BULL. AUSTRAL. MATH. SOC. 41A25, 41A10, 41A36
VOL. 42 (1990) [253–266]

APPROXIMATION BY BOOLEAN SUMS OF LINEAR OPERATORS: TELYAKOVSKIĬ-TYPE ESTIMATES

JIA-DING CAO AND HEINZ H. GONSKA

In the present note we study the question: "Under which general conditions do certain Boolean sums of linear operators satisfy Telyakovskiĭ-type estimates?" It is shown, in particular, that any sequence of linear algebraic polynomial operators satisfying a Timan-type inequality can be modified appropriately so as to obtain the corresponding upper bound of the Telyakovskiĭ-type. Several examples are included.

1. INTRODUCTION

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers. For $f \in C[a, b]$ (real-valued and continuous functions on the compact interval $[a, b]$), let $\|f\| := \max\{|f(t)| : a \leqslant t \leqslant b\}$ denote the Čebyšev norm of f. By c, $\tilde{c}$ we denote positive absolute constants independent of n, f, and $x \in [a, b]$. The constants c and $\tilde{c}$ may be different at different occurrences, even on the same line. Let Π_n be the set of algebraic polynomials of degree $\leqslant n$. For $f \in C[a, b]$, the modulus of continuity of f is defined by

$$\omega(f, \delta) := \sup\{|f(x_1) - f(x_2)| : |x_1 - x_2| \leqslant \delta\}, \quad 0 \leqslant \delta \leqslant b - a.$$

In his well-known paper [26] Timan (see also [27]) proved the following

THEOREM A. *For $n \in \mathbb{N}$ and $f \in C[-1, 1]$ there exists $P_n(f, \cdot) \in \Pi_n$ such that*

$$|f(x) - P_n(f, x)| \leqslant c \cdot \omega\left(f, \left(1 - x^2\right)^{1/2} \cdot n^{-1} + n^{-2}\right), \ |x| \leqslant 1.$$

Telyakovskiĭ[25] improved this to

THEOREM B. *For $n \in \mathbb{N}$ and $f \in C[-1, 1]$ there exists $P_n(f, \cdot) \in \Pi_n$ such that*

$$|f(x) - P_n(f, x)| \leqslant c \cdot \omega\left(f, \left(1 - x^2\right)^{1/2} \cdot n^{-1}\right), \ |x| \leqslant 1.$$

Theorem B provided a partial answer (that is, for the case of arbitrary continuous functions) to a problem posed by Lorentz [17, p.185] in 1963. Quite elementary proofs

Received 26 October 1989

of Theorems A and B were given by Pičugov and Lehnhoff (see [**21, 14, 15**]) who used operators of the following type.

Let, for $f \in C[-1, 1]$ and $n \in \mathbb{N}$,

$$K_{m(n)}(v) := \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cos kv, \quad \text{and}$$
$$G_{m(n)}(f; x) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos(\arccos x + v)) \cdot K_{m(n)}(v) dv.$$

Here, the kernel $K_{m(n)}$ is a trigonometric polynomial of degree $m(n)$ such that

(i) $K_{m(n)}$ is positive and even, and

(ii) $\int_{-\pi}^{\pi} K_{m(n)}(v) dv = \pi$.

This implies that $G_{m(n)}(f, \cdot)$ is an algebraic polynomial of degree $m(n)$. For $s \in \mathbb{N}$, the Masuoka (higher order Jackson) kernels are given by

$$K_{sn-s}(v) = c_{n,s} \left[\frac{\sin(nv/2)}{\sin(v/2)}\right]^{2s},$$

where $c_{n,s}$ is chosen such that $\pi^{-1} \int_{-\pi}^{\pi} K_{sn-s}(v) dv = 1$.

In his second paper Lehnhoff investigated certain Boolean sum modifications of the operators $G_{m(n)}$ in order to prove Theorem B. His research was continued by the second author in [**10, 11**], and by the first in [**1, 2**]. Both investigated Timan-type estimates for the $G_{m(n)}$ and Telyakovskiĭ-type estimates for their modifications $G^{+}_{m(n)}$. In [**3**] the present authors investigated approximation by Boolean sums of *positive linear* operators.

In this article, we study the more general problem of approximation by Boolean sums of *linear* operators. We establish several general results. Our central Theorem 3 shows that, from a Timan-type estimate for linear algebraic polynomial operators A_n, one can always derive a Telyakovskiĭ-type estimate for their Boolean sum modifications A^{+}_{n}. This fact is applied to certain operators $G_{m(n)}$ having the property that $1 - \rho_{1,m(n)} = \mathcal{O}(n^{-2})$, $n \to \infty$. Furthermore, in Section 4 we give applications to Bernstein-Schurer polynomials, Kantorovič polynomials, and Durrmeyer polynomials.

2. Two general theorems

Let $f \in C[a, b]$ and Lf denote the linear function interpolating f at a and b, that is,

$$L(f, x) := \frac{f(b)(x-a) + f(a)(b-x)}{b-a}, \quad a \leqslant x \leqslant b.$$

https://doi.org/10.1017/S0004972700028410 Published online by Cambridge University Press

Let $A\colon C[a, b] \to C[a, b]$ be a linear operator. Then the Boolean sum $L \oplus A$ of L and A is given by

$$A^+ := L \oplus A = L + A - L \circ A.$$

More explicitly,

$$(2.1)\quad A^+(f, x) = A(f, x) + (b-a)^{-1}\{(x-a)[f(b) - A(f, b)] + (b-x)[f(a) - A(f, a)]\}.$$

Note that, if $A(f, a) = f(a)$ and $A(f, b) = f(b)$ for all $f \in C[a, b]$, then $A^+ = A$.

For $k \in \mathbb{N} \cup \{0\}$, let $C^k[a, b]$ denote the space of k-fold continuously differentiable functions. Hence $f \in C^k[a, b]$ means that $f^{(k)} \in C[a, b]$. Using the K-functional technique, one obtains the following:

LEMMA 1. (see, for example, DeVore [7]). *Let $H_n\colon C[a, b] \to C[a, b]$ be a sequence of linear operators, satisfying the following conditions:*

(i) $\|H_n f\| \leqslant c \cdot \|f\|$ *for all* $f \in C[a, b]$.
(ii) *For* $a \leqslant x \leqslant b$, $0 \leqslant \varepsilon_n(x) \leqslant b - a$ *and* $h \in C^1[a, b]$, *one has* $|H_n(h, x) - h(x)| \leqslant c \cdot \varepsilon_n(x) \cdot \|h'\|$.

Then for all $f \in C[a, b]$, $|H_n(f, x) - f(x)| \leqslant c \cdot \omega(f, \varepsilon_n(x))$.

In the following theorem it is shown that Telyakovskiĭ-type estimates for operators of the $L \oplus A$ type hold under quite general conditions.

THEOREM 1. *Let $A_n\colon C[a, b] \to C^1[a, b]$ be a sequence of linear operators, satisfying the following conditions:*

(i) $\|A_n f\| \leqslant c \cdot \|f\|$ *for all* $f \in C[a, b]$,
(ii) $\|d(A_n(h, x))/dx\| \leqslant c \cdot \|h'\|$ *for all* $h \in C^1[a, b]$,
(iii) $|A_n(h, x) - h(x)| \leqslant c \cdot \left(\varepsilon_n \sqrt{(x-a)(b-x)} + \varepsilon_n^2\right) \cdot \|h'\|$ *for all* $h \in C^1[a, b]$, *all* $a \leqslant x \leqslant b$ and $0 \leqslant \varepsilon_n \leqslant 2$.

Then for all $f \in C[a, b]$ *we have*

$$\left|A_n^+(f, x) - f(x)\right| \leqslant c \cdot \omega\left(f, \varepsilon_n \sqrt{(x-a)(b-x)}\right).$$

PROOF: Using the method applied by Lehnhoff in [15], among others, we distinguish three cases:

CASE (A). $\varepsilon_n \leqslant \sqrt{(x-a)(b-x)}$, $a \leqslant x \leqslant b$, which implies $\varepsilon_n^2 \leqslant \varepsilon_n \sqrt{(x-a)(b-x)}$.

https://doi.org/10.1017/S0004972700028410 Published online by Cambridge University Press

If $h \in C^1[a, b]$, from (2.1) and condition (iii) we have

$$
\begin{aligned}
&\left|h(x) - A_n^+(h, x)\right| \\
&\quad \leqslant |h(x) - A_n(h, x)| + \frac{x-a}{b-a}|h(b) - A_n(h, b)| + \frac{b-x}{b-a}|h(a) - A_n(h,a)| \\
&\quad \leqslant c \cdot \left(\varepsilon_n \sqrt{(x-a)(b-x)} + \varepsilon_n^2\right) \cdot \|h'\| + c \cdot \varepsilon_n^2 \cdot \|h'\| + c \cdot \varepsilon_n^2 \cdot \|h'\| \\
&\quad \leqslant c \cdot \left(\varepsilon_n \sqrt{(x-a)(b-x)} + \varepsilon_n^2\right) \cdot \|h'\| \\
&\quad \leqslant c \cdot \varepsilon_n \sqrt{(x-a)(b-x)} \cdot \|h'\|.
\end{aligned} \tag{2.2}
$$

CASE (B). $\sqrt{(x-a)(b-x)} \leqslant \varepsilon_n$, $(a+b)/2 \leqslant x \leqslant b$, hence $(x-a)(b-x) \leqslant \varepsilon_n \sqrt{(x-a)(b-x)}$. From (2.1) we get

$$
\begin{aligned}
h(x) - A_n^+(h, x) = {} & [h(x) - h(b)] - [A_n(h, x) - A_n(h, b)] \\
& + \frac{b-x}{b-a}\{[h(b) - A_n(h, b)] - [h(a) - A_n(h, a)]\}.
\end{aligned} \tag{2.3}
$$

Defining
$$I_n(x) := |A_n(h, x) - A_n(h, b)|,$$

we have from (iii)

$$
\begin{aligned}
&\left|h(x) - A_n^+(h, x)\right| \\
&\quad \leqslant |h(x) - h(b)| + I_n(x) + \frac{b-x}{b-a}\{|h(b) - A_n(h, b)| + |h(a) - A_n(h, a)|\} \\
&\quad \leqslant (b-x)\|h'\| + I_n(x) + \frac{c(b-x)}{b-a} \cdot \varepsilon_n^2 \cdot \|h'\|.
\end{aligned}
$$

By condition (ii),

$$\left\|\frac{d}{dv} A_n(h, v)\right\| \leqslant c \cdot \|h'\|,$$

and thus

$$
\begin{aligned}
I_n(x) &= \left|\int_x^b \left(\frac{d}{dv} A_n(h, v)\right) dv\right| \leqslant \int_x^b \left|\frac{d}{dv} A_n(h, v)\right| dv \\
&\leqslant c(b-x) \cdot \|h'\|.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\left|h(x) - A_n^+(h, x)\right| \\
&\quad \leqslant (b-x) \cdot \|h'\| + c(b-x) \cdot \|h'\| + 4 \cdot \frac{c(b-x)}{b-a}\|h'\| \\
&\quad \leqslant c(b-x) \cdot \|h'\|.
\end{aligned}
$$

Using the fact that

$$1 \leqslant \frac{2(x-a)}{b-a} \quad \text{for} \quad \frac{a+b}{2} \leqslant x \leqslant b,$$

we get

$$(b-x) \leqslant \frac{2(x-a)(b-x)}{b-a}.$$

Thus

$$(2.4) \qquad \left|h(x) - A_n^+(h, x)\right| \leqslant \frac{2c}{b-a}(x-a)(b-x) \cdot \|h'\| \leqslant c \cdot \varepsilon_n \sqrt{(x-a)(b-x)} \cdot \|h'\|.$$

CASE (C). $\sqrt{(x-a)(b-x)} \leqslant \varepsilon_n$, $a \leqslant x \leqslant (a+b)/2$, hence $(x-a)(b-x) \leqslant \varepsilon_n \sqrt{(x-a)(b-x)}$. From (2.1) we get

$$(2.5) \qquad h(x) - A_n^+(h, x) = [h(x) - h(a)] + [A_n(h, a) - A_n(h, x)] + \frac{x-a}{b-a}\{[h(a) - A_n(h, a)] + [A_n(h, b) - h(b)]\}.$$

On account of the fact that $x - a \leqslant 2(x-a)(b-x)/(b-a)$ for $a \leqslant x \leqslant (a+b)/2$, we get again, by means of a method analogous to the one used in Case (B),

$$(2.6) \qquad \left|h(x) - A_n^+(h, x)\right| \leqslant c \cdot \varepsilon_n \sqrt{(x-a)(b-x)} \cdot \|h'\|.$$

Combining (2.2), (2.4), and (2.6) we have, for $n \in \mathbb{N}$ and $a \leqslant x \leqslant b$,

$$(2.7) \qquad \left|h(x) - A_n^+(h, x)\right| \leqslant c \cdot \varepsilon_n \sqrt{(x-a)(b-x)} \cdot \|h'\|.$$

Since $\sqrt{(x-a)(b-x)} \leqslant (b-a)/2$ $(a \leqslant x \leqslant b)$ we obtain

$$0 \leqslant \varepsilon_n \sqrt{(x-a)(b-x)} \leqslant b - a.$$

From (2.1) and condition (i) we have

$$(2.8) \qquad \left\|A_n^+ f\right\| \leqslant c \cdot \|f\| \quad \text{for} \quad f \in C[a, b].$$

Now, from (2.7) and (2.8) and using Lemma 1, we obtain Theorem 1. □

The following is a generalisation of Theorem 5.6 in [**3**].

THEOREM 2. *Let $A_n \colon C[a, b] \to C^1[a, b]$ be a sequence of positive linear operators satisfying the following conditions:*

(i) $A_n(1, x) = 1$, $a \leqslant x \leqslant b$,

(ii) $A_n(|t-x|, x) \leqslant c \cdot \left(\varepsilon_n \sqrt{(x-a)(b-x)} + \varepsilon_n^2\right)$, $a \leqslant x \leqslant b$, $0 \leqslant \varepsilon_n \leqslant 2$,

(iii) $\|dA_n(h, x)/dx\| \leqslant c \cdot \|h'\|$ *for all* $h \in C^1[a, b]$.

https://doi.org/10.1017/S0004972700028410 Published online by Cambridge University Press

Then for all $f \in C[a,b]$,

$$\left|A_n^+(f,x) - f(x)\right| \leqslant c \cdot \omega\left(f, \varepsilon_n\sqrt{(x-a)(b-x)}\right).$$

PROOF: Note that, for $f \in C[a,b]$,

$$|A_n(f,x)| \leqslant |A_n(1,x)|\,\|f\| = \|f\|, \text{ and thus } \|A_n(f)\| \leqslant \|f\|. \tag{2.9}$$

Since A_n is a sequence of positive linear operators and because of conditions (i) and (ii), we have for $h \in C^1[a,b]$

$$\begin{aligned} |A_n(h(t),x) - h(x)| &= |A_n(h(t) - h(x), x)| \\ &\leqslant A_n(|h(t) - h(x)|, x) \\ &\leqslant \|h'\| \cdot A_n(|t-x|, x) \\ &\leqslant c \cdot \left(\varepsilon_n\sqrt{(x-a)(b-x)} + \varepsilon_n^2\right) \cdot \|h'\|. \end{aligned} \tag{2.10}$$

From (2.9), (2.10), condition (iii) and Theorem 1 one arrives at the claim of Theorem 2. ∎

3. APPLICATION: TELYAKOVSKIĬ-TYPE ESTIMATES

In this section we shall demonstrate how estimates of the Telyakovskiĭ-type can be derived from Timan-type inequalities. In the sequel, let

$$\triangle_n(x) := \max\{(1-x^2)^{1/2} \cdot n^{-1}, n^{-2}\}, \ |x| \leqslant 1.$$

We shall need the following auxiliary result from [**3**, Lemma 5.4].

LEMMA 2. *Let* $n \geqslant 1$, $m(n) \in \mathbb{N} \cup \{0\}$ *and* $cn \leqslant m(n) \leqslant \tilde{c}n$ $(n \geqslant 2)$. *If, for* $p_{m(n)} \in \Pi_{m(n)}$ *and* $f \in C^1[-1,1]$,

$$\left|f(x) - p_{m(n)}(x)\right| \leqslant c \cdot \triangle_n(x) \cdot \|f'\|,$$

then

$$\left|f'(x) - p'_{m(n)}(x)\right| \leqslant c \cdot \|f'\|.$$

The main result of this note is contained in

THEOREM 3. *Let* $n \geqslant 1$, $m(n) \in \mathbb{N} \cup \{0\}$ *and* $cn \leqslant m(n) \leqslant \tilde{c}n$ $(n \geqslant 2)$. *Let* $A_n\colon C[-1,1] \to \Pi_{m(n)}$ *be a sequence of linear operators. Suppose that, for all* $f \in C[-1,1]$, *the following Timan-type estimate holds:*

$$|A_n(f,x) - f(x)| \leqslant c \cdot \omega\left(f, (1-x^2)^{1/2} \cdot n^{-1} + n^{-2}\right), \ |x| \leqslant 1. \tag{3.1}$$

Then for A_n^+ *we have the Telyakovskiĭ-type inequality*

$$|A_n^+(f,x)-f(x)| \leqslant c\cdot\omega\left(f,\,(1-x^2)^{1/2}\cdot n^{-1}\right),\ |x|\leqslant 1. \tag{3.2}$$

PROOF: Using elementary properties of the modulus of continuity we obtain from (3.1) for $f\in C[a,b]$

$$|A_n(f,x)-f(x)|\leqslant 2c\cdot\|f\|,$$

hence

$$\|A_nf\|\leqslant c\cdot\|f\|. \tag{3.3}$$

Furthermore, from (3.1) we have for $h\in C^1[a,b]$

$$\begin{aligned}|A_n(h,x)-h(x)| &\leqslant c\left((1-x^2)^{1/2}\cdot n^{-1}+n^{-2}\right)\cdot\|h'\|\\ &\leqslant 2c\Delta_n(x)\cdot\|h'\|.\end{aligned} \tag{3.4}$$

Using Lemma 2 we arrive at

$$\left|\frac{d}{dx}A_n(h,x)-h'(x)\right|\leqslant c\cdot\|h'\|,$$

and hence

$$\left\|\frac{d}{dx}A_n(h,x)\right\|\leqslant c\cdot\|h'\|. \tag{3.5}$$

Combining (3.3), (3.4), (3.5) and Theorem 1 we obtain Theorem 3. ∎

COROLLARY 1. (compare [3, Theorem 5.6]). *Let* $n\geqslant 1$, $m(n)\in\mathbb{N}\cup\{0\}$, *and* $cn\leqslant m(n)\leqslant\tilde{c}n$ $(n\geqslant 2)$. *Furthermore, let* $A_n\colon C[-1,1]\to\Pi_{m(n)}$ *be a sequence of positive linear operators, satisfying the following conditions:*

(i) $A_n(1,x)=1$,
(ii) $A_n(|t-x|,x)=O\left(\sqrt{1-x^2}\cdot n^{-1}+n^{-2}\right),\ n\to\infty$.

Then we have for $f\in C[-1,1]$

$$|A_n^+(f,x)-f(x)|\leqslant c\cdot\omega\left(f,\,\sqrt{1-x^2}\cdot n^{-1}\right),\ |x|\leqslant 1.$$

PROOF: Since A_n is a sequence of positive linear operators and because of conditions (i) and (ii), we have for $f\in C[-1,1]$, using Popoviciu's theorem (see [22]), the pointwise inequality

$$\begin{aligned}|A_n(f,x)-f(x)| &\leqslant 2\cdot\omega(f,A_n(|t-x|,x))\\ &\leqslant c\cdot\omega\left(f,\,\sqrt{1-x^2}\cdot n^{-1}+n^{-2}\right),\ |x|\leqslant 1.\end{aligned}$$

Using Theorem 3 we obtain Corollary 1. ∎

For the Boolean sum modifications of the operators $G_{m(n)}$ introduced in Section 1 we have

COROLLARY 2. (compare [**1, 2**]). *Let* $n \geqslant 1$, $cn \leqslant m(n) \leqslant \tilde{c}n$ $(n \geqslant 2)$ *and* $K_{m(n)}(v) \geqslant 0$. *If* $1 - \rho_{1,m(n)} = \mathcal{O}(n^{-2})$, *then for* $f \in C[-1, 1]$ *there holds the following*

$$\left|G^{+}_{m(n)}(f, x) - f(x)\right| \leqslant c \cdot \omega\left(f, \sqrt{1 - x^2} \cdot n^{-1}\right), \ |x| \leqslant 1.$$

PROOF: Since $1 - \rho_{1,m(n)} = \mathcal{O}(n^{-2})$ we have (see [**1, 2**])

$$\begin{aligned} &\left|f(x) - G_{m(n)}(f, x)\right| \\ &\quad\leqslant 2 \cdot \omega\left(f, \left(1 - \rho_{1,m(n)}\right) \cdot |x| + \sqrt{2} \cdot \sqrt{1 - \rho_{1,m(n)}} \cdot \sqrt{1 - x^2}\right) \\ &\quad= \mathcal{O}\left(\omega\left(f, \sqrt{1 - x^2} \cdot n^{-1} + n^{-2}\right)\right). \end{aligned}$$

Now Theorem 3 immediately yields the estimate for $G^{+}_{m(n)}$. □

Note that it was shown by DeVore in [**6**, p.81] that the relationship $1 - \rho_{1,m(n)} = \mathcal{O}(n^{-2})$, $n \to \infty$, holds true for the Matsuoka kernels K_{sn-s} with $s \geqslant 2$.

REMARK 1. There are many further applications of Theorem 3. Today a large number of different proofs of the Timan theorem are known. As examples we mention the work of Freud and Vértesi [**9**], Freud and Sharma [**8**], Mills and Varma [**19, 29**], Saxena [**23**], Varma [**28**], Vértesi and Kis [**30**], Gonska and Cao [**3**] who constructed linear algebraic polynomial operators W_n satisfying Timan-type estimates. By Theorem 3, for the corresponding operators W_n^{+} it is clear that they give Telyakovskiĭ-type estimates and thus provide a solution to Lorentz' problem for arbitrary continuous functions.

4. FURTHER APPLICATIONS

We give three applications of Theorem 2 for positive linear operators, all of which are related to the classical Bernstein operators.

EXAMPLE 1. Let $\alpha_n \geqslant 0$ and $f \in C[0, 1]$. The Bernstein-Schurer polynomials are defined by (see [**24**])

$$B_n(\alpha_n, f, x) := \sum_{i=0}^{n} f\left(\frac{i}{n + \alpha_n}\right) p_{n,i}(x), \ p_{n,i}(x) := \binom{n}{i} x^i (1 - x)^{n-i}.$$

If $\alpha_n = 0$, then $B_n(\alpha_n, f, x)$ becomes the Bernstein polynomial $B_n(f, x)$.

LEMMA 3. *The following equalities hold:*

$$B_n(\alpha_n, 1, x) = 1, \quad B_n\left(\alpha_n, (t - x)^2, x\right) = \frac{nx(1 - x) + \alpha_n^2 x^2}{(n + \alpha_n)^2}.$$

PROOF: For a proof see Lemma 2.2 and Corollary 2.4 in [**12**].

Since $B_n(\alpha_n, f, 0) = f(0)$, $B_n(\alpha_n, f, 1) = f(n/(n+\alpha_n))$, we have

$$\begin{aligned} B_n^+(\alpha_n, f, x) &= B_n(\alpha_n, f, x) + x[f(1) - B_n(\alpha_n, f, 1)] + (1-x)[f(0) - B_n(\alpha_n, f, 0)] \\ &= B_n(\alpha_n, f, x) + x[f(1) - f(n/(n+\alpha_n))]. \end{aligned}$$

$\square$

THEOREM 4. *Let* $f \in C[0, 1]$, $0 \leqslant x \leqslant 1$, *and* $\beta_n := \max\left(1/\sqrt{n}, \sqrt{\alpha_n}/\sqrt{n}\right)$. *If* $S := \{n \in \mathbb{N}: \alpha_n \leqslant n\}$, *then*

$$\left|B_n^+(\alpha_n, f, x) - f(x)\right| \leqslant c \cdot \omega\left(f, \beta_n\sqrt{x(1-x)}\right), \quad n \in S.$$

PROOF: Note first that $B_n(\alpha_n, f, x)$ is a positive linear operator. Using Lemma 3 we have

$$\begin{aligned} & B_n(\alpha_n, 1, x) = 1, \\ \text{(4.1)} \qquad & B_n\left(\alpha_n, (t-x)^2, x\right) \leqslant \frac{x(1-x)}{n} + \left(\frac{\alpha_n}{n}\right)^2, \end{aligned}$$

hence (see [**13**])

$$\text{(4.2)} \qquad \begin{aligned} B_n(\alpha_n, |t-x|, x) &\leqslant \sqrt{B_n\left(\alpha_n, (t-x)^2, x\right) \cdot B_n(\alpha_n, 1, x)} \\ &\leqslant \sqrt{\frac{x(1-x)}{n} + \left(\frac{\alpha_n}{n}\right)^2} \leqslant \sqrt{\frac{x(1-x)}{n}} + \frac{\alpha_n}{n}. \end{aligned}$$

In view of the definition of β_n we have

$$\beta_n \geqslant \frac{1}{\sqrt{n}}, \ \beta_n \geqslant \frac{\sqrt{\alpha_n}}{\sqrt{n}} n\beta_n^2 \geqslant \frac{\alpha_n}{n},$$

and so

$$\text{(4.3)} \qquad B_n(\alpha_n, |t-x|, x) \leqslant \beta_n\sqrt{x(1-x)} + \beta_n^2.$$

Note also that, for $n \in S$, we have

$$\frac{\sqrt{\alpha_n}}{\sqrt{n}} \leqslant 1 \text{ and } 0 < \beta_n \leqslant 1.$$

Furthermore, from Lorentz [**16**] it is known that

$$\frac{d}{dx}B_n(f, x) = n \cdot \sum_{i=0}^{n-1}\left\{f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right)\right\} p_{n-1,i}(x), \ f \in C[0, 1].$$

https://doi.org/10.1017/S0004972700028410 Published online by Cambridge University Press

Similarly, if $h \in C^1[0, 1]$, we get

$$\frac{d}{dx}B_n(\alpha_n, h, x) = n \cdot \sum_{i=0}^{n-1} \left\{ h\left(\frac{i+1}{n+\alpha_n}\right) - h\left(\frac{i}{n+\alpha_n}\right) \right\} p_{n-1,i}(x)$$
$$= \frac{n}{n+\alpha_n} \sum_{i=0}^{n-1} h'(\xi_{n,i}) p_{n-1,i}(x),$$

hence

$$\left\| \frac{d}{dx}B_n(\alpha_n, h, x) \right\| \leqslant \frac{n}{n+\alpha_n} \cdot \|h'\| \leqslant \|h'\|. \tag{4.4}$$

Combining (4.1), (4.3), (4.4) and Theorem 2 we obtain the claim of Theorem 4. □

Corollary 3. *Let* $\lim_{n\to\infty} \alpha_n/n = 0$. *Then*

$$\lim_{n\to\infty} \left\| B_n^+(\alpha_n, f, x) - f(x) \right\| = 0.$$

Example 2. Let $f \in L[0, 1]$ (the space of Lebesgue integrable functions defined on $[0, 1]$), and $F_1(u) := \int_0^u f(t)dt$. Then the Kantorovič polynomials are defined by (see [16])

$$P_n(f(t), x) := \frac{d}{dx}B_{n+1}(F_1(u), x)$$
$$= (n+1) \cdot \sum_{i=0}^{n} \left\{ \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)dt \right\} \cdot p_{n,i}(x).$$

We also define the function F_2 by $F_2(u) := \int_0^u \left(\int_0^v f(t)dt\right) dv$. Nagel investigated the Kantorovič operators of second order Q_n given by (see [20])

$$Q_n(f(t), x) := \frac{d}{dx}P_{n+1}(F_1(v), x) = \left(\frac{d}{dx}\right)^2 B_{n+2}(F_2(u), x). \tag{4.5}$$

Theorem 5. *Let* $f \in C[0, 1]$ *and* $0 \leqslant x \leqslant 1$; *then*

$$\left| P_n^+(f, x) - f(x) \right| \leqslant c \cdot \omega\left(f, \sqrt{\frac{x(1-x)}{n+1}}\right).$$

Proof: P_n is a sequence of positive linear operators (see [18]), satisfying

$$P_n(1, x) = 1,$$
$$P_n\left((t-x)^2, x\right) = \frac{n-1}{(n+1)^2}x(1-x) + \frac{1}{3(n+1)^2} \tag{4.6}$$
$$\leqslant \frac{x(1-x)}{n+1} + \frac{1}{(n+1)^2},$$

and

$$(4.7)\qquad P_n(|t-x|,\,x) \leqslant \sqrt{P_n\left((t-x)^2,\,x\right)\cdot P_n(1,\,x)} \leqslant \sqrt{\frac{x(1-x)}{n+1}+\frac{1}{n+1}}.$$

Furthermore, if $h \in C^1[0,\,1]$, then $dP_n(h,x)/dx = (d/dx)^2 B_{n+1}\left(\int_0^u h(v)dv,\,x\right)$, where

$$\int_0^u h(v)dv = \int_0^u \left(\int_0^v h'(t)dt\right)dv + \int_0^u h(0)dv = \int_0^u \left(\int_0^v h'(t)dt\right)dv + h(0)u.$$

From (4.5) we have

$$\frac{d}{dx}P_n(h,\,x) = \left(\frac{d}{dx}\right)^2 B_{n+1}\left[\int_0^u \left(\int_0^v h'(t)dt\right)dv,\,x\right] + h(0)\left(\frac{d}{dx}\right)^2 B_{n+1}(u,\,x) = \left(\frac{d}{dx}\right)^2 B_{n+1}\left[\int_0^u \left(\int_0^v h'(t)dt\right)dv,\,x\right] = Q_{n-1}(h',\,x).$$

Nagel [20] proved that the Q_n are positive linear operators and that

$$Q_n(1,\,x) = 1-(n+2)^{-1}.$$

Hence

$$|Q_{n-1}(h',\,x)| \leqslant Q_{n-1}(1,\,x)\cdot\|h'\| \leqslant \|h'\|,$$

and, consequently,

$$(4.8)\qquad \left\|\frac{d}{dx}P_n(h,\,x)\right\| \leqslant \|h'\|.$$

Combining (4.6), (4.7), (4.8), and using Theorem 2, we obtain Theorem 5. □

EXAMPLE 3. For $f \in L[0,\,1]$, the so-called Durrmeyer operators [5] are given by

$$M_n(f,\,x) := (n+1)\cdot\sum_{i=0}^{n}\left\{\int_0^1 p_{n,i}(t)f(t)dt\right\}\cdot p_{n,i}(x).$$

For their modifications we have

https://doi.org/10.1017/S0004972700028410 Published online by Cambridge University Press

Theorem 6. *Let $f \in C[0, 1]$, $0 \leqslant x \leqslant 1$; then*

$$\left|M_n^+(f, x) - f(x)\right| \leqslant c \cdot \omega\left[f, \frac{\sqrt{x(1-x)}}{\sqrt{n+2}}\right].$$

Proof: It was shown in [5] that M_n is a sequence of positive linear operators for which

(4.9)

$$\begin{aligned} M_n(1, x) &= 1, \\ M_n\left((t-x)^2, x\right) &= (n+1) \cdot \frac{2nx(1-x) - 6x(1-x) + 2}{(n+1)(n+2)(n+3)} \\ &\leqslant \frac{2nx(1-x)}{(n+2)(n+3)} + \frac{2}{(n+2)(n+3)} \leqslant 2\left(\frac{x(1-x)}{n+2} + \frac{1}{(n+2)^2}\right), \end{aligned}$$

and consequently,

(4.10)

$$M_n(|t-x|, x) \leqslant \sqrt{M_n\left((t-x)^2, x\right) \cdot M_n(1, x)} \leqslant \sqrt{2} \cdot \left(\frac{\sqrt{x(1-x)}}{\sqrt{n+2}} + \frac{1}{n+2}\right).$$

If $h \in C^1[0, 1]$, we also have from [5] that

$$\begin{aligned} &\int_0^1 p_{n,i}(t)dt = (n+1)^{-1}, \quad i = 0, 1, \ldots, n, \\ &\frac{d}{dx}M_n(h, x) = n \cdot \sum_{i=0}^{n-1} p_{n-1,i}(x) \cdot \int_0^1 h'(t)p_{n+1,i+1}(t)dt, \\ &\left\|\frac{d}{dx}M_n(h, x)\right\| \leqslant \|h'\| \cdot n \cdot \sum_{i=0}^{n-1} p_{n-1,i}(x) \cdot \int_0^1 p_{n+1,i+1}(t)dt \\ &\qquad = \frac{n}{n+2} \cdot \|h'\| \leqslant \|h'\|. \end{aligned} \tag{4.11}$$

(4.9) through (4.11) and Theorem 2 now imply Theorem 6. ∎

Remark 2. The present authors proved in [4] that $\omega(f, \delta)$ in Theorems 5 and 6 may be replaced by $\omega_2(f, \delta)$, the second order modulus of continuity of f.

References

[1] Jia-Ding Cao, 'Generalizations of Timan's theorem, Lehnhoff's theorem and Teljakowskiĭ's theorem (Chinese)', *Kexue Tongbao* 15 (1986), 1132–1135.

[2] Jia-Ding Cao, 'Generalizations of Timan's theorem, Lehnhoff's theorem and Teljakowskii's theorem', *Schriftenreihe des Fachbereichs Mathematik SM-DU* **106** (1986).

[3] Jia-Ding Cao and H.H. Gonska, 'Approximation by Boolean sums of positive linear operators III: Estimates for some numerical approximation schemes', *Numer. Funct. Anal. Optim.* **10** (1989), 643–672.

[4] Jia-Ding Cao and H.H. Gonska, 'Pointwise estimates for modified positive linear operators', *Portugal. Math.* **46** (1989), 401–430.

[5] M.M. Derrienic, 'Sur l'approximation des fonctions intégrables sur [0, 1] par des polynômes de Bernstein modifiés', *J. Approx. Theory* **31** (1981), 325–343.

[6] R.A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators* (Springer–Verlag, Berlin, Heidelberg, New York, 1972).

[7] R.A. DeVore, 'Degree of approximation', in *Approximation Theory II: Proc. Int. Sympos. Austin, Editor G.G. Lorentz et al*, pp. 117–161 (Academic Press, New York, 1976).

[8] G. Freud and A. Sharma, 'Some good sequences of interpolatory polynomials', *Canad. J. Math.* **26** (1974), 233–246.

[9] G. Freud and P. Vértesi, 'A new proof of A.F. Timan's approximation theorem', *Studia Sci. Math. Hungar.* **2** (1967), 403–414.

[10] H.H. Gonska, 'On Pičugov-Lehnhoff operators', *Schriftenreihe des Fachbereichs Mathematik SM-DU* **86** (1985).

[11] H.H. Gonska, 'Modified Pičugov-Lehnhoff operators', in *Approximation Theory V: Proc. Int. Sympos. College Station,* Editor C.K. Chui et al, pp. 355–358, 1986. (Acad. Press, New York).

[12] H.H. Gonska and J. Meier, 'Quantitative theorems on approximation by Bernstein-Stancu operators', *Calcolo* **21** (1984), 317–335.

[13] P.P. Korovkin, *Linear Operators and Approximation Theory* (Delhi, Hindustan, 1960).

[14] H.G. Lehnhoff, 'A simple proof of A.F. Timan's theorem', *J. Approx. Theory* **38** (1983), 172–176.

[15] H.G. Lehnhoff, 'A new proof of Teljakowskii's theorem', *J. Approx. Theory* **38** (1983), 177-181.

[16] G.G. Lorentz, *Bernstein Polynomials* (University of Toronto Press, Toronto, 1953).

[17] G.G. Lorentz, 'Query in 'New and Unsolved Problems'', in *On Approximation Theory: Proc. Conference Oberwolfach,* Editors P.L. Butzer and J. Korevaar, pp. 179–199 (Birkhäuser, Basel, 1972 (2nd ed.)).

[18] C.P. May, 'A saturation theorem for modified Bernstein polynomials in L_p—-spaces', *SIAM J. Math. Anal.* **10** (1979), 321–330.

[19] T.M. Mills and A.K. Varma, 'A new proof of A.F. Timan's approximation theorem', *Israel J. Math.* **18** (1974), 39–44.

[20] J. Nagel, 'Kantorovič operators of second order', *Monatsh. Math.* **95** (1983), 33–44.

[21] S.A. Pičugov, 'Approximation of continuous functions on a segment by linear methods', (Russian), *Mat. Zametki* **24** (1978), 343–348.

[22] T. Popoviciu, 'Über die Konvergenz von Folgen positiver linearer Operatoren', *An. Ştiinţ.*

Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) **17** (1971), 123–132.

[23] R.B. Saxena, 'The approximation of continuous functions by interpolatory polynomials', *Proc. Math. Inst. Bulgar. Akad. Sci.* **12** (1970), 97–105.

[24] F. Schurer, 'On linear positive operators in approximation theory', *Dissertation* (Delft University of Technology, Delft, The Netherlands, 1965).

[25] S.A. Telyakovskiĭ, 'Two theorems on the approximation of functions by algebraic polynomials', (Russian), *Mat. Sb. (N.S.)* **70** (1966), 252–265.

[26] A.F. Timan, 'Strengthening of Jackson's theorem on best approximation of continuous functions given on a finite interval of the real axis', (Russian), *Dokl. Akad. Nauk SSSR* **78** (1951), 17–20.

[27] A.F. Timan, *Theory of Approximation of Functions of a Real Variable* (Macmillan, New York, 1963).

[28] A.K. Varma, 'A new proof of A.F. Timan's approximation theorem II', *J. Approx. Theory* **18** (1976), 57–62.

[29] A.K. Varma and T.M. Mills, 'On the summability of Lagrange interpolation', *J. Approx. Theory* **9** (1973), 349–356.

[30] P. Vértesi and O. Kis, 'On a new interpolation process', *Ann. Univ. Sci. Budapest* **10** (1967), 117–128.

Department of Mathematics
Fudan University
Shanghai
Peoples Republic of China

Department of Computer Science
European Business School
D–6227 Oestrich–Winkel
West Germany