A CHARACTERIZATION OF CERTAIN PTOLEMAIC GRAPHS

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1. Introduction. With every connected graph G there is associated a metric space $M(G)$ whose points are the vertices of the graph with the distance between two vertices a and b defined as zero if $a = b$ or as the length of any shortest arc joining a and b if $a \neq b$. A metric space M is called a *graph metric space* if there exists a graph G such that $M = M(G)$, i.e., if there exists a graph G whose vertex set can be put in one-to-one correspondence with the points of M in such a way that the distance between every two points of M is equal to the distance between the corresponding vertices of G.

Necessary and sufficient conditions are given in §3 that a metric space be a graph metric space.

It is natural to impose various conditions on graph metric spaces and to determine the properties which the corresponding graphs possess. In this paper we investigate the graphs whose associated metric spaces are ptolemaic, a metric space (M, d) being called ptolemaic if for each four points $x, y, z, w \in M$, the three numbers $d(x, y) \cdot d(z, w)$, $d(x, z) \cdot d(y, w)$, and $d(x, w) \cdot d(y, z)$ satisfy the triangle inequality (**1**, p. 79). Calling such graphs ptolemaic, we characterize (in §4) the ptolemaic graphs satisfying a certain additional condition, called the weakly geodetic property, as those graphs for which any two vertices on the same circuit are adjacent.

2. Definitions and notation. All graphs considered here are *ordinary graphs*, i.e., finite undirected graphs containing no loops or multiple edges. For the basic definitions and general information on this subject, see (**4** and **5**). We also assume throughout this paper that all graphs are connected.

By an *arc* of length n is meant a finite sequence of edges $(a_0, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n)$, where the vertices $a_0, a_1, \ldots, a_n$ are distinct from one another. If $n \geqslant 3$ and $a_0 = a_n$ but all other vertices are distinct, then this sequence is called a *circuit* of length n and is denoted by

$$C = (a_0, a_1, \ldots, a_{n-1}, a_n = a_0).$$

Any edge (a_i, a_j) with non-consecutively numbered vertices of C as end-points, i.e., with i and j such that $i - j \not\equiv \pm 1 \pmod{n}$, will be called a *diagonal* of C.

If we denote the distance between two vertices a and b by ab, then we can say a graph is *ptolemaic* if for every four of its vertices p, q, r, and s, the numbers $pq \cdot rs$, $pr \cdot qs$, and $ps \cdot qr$ satisfy the triangle inequality.

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Should a graph have the property that there exists a unique shortest arc joining each pair of its vertices, it is said to be *geodetic*. A weaker property to be considered here is defined by assuming only that a unique shortest arc exists joining a pair of vertices a and b whenever $ab = 2$, and any graph which enjoys this weaker property will be called *weakly geodetic*. An example of a geodetic graph is a tree; a simple circuit of length six is an example of a weakly geodetic graph which is not geodetic.

We shall use a definition of "lobe graph" which can be shown to be equivalent to the one proposed by Ore (**5**): The *lobe graph* of a graph G *determined by the edge* E is the subgraph of G consisting of E and all edges F in G such that E and F belong to some circuit in G. It is clear that every edge of G belongs to one and only one lobe graph of G.

A graph in which every lobe graph is either a single circuit or a single edge has been called a *Husimi tree* (cf. **3**). Modifications of such graphs prove to be relevant to ptolemaic graphs.

A vertex v of G is called a *separating vertex* (or *cut point*) of G if the removal of v and all edges incident with it disconnects G. A vertex is a separating vertex if and only if it belongs to two or more lobe graphs of G.

Finally, we define a *terminal lobe graph* as one which has only one separating vertex in G.

3. Graph metric spaces. Let M be a metric space with metric d. If $a, b \in M$, then $x \in M$ is said to be *between* a and b if $d(a, x) > 0$, $d(x, b) > 0$, and $d(a, b) = d(a, x) + d(x, b)$.

We now give a characterization of graph metric spaces.

THEOREM. *A finite metric space* (M, d) *is a graph metric space if and only if* (*i*) *the distance between every two points of* M *is an integer and* (ii) *if* $a, b \in M$ *and* $d(a, b) \geqslant 2$, *then there exists a point* $x \in M$ *such that* x *is between* a *and* b.

Proof. It is clear that every graph metric space satisfies (i) and (ii) above. Let M be a metric space satisfying (i) and (ii), and consider the graph G whose vertex set can be put in one-to-one correspondence with the points of M and where two vertices of G are joined by an edge if and only if the distance between the corresponding points of M is one. It is a routine matter now to check the fact that M is the associated metric space of G.

The familiar metric space (M, d) having $d(x, y)$ equal to 1 or 0 according to whether $x \neq y$ or $x = y$ is now easily seen to be a graph metric space; in fact, (M, d) could be considered as the associated metric space of a complete graph.

4. A characterization of certain ptolemaic graphs. It is a simple matter to show that if $M = M(G)$ is a graph metric space which can be embedded in some Euclidean space, then either G is an arc (and can be embedded in E^1)

or is a complete graph of order n (and can be embedded in E^{n-1}). A less stringent restriction is imposed by assuming that M is ptolemaic. We shall see that a ptolemaic graph having the weakly geodetic property relaxes the condition that a graph be complete and forces only the lobe graphs of the graph to be complete.

Suppose that a graph G has the property that each of its lobe graphs is a complete subgraph of G. We shall call such a graph a *completed Husimi tree*. One characterization of these graphs has been given by Harary **(2)**. We now prove our main result.

THEOREM. *A weakly geodetic graph is ptolemaic if and only if it is a completed Husimi tree.*

Proof. Let G be a completed Husimi tree, and suppose p, q, r, and s are any four vertices of G. Since the triangle inequality is obviously satisfied for the numbers $pq\cdot rs$, $pr\cdot qs$, and $ps\cdot qr$ whenever two or more of the four vertices coincide, it may be assumed that p, q, r, and s are distinct. We employ induction on the number n of vertices in G (the result being trivial for the case $n = 1$), and assume that all completed Husimi trees having fewer than n vertices are ptolemaic.

If p, q, r, and s are all contained in the same lobe graph, then the distance between any two of them is equal to one, and the result follows immediately; hence we may assume that the four vertices do not all belong to a single lobe graph. We may also assume that some one vertex of the four, say p, is a non-separating vertex contained in a terminal lobe graph L of G, for otherwise a terminal lobe graph of G can be deleted with the separating vertex excluded without removing any of p, q, r, and s, and the desired result follows from the induction hypothesis. Let v be the separating vertex of L in G. If u is any vertex of G different from v, then either $up = 1$ if u lies in L, or, if not, $up = uv + 1$ since every arc joining u and p must pass through v. There are three cases to consider.

Case 1. p and two of the three vertices q, r, s are contained in L, say q and r. Then $pq = pr = qr = 1$ and either $ps = qs = rs$ if neither q nor r is v, or else $ps = qs = rs + 1$ if $r = v$, say. In any case, the triangle inequality is satisfied.

Case 2. p and one other vertex of q, r, s are in L, say q. If $q \neq v$, then $pq = 1$, $pr = qr$, and $ps = qs$. The first inequality

$$pq\cdot rs \leqslant pr\cdot qs + ps\cdot qr$$

is obvious while

$$pr\cdot qs < rs + pr\cdot qs = pq\cdot rs + ps\cdot qr,$$

and the third inequality follows like the second.

If $q = v$, then $pq = 1$, $pr = qr + 1$, and $ps = qs + 1$. Here we have $pq\cdot rs = rs$, $pr\cdot qs = qr\cdot qs + qs$, and $ps\cdot qr = qr\cdot qs + qr$, which are easily seen to satisfy the triangle inequality.

Case 3. None of the vertices q, r, s lie in L. We see that $pq = qv + 1$, $pr = rv + 1$, and $ps = sv + 1$. The removal of L with v excluded and an application of the induction hypothesis on q, r, s, and v yields the fact that the three numbers

$$qr \cdot sv = ps \cdot qr - qr, \; qs \cdot rv = pr \cdot qs - qs, \text{ and } qv \cdot rs = pq \cdot rs - rs$$

satisfy the triangle inequality. Since also $rs \leqslant qs + qr$, adding this to

$$pq \cdot rs - rs \leqslant (pr \cdot qs - qs) + (ps \cdot qr - qr),$$

we obtain $pq \cdot rs \leqslant pr \cdot qs + ps \cdot qr$. The other two combinations involved with the triangle inequality are obtained in like manner.

This completes the proof that the condition is sufficient. For the proof of the necessity, let G be a weakly geodetic, ptolemaic graph. If G has no circuits, then it is a tree and is therefore a completed Husimi tree. Otherwise, we use induction on the length n (the number of edges) of the circuits in G to show that all diagonals of all circuits in G are present, thereby proving that G is a completed Husimi tree. For $n = 4$ (the first case in which a circuit can have diagonals), it is clear that such circuits contain all their diagonals from the fact that G is weakly geodetic. Suppose that all circuits of G of length less than n, $n \geqslant 5$, contain all their diagonals. Consider

$$C = (a_0, a_1, a_2, \ldots, a_n = a_0),$$

a circuit of G of length n. If C contains a diagonal $D = (a_r, a_s)$, $r < s$, then D divides C into two circuits of length less than n, namely

$$C_1 = (a_0, a_1, \ldots, a_r, a_s, a_{s+1}, \ldots, a_n = a_0)$$

and

$$C_2 = (a_r, a_{r+1}, \ldots, a_{s-1}, a_s, a_r).$$

By hypothesis, C_1 and C_2 contain all their diagonals. Let a_i be on C_1 and a_j be on C_2 and both be different from a_r or a_s. Then

$$C_3 = (a_i, a_r, a_j, a_s, a_i)$$

is a circuit of length four and therefore contains all its diagonals. In particular, the edge (a_i, a_j) is present in G and is a diagonal of C. It follows that C contains all its diagonals, that the induction carries, and that G is a completed Husimi tree.

It remains to be shown that C must contain a diagonal such as D. Suppose, on the contrary, that C contains no diagonals at all. First we claim that for any two vertices a_r and a_s, $r < s$, on C, $a_r a_s$ is equal to the minimum of the lengths of the two arcs

$$P_1 = (a_r, a_{r+1}, \ldots, a_s) \text{ and } P_2 = (a_s, a_{s+1}, \ldots, a_0, \ldots, a_r).$$

For, if there were an arc P_3 of shorter length from a_r to a_s, then of necessity, P_1 and P_3, for example, would form a circuit C' of length less than or equal to the sum of the lengths of P_1 and P_3, which is less than n, and which contains at least two vertices a_i and a_j of P_1 with $i - j \not\equiv \pm 1 \pmod{n}$. Thus C' must contain all its diagonals, so either $D' = (a_i, a_j)$ is an edge of C' or a diagonal of C', but in either case, D' is a diagonal of C, and this is a contradiction. This proves the assertion. Taking $p = a_0$, $q = a_1$, $r = a_2$, and $s = a_k$ with $k = [\frac{1}{2}n] + 1$, it follows that

$$pq \cdot rs = 1 \cdot (k - 2) = [\tfrac{1}{2}n] - 1,$$

$$pr \cdot qs = 2(k - 1) = 2[\tfrac{1}{2}n] \geqslant n - 1,$$

and

$$ps \cdot pr = (n - k) \cdot 1 = n - [\tfrac{1}{2}n] - 1,$$

so that

$$pq \cdot rs + ps \cdot qr = n - 2 < n - 1 \leqslant pr \cdot qs,$$

which violates the fact that G is ptolemaic. Therefore, C must contain at least one diagonal, hence all diagonals, and so G is a completed Husimi tree.

Since a completed Husimi tree is a geodetic graph, we arrive immediately at the following corollary:

COROLLARY. *A weakly geodetic ptolemaic graph is a geodetic graph.*

REFERENCES

1. L. Blumenthal, *Distance geometry* (Oxford, 1953).
2. Frank Harary, *A characterization of block-graphs*, Can. Math. Bull., *6* (1963), 1–6.
3. Frank Harary and Robert Z. Norman, *The dissimilarity characteristic of Husimi trees*, Ann. Math. (2), *58* (1953), 134–41.
4. E. A. Nordhaus and B. M. Stewart, *Triangles in an ordinary graph*, Can. J. Math., *15* (1963), 33–41.
5. O. Ore, *Theory of graphs* (Providence, 1962).

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