element (up to associates) in G and every $g \in G$ is of the form $u * \pi_1{}^{n_0}$ where u is a unit of G and n_0 is the minimum n such than $a_n \neq 0$. In contrast to this situation we have the following.

THEOREM 4. *If $S(\leqq)$ is a locally finite partially ordered set which contains intervals C_1 and C_2 which are chains of length 1 and 2 respectively, and if S also contains an interval isomorphic to $C_1 \oplus C_1$, then $G(+, *)$ contains infinitely many irreducible elements, no two of which are associates.*

Proof. We claim that the set

$$P = \{\pi_1 + a\pi_2 : a \in K\}$$

is a set of nonassociated irreducible elements. Suppose that

$$\pi_1 + a\pi_2 = g_1 * g_2$$

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where g_1 and g_2 are nonunits of G. Then (recall that nonunits of G vanish on unit intervals of S)

$$1 = (\pi_1 + a\pi_2)(C_1) = (g_1 * g_2)(C_1) = 0,$$

a contradiction. Thus $\pi_1 + a\pi_2$ is irreducible. Now suppose that two elements $\pi_1 + a\pi_2$ and $\pi_1 + b\pi_2$ of P are associates, say

$$\pi_1 + a\pi_2 = g * (\pi_1 + b\pi_2)$$

where g is a unit of G. Then using I_0 to denote a unit interval and evaluating this equation successively at C_1, C_2 and $C_1 \oplus C_1$ we obtain

$$1 = g(I_0),\ a = bg(I_0) + g(C_1) \text{ and } 0 = 2g(C_1).$$

Thus $a = b$ and the proof is complete.

4. Applications to regular convolution rings. Let $A(+, *_C)$ be a regular convolution ring as defined by Narkiewicz [**2**], i.e., A is the set of arithmetic functions, $+$ is pointwise addition and $*_C$ is a convolution product defined as follows. Let C be a mapping from the natural numbers N to the finite subsets of N such that $C(n)$ is a subset of the set of divisors of n for each $n \in N$. Then for $\alpha, \beta \in A$ and $n \in N$,

$$(\alpha *_C \beta)(n) = \sum (\alpha(d)\beta(n/d) : d \in C(n)).$$

Then Narkiewicz calls $A(+, *_C)$ regular if it is a commutative ring with unity, the multiplicative functions form a semigroup under $*_C$ and the "Möbius" function assumes only the values 0 or -1 at a prime power $p^k > 1$. As Narkiewicz showed [**2**, Theorem II], every regular convolution is determined by selecting for each prime $p \in N$ a collection of arithmetic progressions of the type $\{m, 2m, 3m, \ldots\}$ (finite or infinite) which partition N and then defining for each power p^k, $C(p^k) = \{1, p^m, p^{2m}, \ldots, p^{rm}\}$ where $k = rm$ and $\{m, 2m, \ldots\}$ is the progression in which k appears.

Let $\leqq_C$ be the relation on N defined by: $a \leqq_C b$ if and only if $a \in C(b)$. Then $N(\leqq_C)$ is a locally finite partially ordered set and can be expressed as a direct sum of trees as Scheid has shown [**3**]. For each prime $p \in N$ let $T_p = \{1, p, p^2, \ldots\}$. Then $T_p(\leqq_C)$ is a tree and

$$N(\leqq_C) = T_2 \oplus T_3 \oplus T_5 \oplus \ldots \oplus T_{p_n} \oplus \ldots \tag{20}$$

where p_n is the nth prime. Let $F(+, *)$ be the incidence algebra on $N(\leqq_C)$ and $G(+, *)$ be the subalgebra defined previously. Then $A(+, *_C)$ can be imbedded in $F(+, *)$ under the mapping $\alpha \leftrightarrow \alpha'$ where α' satisfies $\alpha'(m, n) = \alpha'(1, n/m) = \alpha(n/m)$ for all $m, n \in N$ such that $m \in C(n)$. We shall denote the image of $A(+, *_C)$ in $F(+, *)$ by $A(+, *)$. Then it is easily seen that $G(+, *) \subset A(+, *)$ and the following theorems are then corollaries of the theorems in section 3. We shall use the notation $\lambda(p^k) = \lambda(1, p^k)$, i.e., the length of the prime power p^k is the length of the chain $[1, p^k]$ in the tree T_p.

THEOREM 5. *If $N(\leqq_C)$ is given by* (20) *and for each prime power p^k, there are infinitely many primes q such that $\lambda(q^l) \geqq \lambda(p^k)$ for some power q^l, then $G(+, *)$ is a unique factorization domain.*

THEOREM 6. *If $N(\leqq_C)$ is given by* (20) *and there are positive integers m and n such that $\lambda(T_{p_i}) = \infty$ for n values of i, $\lambda(T_{p_i}) \leqq m$ for all other i and $\lambda(T_{p_i}) = m$ for infinitely many i, then $A(+, *)$ contains a subalgebra $H(+, *)$ which is a unique factorization domain isomorphic to K_{m+n} and contains $G(+, *)$.*

THEOREM 7. *If $N(\leqq_C)$ is given by* (20) *and if $\lambda(T_{p_i}) \geqq 2$ for some i then $G(+, *)$ contains infinitely many nonassociated irreducible elements.*

The only thing requiring a proof in the above theorems is that $H \subset A$ in Theorem 6. Going back to the definition of H in Section 3, this amounts to showing that the π_j''s and the e''s lie in A and it is not difficult to verify this.

An interesting special case is when each Tp_n is an infinite chain and $\leqq_C$ is the usual divisibility order on N. Then $A(+, *)$ is the Dirichlet convolution ring which Cashwell and Everett have shown to be isomorphic to K_ω. Thus by Theorem 5, $A(+, *)$ is isomorphic to its proper subring $G(+, *)$.

Another case of interest is when $\lambda(Tp_n) = 1$ for all $n \geqq 1$. Then $A(+, *)$ is the unitary convolution ring of arithmetic functions and $G(+, *) \cong K_1$.

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