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REFLEXIVE IDEALS AND INJECTIVE MODULES OVER NOETHERIAN v-H ORDERS

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The class of prime Noetherian v-H orders is a class of Noetherian prime rings including the commutative integrally closed Noetherian domains, and the hereditary Noetherian prime rings, and designed to mimic the latter at the level of height one primes. We continue recent work on the structure of indecomposable injective modules over Noetherian rings by describing the structure of such a module E over a prime Noetherian v-H order R in the case where the assassinator P of E is a reflexive prime ideal. This description is then applied to a problem in torsion theory, so generalising work of Beck, Chamarie and Fossum.

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1. Introduction

There has been considerable interest recently in studying the structure of injective modules over non-commutative Noetherian rings using the techniques of the theory of prime links; see, for example [**2, 6, 8, 10**]. Here we make a contribution to this theory by discussing those indecomposable injective modules over a prime Noetherian v-H order with enough v-invertible ideals that have an associated prime ideal that is reflexive. Precise definitions and basic properties of the class of rings we are studying are given in Section 2, but to give an idea of the scope of the results one should keep in mind the following analogy: v-H orders stand in the same relation to maximal orders as hereditary Noetherian prime rings do to Dedekind prime rings. In fact, at the level of height one prime ideals the behaviour of v-H orders should be, up to technical details, similar to that of hereditary Noetherian prime rings. This can be seen, for example, in the structure theorem for injectives given in Theorem 4.1.

In Section 2 we give the definitions and basic properties of the rings that interest us here. In Section 3 we recall the definitions of certain technical conditions—density, stability, and the strong second layer condition—and show that these hold for the reflexive prime ideals we are looking at. These conditions ensure that the injective modules we wish to study have a reasonable structure. In Section 4 the structure of injective modules is adapted from [**2**], yielding Theorem 4.1. This structure theory is then applied to a problem concerning the stability of a certain torsion theory (Theorem 4.2), thus generalising results of Beck [**1**], Chamarie [**4**] and Fossum [**5**] on maximal orders. The paper ends with two examples that set Theorem 4.2 in context.

For any unexplained notation, we refer the reader to [**13**].

2. Prime Noetherian *v-H* orders

Our primary reference for this class of rings is **[11]**; the definitions and properties listed in the following paragraph can all be found there.

Let R be a prime Noetherian ring with simple Artinian quotient ring Q. For a right [resp. left] R-ideal I in Q, set $I^* = \{q \in Q : qI \subseteq R\}$ [resp. $I^+ = \{q \in Q : Iq \subseteq R\}$]. Recall that I is said to be *right reflexive* [resp. *left reflexive*] if $I^{*+} = I$ [resp. $I^{+*} = I$]. A (two-sided) ideal I is *reflexive* provided both these equalities hold; and in this case I is called *v-invertible* if $(II^+)^{*+} = R = (I^*I)^{+*}$. The reflexive ideal I is v-invertible if and only if $O_r(I) = O_l(I) = R$. From this it follows easily that, for I v-invertible,

$$I^* = I^+ = \{q \in Q : IqI \subseteq I\};$$

this set will be denoted by I^{-1}. If A and B are v-invertible ideals, then so is $(AB)^{*+}$; hence one obtains the structure of an abelian group $(F(R), \cdot)$ on the set of v-invertible ideals and their inverses.

Notice that, if R is a maximal order, then every reflexive ideal I is v-invertible **[13,** Proposition 5.1.4].

Definition 2.1. By **[11,** Theorem 1.13(2) and Theorem 2.23], we may state that R is a (*prime, Noetherian*) *v-H order with enough v-invertible ideals* if and only if R is prime Noetherian and

(i) each maximal v-invertible ideal of R is semiprime;

(ii) each maximal v-invertible ideal P is localisable, and R_P (the Ore localisation at the semiprime, localisable ideal P) is a semilocal hereditary Noetherian prime ring with unique maximal invertible ideal $J(R_P)$, while

(iii) $S(R) = \{q \in Q : qI \subseteq R, 0 \neq I \lhd R\}$ is a prime maximal order with no proper reflexive ideals, satisfying the ascending chain condition on closed right or left ideals;

(iv) $R = \cap R_P \cap S(R)$ where P ranges over the set of all maximal v-invertible ideals of R;

(v) for every ideal A of R,

$$A^{+*} = \cap AR_P \cap S(R) = A^{*+};$$

and (vi) each regular element of R is a unit in all but finitely many of the rings R_P.

Note that, in view of **[4,** Prop.1.10] the prime Noetherian ring R is a maximal order if and only if it satisfies (i),…,(vi) and every maximal v-invertible ideal is prime (or, equivalently, every reflexive ideal is v-invertible).

Let R be a prime Noetherian v-H order with enough v-invertible ideals, and let I be an ideal of R. We shall need the following generalisations to this setting of well-known properties of maximal orders:

2.2 **[11,** Lemma 1.2]. *I is right reflexive if and only if I is left reflexive.*

2.3. *Suppose that I is v-invertible, and set $S=II^{-1}$ and $T=I^{-1}I$, so that S and T are ideals of R.*

(a) [**11**, Lemma 2.5]. *$I\subseteq S\cap T$, and $S\cap T\cap\mathscr{C}(J)\neq\phi$ for every v-invertible ideal J of R. Suppose in addition that I is semiprime. Then*

(b) [**11**, Propositions 2.1 and 2.7]. *I is right and left classically localisable; and*

(c) *for all $n\geqq 1$,*

$$\begin{aligned}(I^n)^{*+}&=(I^n)^{+*}\\ &=I^nR_I\cap R\\ &=\{r\in R: rd\in I^n,\ \text{for some } d\in\mathscr{C}(I)\}\\ &=\{r\in R: dr\in I^n,\ \text{for some } d\in\mathscr{C}(I)\}.\end{aligned}$$

The first equality follows from 2.1(v). The second equality also follows from 2.1 (v) once one notes that $IR_P=R_P$ for each reflexive ideal P that does not contain I, and that $R_I=\cap R_P$, where P ranges over the primes minimal over I. The third and fourth equalities follow immediately from standard results on localisation.

We shall denote the ideal $(I^n)^{*+}$ by $I^{(n)}$. Repeated use of [**11**, Lemma 1.3] shows that $I^{(n)}=I\cdot I\cdot\ldots\cdot I$ (n times), where $\cdot$ denotes the group multiplication in the set $F(R)$ of v-invertible ideals.

2.4. *Let P be a maximal reflexive ideal of R. Then P is prime by* [**11**, Lemma 1.4]. *Also, either P is v-invertible or there exist maximal reflexive ideals $P=P_1,P_2,\ldots,P_n$ such that $X=P_1\cap\cdots\cap P_n$ is v-invertible, by* [**11**, Lemmas 1.5, 1.9 and 1.11]. *In either case, P generates a maximal ideal in a localisation R_X of R that is a hereditary Noetherian prime ring, by* [**11**, Proposition 2.7]. *Since HNP rings have Krull dimension one, this forces P to have height one.*

In summary, maximal reflexive ideals of R are prime, and if P is a reflexive prime ideal of R then P has height one.

2.5. Although the localisation result alluded to in (2.4) is established without using the modern terminology as presented in [**8**], we need this terminology to develop the structure theory of injective modules that is presented in Section 4, so we give a brief summary of it here.

A prime ideal Q of a Noetherian ring R is *right linked* to a prime P if there exists an ideal A of R with $QP\subseteq A\subsetneqq P\cap Q$ such that $P\cap Q/A$ is torsion free as a left R/Q-module and as a right R/P-module. If such a link exists, we write $Q\rightsquigarrow P$; see [**8**, page 135]. A set $\mathscr{P}$ of primes of R is *right stable* if, whenever $P\in\mathscr{P}$ and $Q\rightsquigarrow P$, then $Q\in\mathscr{P}$. Right and left stable sets of primes are called *stable*. The (*right*) *clique of P* is the smallest (right) stable set of prime ideals of R containing P.

A prime ideal P of R is said to satisfy the *right strong second layer condition* if, whenever Q is a prime strictly contained in P and U and V are, respectively, a P-prime and a Q-prime module, then there is no essential extension of U by V. If P satisfies the right *and* the left strong second layer condition; it is said to satisfy the *strong second layer condition.* A set $\mathscr{P}$ of primes satisfies this condition if each member of $\mathscr{P}$ does. For a detailed discussion of these concepts, see [**8**].

Let P be a reflexive prime ideal of R, and let I be maximal among v-invertible ideals in P. By (2.1), I is a maximal v-invertible ideal of R, and either $I=P$, or $I=\bigcap_{i=1}^{n} M_i$ where $P=M_1,M_2,\dots,M_n$ form a *cycle* of reflexive ideals: $(M_i^2)^{*+}=M_i$ for all i. It follows easily from [**11**, Proposition 2.7] and the standard theory of HNP rings as presented in [**8**, Appendix], for example, that clique(P) is $\{M_1,\dots,M_n\}$, with links $M_i \rightsquigarrow M_{i+1}$ (mod n), and no other links.

3. Properties of reflexive primes

Let P be a prime ideal of a Noetherian ring R. An R-module U is said to be a *P-prime module* if P is the annihilator of each nonzero submodule of U. In studying injective modules a basic problem is to understand the possible non-split extensions of a P-prime module U by a Q-prime module V. In particular, one needs to know whether it is possible to have a non-split extension of a torsion R/P-module by a torsion free R/Q-module (or vice versa). In order to deal with problems such as these three conditions are discussed in this section—the density condition, stability and the (strong) second layer condition. In considering the first of these it is convenient to widen slightly the definition given in [**8**, page 176]:

Definition. Let R and U be semiprime Noetherian rings, and let B be an R-U-bimodule, torsion free and finitely generated on each side. Then B satisfies the *density condition* if the following property, and its left-handed version, hold: Let E be an essential right submodule of B. Then there exists a regular element d of R such that $dB\subseteq E$.

The relevance of the density condition to questions of representation theory is made plain in Corollary 3.5 below. For the proof of our result on the density condition, we need an easy lemma.

Lemma 3.1. *Let R be a prime Noetherian ring. Let A and B be right ideals of R, and let I be a v-invertible ideal of R. Set $T=I^{-1}I$. Then*

$$(AI\cap BI)T\subseteq (A\cap B)I.$$

Proof. Set $S=II^{-1}$. Then $(AI\cap BI)T=(AI\cap BI)I^{-1}I\subseteq(AII^{-1}\cap BII^{-1})I=(AS\cap BS)I\subseteq(A\cap B)I.$

Theorem 3.2 *Let R be a prime Noetherian v-H order with enough v-invertible ideals. Let I be a semiprime v-invertible ideal of R. Then $I/I^{(2)}$ satisfies the right and left density conditions.*

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Proof. We shall show that $B:=I/I^{(2)}$ satisfies the right density condition. Thus, let E be a right ideal of R such that $I^{(2)}\subseteq E$ and $E/I^{(2)}$ is an essential right submodule of B; we must find an element $d\in\mathscr{C}(I)$, with $dI\subseteq E$. Set $T=I^{-1}I$.

Note that $E\subseteq I$ and so $EI^{-1}\subseteq II^{-1}\subseteq R$. Suppose that

$$d\in EI^{-1}\cap\mathscr{C}(I). \tag{1}$$

Then $dI\subseteq EI^{-1}I=ET\subseteq E$, and the proof is complete. It remains to prove (1).

By [**13**, Proposition 2.3.5] it is enough to show that $(EI^{-1}+I)/I$ is an essential right ideal of R/I. Accordingly, let F be a right ideal of R with $EI^{-1}\cap F\subseteq I$. By Lemma 3.1, (with $A=EI^{-1}$ and $B=F$),

$$(EI^{-1}I\cap FI)T\subseteq(EI^{-1}\cap F)I\subseteq I^2.$$

But $T\cap\mathscr{C}(I)\neq\phi$, by (2.3)(a), so the above shows that

$$ET\cap FI\subseteq I^{(2)}, \tag{2}$$

in view of (2.3)(c).

We claim that

$$(ET+I^{(2)})/I^{(2)} \textit{ is essential in } I/I^{(2)}. \tag{3}$$

For, let X be a right ideal of R with $I^{(2)}\subsetneqq X\subseteq I$. Since $E/I^{(2)}$ is essential in $I/I^{(2)}$,

$$I^{(2)}\subsetneqq E\cap X. \tag{4}$$

Now there exists $c\in\mathscr{C}(I)\cap T$ by (2.3)(a), and

$$(E\cap X)c\subseteq Ec\cap Xc\subseteq ET\cap X.$$

If $ET\cap X\subseteq I^{(2)}$, then it follows from the above and (2.3)(c) that $E\cap X\subseteq I^{(2)}$, contradicting (4). Hence, $ET\cap X\nsubseteq I^{(2)}$, and (3) is proved.

By (3) and (2), $FI\subseteq I^{(2)}$, and so $FII^{-1}\subseteq I^{(2)}I^{-1}\cap R\subseteq IR_I\cap R=I$. But $II^{-1}\cap\mathscr{C}(I)\neq\phi$ by (2.3)(a), and so $F\subseteq I$. That is, $(EI^{-1}+I)/I$ is essential in R/I, as required.

The next lemma shows that v-invertible ideals satisfy a weak form of the Artin–Rees property. The proof is adapted from the corresponding result for maximal orders, due to M. Chamarie [**12**, Prop. IV.2.13].

Lemma 3.3. *Let R be a prime Noetherian v-H order, with enough v-invertible ideals and let I be a semiprime v-invertible ideal of R. Let E be a right ideal of R. Then there exists a positive integer n and an ideal B of R such that $B\cap\mathscr{C}(I)\neq\phi$ and*

$$(E\cap I^n)B\subseteq EI.$$

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Proof. Note that, for each $k \geqq 1$, $I^{(k)}$ is v-invertible, by (2.3)(c), and $(E \cap I^{(k)})(I^{(k)})^{-1}$ is a right ideal of R. Thus, there exists a positive integer n such that

$$\sum_{k \geqq 1} (E \cap I^{(k)})(I^{(k)})^{-1} = \sum_{k=1}^{n} (E \cap I^{(k)})(I^{(k)})^{-1}.$$

Now $(E \cap I^{(n+1)})(I^{(n+1)})^{-1} I^{(n+1)}$ is contained in $\sum_{k=1}^{n} (E \cap I^{(k)})(I^{(k)})^{-1} I^{(n+1)}$. Also, for $k \leqq n$,

$$(I^{(k)})^{-1} I^{(n+1)} \subseteq (I^{(k)})^{-1} \cdot I^{(n+1)} \subseteq I.$$

Therefore, $\sum_{k=1}^{n} (E \cap I^{(k)})(I^{(k)})^{-1} I^{(n+1)} \subseteq EI$, and so

$$(E \cap I^{(n+1)})(I^{(n+1)})^{-1} I^{(n+1)} \subseteq EI.$$

Set $B = (I^{(n+1)})^{-1} I^{(n+1)}$. By (2.3)(a), $B \cap \mathscr{C}(I) \neq \phi$, so the proof is complete.

Theorem 3.4. *Let R be a prime Noetherian v-H order with enough v-invertible ideals.*

(*i*) *The set $\mathscr{P}$ of reflexive prime ideals of R is stable.*

(*ii*) *Each clique of reflexive prime ideals consists of the prime ideals minimal over a maximal v-invertible ideal.*

(*iii*) *$\mathscr{P}$ satisfies the strong second layer condition.*

Proof. Once (iii) is proved, the remainder of the theorem will follow at once from (2.5). Accordingly, let $P \in \mathscr{P}$, and note that P has height one, by (2.4). Thus the only way that the right strong second layer condition could fail would be if there were a module M that was an essential extension of a P-prime module U by a Q-prime module V with Q the zero ideal. So if we can show that any finitely generated essential extension M of a P-prime module U is annihilated by a power of the maximal v-invertible ideal I contained in P then such a configuration is impossible and (iii) is proved. Since M is finitely generated and R is prime, we need consider only the case where M is cyclic, say $M \cong R/C$, for a right ideal C of R. Under this isomorphism, let $U \cong E/C$. Let I be as above.

By Lemma 3.3 there exist $n \geqq 1$, and an ideal B of R, with $B \nsubseteq P$, such that $(E \cap I^n)B \subseteq EI$. In other words,

$$(MI^n \cap U)B \subseteq UI = 0.$$

Since all non-zero submodules of U have annihilator P, it follows that $MI^n \cap U = 0$. But U is essential in M, so this forces $MI^n = 0$, as required.

Our interest in the density condition is explained by the following corollary, whose deduction from Theorem 3.2 is adapted from an argument of Jategaonkar [8, Proposition 6.3.1].

Corollary 3.5. *Let P, Q be (not necessarily distinct) reflexive prime ideals of R, a prime Noetherian v-H order with enough v-invertible ideals. Let U be a P-prime module that is torsion as an R/P-module and let V be a Q-prime module that is torsion free as an R/Q-module. Then there is no essential extension of U by V; (that is, there is no exact sequence $0\to U\to M\to V\to 0$ with U an essential submodule of M).*

Proof. Let I be the maximal v-invertible ideal contained in P. By (2.5) and (2.3)(b), I is a (non-zero) semiprime, classically localisable ideal. Suppose that a module M exists having the properties delineated above. Clearly, replacing U and V by submodules if necessary, we may assume that M is finitely generated. As in the proof of Theorem 3.4, $MI^n=0$ for some positive integer n, since $UI=0$ and U is essential in M.

Since $Ann_R(V)=Q, I^n\subseteq Q$, and hence $I\subseteq Q$ since Q is prime. Note that Q is minimal over I, since Q has height one. Now $MI^2\subseteq MQP=0$. Let A be the right annihilator of $I^{(2)}/I^2$. Since $I^{(2)}/I^2$ is finitely generated as a left module and, by (2.3)(c), is $\mathscr{C}(I)$-torsion on the right, $A\cap\mathscr{C}(I)\neq\phi$. Now $\mathscr{C}(I)\subseteq\mathscr{C}(P)$, so A is not contained in P. But $MI^{(2)}A=0$; since the P-prime module U is essential in M, we deduce that $MI^{(2)}=0$.

Let $u\in U$ and choose $c\in\mathscr{C}(P)$ with $uc=0$, so that $u(cR+P)=0$. Now $(cR+P)/I$ is an essential right ideal of the semiprime Noetherian ring R/I, and so contains a regular element. Thus U is a $\mathscr{C}(I)$-torsion module. Set $U'=\{u\in M : uI=0\}$. Then U' is a torsion R/I-module, by [**8**, Proposition 2.2.2(a)], since U is an essential submodule of U'. Now $\mathscr{C}(I)\subseteq\mathscr{C}(Q)$, so U'/U is a $\mathscr{C}(Q)$-torsion submodule of the torsion-free R/Q-module $V\cong M/U$. Hence $U'=U$. Let $E=E(U)$, the $R/I^{(2)}$-injective hull of U, and write $B=I/I^{(2)}$. Applying $Hom_R(-,E)$ to the exact sequence $0\to B\to R/I^{(2)}\to R/I\to 0$ yields

$$0\to Ann_E(I)\to E\to Hom_R(B,E)\to 0.$$

As we have observed above, $F:=Ann_E(I)$ is $\mathscr{C}(I)$-torsion. Now $M\subseteq E$, and $M\cap F=U$, so that V is an R/I-torsion free submodule of $Hom_R(B,E)$. This last module is just $Hom_R(B,F)$. If $g\in Hom_R(B,F)$, then $\ker g$ is an essential right submodule of B, since $B/\ker g\cong \operatorname{im} g\subseteq F$ is a torsion R/I-module, and B is torsion-free. Since B satisfies the density condition, there exists $d\in\mathscr{C}(I)$ with $dB\subseteq\ker g$. That is, $gd=0$, so $Hom_R(B,F)$ is R/I-torsion. This contradiction shows that no such module M can exist.

4. Applications and examples

Our first result concerns the structure of certain indecomposable injective modules. To state it, it is convenient to recall the following notation; see [**8**, §9.1] or [**2**, §5] for details. Let $\mathscr{P}$ be a set of prime ideals of the Noetherian ring R which is (right) stable and satisfies the (right) strong second layer condition, as defined in Section 3. Assume also that if $P,Q\in\mathscr{P}$ and $P\subseteq Q$, then $P=Q$. Let E be an indecomposable injective right R-module with assassinator P, where $P\in\mathscr{P}$. The *fundamental series* $\{E_n : n\geqq 0\}$ of submodules of E is constructed by setting $E_0=0$, and defining E_n/E_{n-1} to be the sum of all the finitely generated submodules of E/E_{n-1} whose annihilator is a prime ideal, for $n>0$. Then $E=\bigcup_{n\geq 0}E_n$, and E_n/E_{n-1} is a direct sum of Q-prime modules, for various

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prime ideals Q. The primes Q occurring as annihilators of these summands of E_n/E_{n-1} are each linked by a chain $Q \rightsquigarrow P_{n-1} \rightsquigarrow \cdots \rightsquigarrow P_1 = P$ of exactly $(n-1)$ second layer links to P, [**2**, Lemma 5.4], [**10**].

When R is as in Section 3 and $\mathscr{P}$ is the set of reflexive primes of R, we can put more flesh on the bones laid out above.

Theorem 4.1. *Let R be a prime Noetherian v-H order with enough v-invertible ideals. Let P be a reflexive prime ideal of R, with clique of P the cycle $\{P = M_1, \ldots, M_n\}$ (in correct cyclical order). Let E be an indecomposable injective right R-module with assassinator P.*

(i) *$Ann_R(E_t/E_{t-1}) = M_{t(\text{modulo } n)}$, for all $t \geqq 1$.*

(ii) *If $Ann_E(P)$ is a torsion free R/P-module (that is, if E is a summand of $E_R(R/P)$, then E_t/E_{t-1} is the irreducible module over the simple Artinian quotient ring of $R/M_{t(\text{modulo } n)}$, for all $t \geqq 1$.*

(iii) *If $Ann_E(P)$ is a torsion R/P-module, then E_t/E_{t-1} is a torsion $R/M_{t(\text{modulo } n)}$-module, for all $t \geqq 1$.*

Proof. Let I be the maximal v-invertible ideal in P, so $I = \bigcap_{i=1}^n M_i$ by (2.5). By (2.3)(c) and (2.5), I is classically localisable, the localised ring R_I being an *HNP* ring with maximal invertible ideal $J(R_I) = IR_I$, and with $\{PR_I, M_2R_I, \ldots, M_nR_I\}$ the unique cycle of maximal ideals of R_I.

(ii) Suppose that $Ann_E(P)$ *is* R/P-torsion free. If E contained $\mathscr{C}(I)$-torsion elements, then so would its essential submodule $Ann_E(P)$, contradicting the hypothesis, since $\mathscr{C}(I) \subseteq \mathscr{C}(P)$. Hence E is $\mathscr{C}(I)$-torsion free, and so is easily seen to admit a structure as an R_I-module. In fact, as such it is the injective hull of the irreducible R_I/PR_I-module, and the R_I-submodules E_t constitute the socle series—again, this is routine to confirm. The nature of the layers E_t/E_{t-1} can now be deduced from the well-known description of injective indecomposables over an *HNP* ring with enough invertible ideals; see, for example, [**7**, Theorem 22].

(i) This follows at once from (ii) and the fact that the annihilator of each layer is an invariant of P, rather than simply of E [**2**, Corollary 5.9].

(iii) This follows from (i) and Corollary 3.5.

It would be interesting to know whether, in the setting of Theorem 4.1(iii), the layers E_t/E_{t-1} are uniform, or injective, as $R/M_{t(\text{modulo } n)}$-modules. (Both are true when I is polynormal, and the latter is true when I is invertible—see [**2**, §6].)

We now apply Theorem 4.1 to a question concerning the stability of certain torsion theories. We continue to assume that R is a prime Noetherian ring. Let Q denote the simple Artinian quotient ring of R, and put

$$S(R) = \{q \in Q \colon qI \subseteq R, 0 \neq I \lhd R\}.$$

Put $E = Q \oplus E_R(S(R)/R)$, and let τ be the torsion theory cogenerated by E. Recall that a torsion theory ρ is *stable* if the injective hull of a ρ-torsion module is ρ-torsion.

Theorem 4.2. *Let R be a prime Noetherian v-H order with enough v-invertible ideals, and define τ as above. Then τ is stable.*

Proof. Observe that E is a direct sum of Q and copies of $E(U_P)$, where U_P is a uniform right ideal of R/P, and P ranges over the reflexive prime ideals of R. Let V be a non-zero τ-torsion R-module, so that there are no non-zero homomorphisms from V to E; we must show that the same is true of $E(V)$.

Suppose, then, that $E(V)$ has a subfactor isomorphic to U_P, a uniform right ideal of the factor of R by the reflexive prime P. Thus, taking X to be an appropriate cyclic submodule of an indecomposable summand of $E(V)$, there exists a short exact sequence

$$0 \to V' \to X \to U_P \to 0 \tag{1}$$

where X is cyclic and uniform, and V' is an essential extension of V. By a routine argument using a critical composition series of V', we may assume that V' is a Q-prime τ-torsion module for some prime ideal Q.

Suppose first that

$$Q \neq 0. \tag{2}$$

Now $V' \subseteq Ann_X(Q)$. Suppose that this inclusion is strict. Thus, $Q \subseteq P$. The height of P is 1 by (2.4), and so, in view of (2), $Q = P$. Replacing X by $Ann_X(Q)$, (and noting that, still, by our supposition, $U_P \neq 0$), (1) becomes a sequence of R/P-modules, so X cannot be uniform since a finite direct sum of copies of U_P contains a non-zero free R/P-module.

Suppose now that (2) holds, but $V' = Ann_X(Q)$. Again replacing X by a submodule if necessary, we may assume that $Ann_R(X)$ is maximal amongst annihilators of submodules not contained in V'. Then [**8**, Lemma 6.1.2] applies, and we deduce that *either* (a) $P \rightsquigarrow Q$, *or* (b) $Ann(X) = P \subsetneq Q$.

If (a) holds, then Q is a reflexive prime by Theorem 3.4. Since V' is τ-torsion, it must be a torsion R/Q-module, so Theorem 4.1(iii) shows that $X/V' \cong U_P$ is R/P-torsion. This contradiction rules out (a).

In case (b), then (as for $Q = P$), (1) is a sequence of R/P-modules, and a contradiction follows in the same way as before.

We are thus left with the case where $Q = 0$. Here Proposition 4.3 below applies to show that a sequence (1) cannot exist.

Hence if V is τ-torsion, then so is $E(V)$, and the theorem is proved.

Proposition 4.3. *Let R be a prime Noetherian v-H order with enough v-invertible ideals. Let P be a reflexive prime ideal of R and let I be the maximal v-invertible ideal contained in P. Let*

$$0 \to V \to X \to U \to 0$$

be an exact sequence of finitely generated R-modules, where (a) V is 0-prime, (b) U is a uniform right ideal of R/P, and (c) $Hom_R(V, E(R/I)) = 0$. Then X is not a uniform module.

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Proof. Assume that X is uniform. Choose $x \in X \backslash V$. The exact sequence $0 \to xR \cap V \to xR \to (xR+V/V) \to 0$ inherits all the above conditions and xR is uniform. Thus, without loss of generality, assume that $X = xR$ and aim to produce a contradiction. Note the inclusions $V \supseteq XI \supseteq VI \supseteq XI^2$.

Now $Hom_R(V/XI^2, E(R/I)) = 0$, by (c). Thus, if X/XI^2 is an essential extension of V/XI^2 then the density condition for I, as given by Theorem 3.2, implies that X/V is R/I-torsion, a contradiction. Hence, there exists a submodule Y such that $Y \nsubseteq V$ and $Y \cap V = XI^2$. Since $YI \subseteq V$ this forces $YI \subseteq XI^2$.

In order that multiplication by I^{-1} makes sense, it is convenient to move into the ring R. Set $L = \{r \in R : xr = 0\}$, $E = \{r \in R : xr \in V\}$, and $F = \{r \in R : xr \in Y + V\}$. Thus $L \subseteq E \subsetneqq F$, $xE = V$ and $xF = Y + V$. Now $R/E \cong X/V = U$ is isomorphic to a uniform right ideal of R/P and so is R/I-torsion free. Therefore

$$E \cap \mathscr{C}(I) = \phi. \tag{1}$$

However, R/F is a proper image of U and so is R/I-torsion; that is,

$$F \cap \mathscr{C}(I) \neq \phi. \tag{2}$$

The inclusions $xFI \subseteq (Y+V)I \subseteq XI^2 + VI \subseteq VI = xEI$ imply that $FI \subseteq EI + L$. Therefore, $FII^{-1} \subseteq EII^{-1} + LI^{-1}$ and so, more precisely, $FII^{-1} \subseteq EII^{-1} + (LI^{-1} \cap R)$. If $LI^{-1} \cap R \neq L$ then choose $a \in (LI^{-1} \cap R) \backslash L$. Since $aI \subseteq LI^{-1}I \subseteq L$, we have $xa \neq 0$ while $xaI = 0$, contradicting hypothesis (a). Thus $LI^{-1} \cap R = L$ and so $FII^{-1} \subseteq EII^{-1} + L \subseteq E$. However, $FII^{-1} \cap \mathscr{C}(I) \neq \phi$ by (2), while $E \cap \mathscr{C}(I) = \phi$, by (1), a contradiction.

Remarks 4.4. We sketch here another view of the preceding two results. Throughout, assume that R is a prime Noetherian v-H order with enough v-invertible ideals.

(i) Let I be a maximal v-invertible ideal of R. For an R-module X, denote by $\tau_I(X)$ the $\mathscr{C}(I)$-torsion submodule of X. Arguments similar to those used to deduce Theorem 4.2 from Theorem 4.1 and Proposition 4.3 will also yield the conclusion that *the torsion theory defined by* τ_I *is stable.* Moreover it is easy to check that

$$\tau = \bigcap_I \tau_I,$$

the intersection being over all the maximal v-invertible ideals of R. Since an intersection of stable torsion theories is clearly stable, we retrieve the conclusion of Theorem 4.2. (Other settings where localisation preserves essential extensions are discussed in [6].)

(ii) Let E be an injective R-module, say $E = E(X)$. One can deduce from the preceding results that $\tau_I(E)$ consists of all indecomposable summands of E which are not *either* (a) R-torsion free (i.e. isomorphic to the irreducible right ideal of $Q(R)$) *or* (b) isomorphic to a summand of $E(R/I)$. One can therefore conclude that

$$E(X) \otimes_R R_I = E(X \otimes_R R_I) = E/\tau_I(E)$$

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is the largest summand of $E(X)$ consisting of modules of types (a) and (b). Thus, $E/\tau(E)$ is the direct sum of all summands of E which are summands of $Q(R)$, or of $E(R/I)$, for I v-invertible. For fully bounded maximal orders, these remarks are obtained as [**4**, Corollaire 5.8].

(iii) A final consequence of the stability of τ, due to Chamarie in the case of bounded maximal orders, is also accessible in the present context. Let Mod R denote the category of right R-modules; then *the quotient category* Mod R/τ *has injective dimension at most one*. Theorem 4.2 and (ii) above can be used exactly as in the proof of [**4**, Théorème 5.9] to obtain this conclusion.

As already remarked, for the case of a bounded maximal order, Theorem 4.2 is a result of M. Chamarie [**4**, Corollaire 5.5]. This improved on earlier work of R. Fossum on orders satisfying a polynomial identity [**5**]; the commutative case is due to Beck [**1**]. (In all these cases, the Noetherian hypothesis can be weakened.) One might first suspect that the "correct" generalisation of Chamarie's result would give the stability of the torsion theory ρ cogenerated by $E(R)\oplus E(Q/R)$—but, as the following example shows, ρ is *not* stable in general.

Example 4.5. Let U be the enveloping algebra of $sl(2,\mathbb{C})$, let A be the minimal primitive ideal of U contained in the augmentation ideal I of U, and set $R=U/A$ and $M=I/A$. Then R is a prime Noetherian maximal order [**9**, Corollary 2.10] having M as its unique proper ideal. Let Q be the quotient division ring of R.

Claim. *The torsion theory ρ cogenerated by $Q\oplus E(Q/R)$ is not stable.*

By [**14**], R has Krull dimension 1. Let J be a maximal right ideal of R, $J\neq M$. By [**15**, Theorem 2.6] J is projective. It follows that J is reflexive [**13**, 5.1.7]; in particular, Q/R contains a submodule isomorphic to R/J. On the other hand, suppose T/R is a right R/M-module contained in Q/R. Then $TM^2\subseteq M$. But $I^2=I$, so $M^2=M$, and therefore $TM\subseteq M$. Thus $T\subseteq O_l(M)$, and $O_l(M)$ equals R since R is a maximal order; otherwise put, $T/R=0$. We conclude that the ρ-torsion modules are simply the direct sums of copies of R/M. But R/M is *not* an injective R-module; indeed a non-split extension of R/M is afforded by the dual of the Verma module with R/M as irreducible image. Thus ρ is *not* stable.

Note that, in this example, τ is just the Goldie torsion theory.

Since the ring of Example 4.5 fails to satisfy the second layer condition, one might speculate that ρ is always stable for rings with the second layer condition. It seems very unlikely that this should be true—indeed the Weyl algebras $A_n(\mathbb{C})$ (for $n\geqq 2$) probably afford counterexamples—but we are unable to confirm this.

Although Theorem 4.2 demonstrates that, in the bounded case, Chamarie's hypothesis that R be a maximal order is stronger than necessary, (and indeed it is not hard to see that τ is stable when R is a commutative Noetherian domain of Krull dimension 2), the conclusion of Theorem 4.2 is *not* valid for all two-dimensional prime Noetherian PI rings. This is shown by our final example.

Example 4.6. Let k be a field, let $T=k[X,Y]$, let I be the ideal of T with constant term 0, and

$$R=\begin{bmatrix} T & T \\ I & T \end{bmatrix}.$$

Let $F=M_2(k(X,Y))$, the quotient ring of R.

Claim. *The torsion theory τ cogenerated by $F\oplus E(F/R)$ is not stable.*

Indeed, set

$$P=\begin{bmatrix} I & T \\ I & T \end{bmatrix} \text{ and } Q=\begin{bmatrix} T & T \\ I & I \end{bmatrix}.$$

Then, as right modules, there is a non-split extension of R/Q by R/P, since $PQ=M_2(I)\subsetneqq P\cap Q$. But $(R/Q)_{|R}$ is τ-torsion while $(R/P)_{|R}\cong(M_2(T)/R)_{|R}$ is τ-torsion free.

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