SOME RESULTS ON COHERENT RINGS II

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According to Bourbaki [**1**, pp. 62–63, Exercise 11], a left (resp. right) A-module M is said to be pseudo-coherent if every finitely generated submodule of M is finitely presented, and is said to be coherent if it is both pseudo-coherent and finitely generated. This Bourbaki reference contains various results on pseudo-coherent and coherent modules. Then, in [**1**, p. 63, Exercise 12], a ring which as a left (resp. right) module over itself is coherent is said to be a left (resp. right) coherent ring, and various results on and examples of coherent rings are presented. The result stated in [**1**, p. 63, Exercise 12*a*] is a basic theorem of [**2**] and first appeared there. A variety of results on and examples of coherent rings and modules are presented in [**3**].

In this note, all rings contain an identity, all modules are unitary, and all ring homomorphisms " preserve " identities. If the underlying ring is non-commutative, all definitions and results will be given for the left side; the " right side " case will be immediate.

The first results presented here concern a ring A with an ideal I which as a left ideal is finitely generated and an A/I-module M. They are used to derive necessary and sufficient coherence conditions on A/I and I for A to be left coherent. This theorem is used to show that the direct product of finitely many left coherent rings is left coherent and another application of this theorem is sketched.

A result of [**3**] states that, if S is a multiplicative system in the commutative coherent ring A, then A_S must also be coherent. Here we show that, if every localization at a maximal ideal of a semi-local ring is coherent, then A is also coherent. Then an example of a commutative non-coherent ring is given whose localization at any maximal ideal is noetherian and hence coherent. Finally, some results on coherent modules over commutative rings are presented.

LEMMA 1. *Let A be a ring, let I be a two-sided ideal of A which is finitely generated as a left ideal, and let M be a finitely generated left A/I-module. Then M is a finitely presented left A-module under pull back along the canonical ring homomorphism $A \xrightarrow{p} A/I$ if and only if M is a finitely presented left A/I-module.*

Proof. Suppose that $M = \sum_{i=1}^{n} (A/I)m_i$; hence we obtain an exact sequence of left A/I-modules:

$$0 \to K \xrightarrow{h} (A/I)^n \xrightarrow{\pi} M \to 0,$$

where $\pi(a_1+I, \ldots, a_n+I) = \sum_{i=1}^{n} (a_i+I)m_i$. We have also the exact sequences of left A-modules

$$0 \to I^n \to A^n \xrightarrow{\alpha} (A/I)^n \to 0,$$

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where $\alpha(a_1, \ldots, a_n) = (a_1+I, \ldots, a_n+I)$ and

$$0 \to L \xrightarrow{k} A^n \xrightarrow{\pi\alpha} M \to 0.$$

Thus we get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & I^n & \xrightarrow{\text{id}} & I^n & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \xrightarrow{k} & A^n & \xrightarrow{\pi\alpha} & M & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \text{id}} & & \\
0 & \to & K & \xrightarrow{h} & (A/I)^n & \xrightarrow{\pi} & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where id always denotes the identity map. Viewing M, $(A/I)^n$ and K as A-modules by pull back along $A \xrightarrow{p} A/I$, we see that all maps are obviously A-homomorphisms of left A-modules. Moreover, all rows and columns are exact. If M is a finitely presented left A/I-module, then K is a finitely generated left A/I-module and hence a finitely generated left A-module. Since I^n is a finitely generated left A-module, the exactness of the first column implies that L is a finitely generated left A-module, whence M is a finitely presented left A-module. Conversely, if M is a finitely presented left A-module, then L and hence K are finitely generated left A-modules. Thus K is a finitely generated left A/I-module and so M is a finitely presented left A/I-module.

COROLLARY 1.1. *If I is a two-sided ideal in the ring A such that I is finitely generated as a left ideal, and if M is a left A/I-module, then M is a pseudo-coherent (resp. coherent) left A/I-module if and only if M is a pseudo-coherent (resp. coherent) left A-module under pull back along the canonical ring homomorphism $A \xrightarrow{p} A/I$.*

THEOREM 2. *If A is a ring with a two-sided ideal I which is finitely generated as a left ideal, then A is left coherent when and only when A/I is a left coherent ring and I is a coherent left A-module.*

Proof. Consider the exact sequence of left A-modules: $0 \to I \to A \xrightarrow{p} A/I \to 0$. If A is a coherent ring, then I is a left coherent module and A/I is a left coherent A-module by Exercise 11*a* on p. 62 of [**1**]. Consequently A/I is a left coherent ring. For the converse, if A/I is a left coherent ring, then A/I is a left coherent A-module. Assuming that I is a left coherent A-module we conclude that A is a left coherent ring by Exercise 11*a* on p. 62 of [**1**].

COROLLARY 2.1. *Let $\{A_i \mid 1 \leqq i \leqq n\}$ be a finite set of left coherent rings; then the product ring $R = \prod_{i=1}^{n} A_i$ is also left coherent.*

Proof. By induction it suffices to treat the case in which $n = 2$. Let $\mathfrak{a}_1 = \{(\alpha, 0) \mid \alpha \in A_1\}$ and $\mathfrak{a}_2 = \{(0, \beta) \mid \beta \in A_2\}$. Now A_1 left coherent implies that $\mathfrak{a}_1$ is a left coherent $R/\mathfrak{a}_2$-module. Hence $\mathfrak{a}_1$ is a left coherent R-module. But $R/\mathfrak{a}_1 \cong A_2$ is a left coherent ring and thus R is left coherent.

By the same method, it is easily seen that, if A is a commutative coherent ring with coherent module M, then the ring $A^* = A \oplus M$, where $(a, m)(a', m') = (aa', a'm + am')$ is also a commutative coherent ring.

COROLLARY 2.2. *Let A be a commutative (quasi) semi-local ring with maximal ideals $m(1), m(2), \ldots, m(n)$ such that each localization $A_{m(i)}$ at $m(i)$ for $1 \leqq i \leqq n$ is coherent; then A must itself be coherent.*

Proof. Since the product ring $\prod_{i=1}^{n} A_{m(i)}$ is coherent and is a faithfully flat A-module by Proposition 10 on p. 111 of [1], we conclude that A is coherent by means of Corollary 2.1 of [3].

However, the assumption that the number of maximal ideals is finite is essential for

THEOREM 3. *There exists a commutative ring A which is not coherent such that, for each maximal ideal m of A, the localization A_m at m is noetherian and consequently coherent. (Each A_m may even be an integral domain.)*

Proof. We utilize the results of [4]. Accordingly, let $\{R_\lambda\}$ be a set of noetherian local rings which contain a common field K (for example the R_λ's may be power series rings $K[[x]]$). Let A be an infinite set for which there is a map ϕ onto the set $\{R_\lambda\}$ and let B be another infinite set. Let $C = A \times B$ and let Ω be the set of functions defined on the disjoint union $A \cup C$ such that if $a \in A$ then $f(a) \in \phi(a)$, and if $c \in C$ then $f(c) \in K$. Let M be the subset of Ω consisting of those f such that $f(c) = 0$ for every $c \in C$, $f(a) = 0$ for all but a finite number of elements a of A, and $f(a)$ is in the maximal ideal of $\phi(a)$ for every $a \in A$. Let K^* be the subset of Ω consisting of those f such that $f(a) = 0$ for all $a \in A$ and $f(c) = 0$ for all but a finite number of elements $c \in C$. Elements $k \in K$ are identified with elements $f \in \Omega$ such that $f(x) = k$ for every $x \in A \cup C$. For each $a \in A$ let e_a denote the element of Ω such that $e_a(x) = 1$ if $x \in \{a\} \cup (\{a\} \times B)$ and zero otherwise, and for each $c \in C$ let e_c denote the element of Ω such that $e_c(x) = 1$ if $x = c$ and zero otherwise. Finally, let T denote the commutative subring of Ω generated by M, K^*, K and the Ke_a. Then, as shown by Nagata in [4], $T = K + K^* + M + \sum Ke_a$ and is such that the total quotient ring of T is T itself; also the localization T_m of T at any maximal ideal m is isomorphic either to K or to one of the R_λ, and for each R_λ there is a maximal ideal $m(\lambda)$ of T such that $T_{m(\lambda)} \cong R_\lambda$. Thus it suffices to show that T is not coherent. In fact, let $a' \in A$ be some fixed element of A and let $\mu \in M$ be such that $\mu(a') \neq 0$ but $\mu(x) = 0$ for all $x \neq a'$; then we will complete the proof by showing that $\mathfrak{c} = \text{ann}(\mu) = \{f \in T \mid f\mu = 0\}$ is a non-infinitely generated ideal of T. Now $f = k + k^* + m + \sum k_a e_a \in \mathfrak{c}$ ($k_a = 0$ except for a finite number of $a \in A$) when and only when

$$f(a')\mu(a') = k\mu(a') + m(a')\mu(a') + k_{a'}\mu(a') = 0.$$

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But $k+k_{a'} \neq 0$ implies that $\mu(a') \in N_{\phi(a')}\mu(a')$, where $N_{\phi(a')}$ is the maximal ideal of $\phi(a')$; this cannot occur. Hence

$$f = k+k^* + m + \sum k_a e_a \in \mathfrak{c}$$

if and only if $k_{a'} = -k$ and $m(a')\mu(a') = 0$, and we conclude that

$$\mathfrak{c} = K(1-e_{a'}) + K^* + M' + \sum_{a \neq a'} Ke_a,$$

where $M' = \{m \in M \mid m(a')\mu(a') = 0\}$. Assume that $\mathfrak{c} = \text{ann}(\mu)$ is generated as an ideal by a finite set $\{t_1, \ldots, t_r\}$; therefore $\mathfrak{c}$ is certainly generated by $\{1-e_{a'}, t_1, \ldots, t_r\}$, where we may assume that

$$t_i = k_i^* + m_i' + \sum_{j=1}^{s} k^i_{a(j)} e_{a(j)},$$

where $a(j) \neq a'$ and $m_i' \in M'$. Since $\{k_1^*, \ldots, k_r^*\}$ has only finite support, $\mathfrak{c}$ must vanish on all but a finite subset of $\{a'\} \times B$; but $\mathfrak{c} \supseteq K^*$, which gives the desired contradiction.

We conclude with some results on coherent modules over commutative rings.

THEOREM 4. *If a commutative ring A has a faithful coherent module M, then A must be coherent.*

Proof. Let $M = \sum_{i=1}^{n} am_i$; then the mapping $f: A \to \prod_{i=1}^{n} (Am_i)$ defined by $f(a) = (am_1, \ldots, am_n)$ is an A-homomorphism of A-modules, which is an injection. But $\prod_{i=1}^{n} (Am_i)$ is the direct sum of the coherent modules (Am_i) for $1 \leqq i \leqq n$, and hence by Exercise 11*c* on p. 62 of [1] is coherent. Thus A is a coherent ring.

COROLLARY 4.1. *If M is a coherent module over the commutative ring A, then*

$$(0: M) = \{a \in A \mid aM = (0)\}$$

is a finitely generated ideal of A and $A/(0:M)$ is a coherent ring.

Proof. Let $M = \sum_{i=1}^{n} Am_i$ and consider the A-homomorphism of A-modules $f: A \to M^n$, where $f(a) = (am_1, \ldots, am_n)$. Since M^n is a coherent A-module, we conclude that $\ker(f) = (0:M)$ is a finitely generated ideal. Finally, M is a coherent faithful $A/(0:M)$-module.

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