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CENTRAL IDEMPOTENTS IN p-ADIC GROUP RINGS

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Abstract

We provide character-free proofs of some results on idempotents in p-adic group rings, centering around Brauer's Second Main Theorem on Blocks.

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The main purpose of this paper is to provide character-free proofs of some (known) results on central idempotents in p-adic group rings of finite groups. The results we have in mind are all directly or indirectly related to Brauer's Second Main Theorem on Blocks. Thus we also prove a character-free version of the Second Main Theorem, using ideas of Puig [18, 19].

In the following, $\mathcal{O}$ will denote a complete discrete valuation ring with algebraically closed residue field $\mathbb{F}$ of prime characteristic p, and $\alpha \mapsto \bar{\alpha}$ will denote the standard epimorphism $\mathcal{O} \to \mathbb{F}$.

Unless stated otherwise, the algebras we will consider are all associative with identity element, and free of finite rank over their ring of coefficients ($\mathcal{O}$ or $\mathbb{F}$). For such an algebra A, we denote by JA its Jacobson radical, by ZA its center, by UA its group of units and by $[A, A]$ its ZA-submodule consisting of all finite sums of elements of the form $[a, b] = ab - ba$ $(a, b \in A)$.

For a finite group G, $\mathcal{O}G$ and $\mathbb{F}G$ denote the group algebras of G over $\mathcal{O}$ and $\mathbb{F}$, respectively. There is a useful map $\lambda : \mathcal{O}G \to \mathcal{O}$ defined in the following way: If $a = \sum_{g \in G} \alpha_g g \in \mathcal{O}G$ with $\alpha_g \in \mathcal{O}$ for $g \in G$ then $\lambda(a) = \alpha_1$. It is well-known that λ vanishes on $[\mathcal{O}G, \mathcal{O}G]$.

We will have to use properties of G-algebras. We therefore recall that a *G-algebra* over $\mathcal{O}$ is a pair consisting of an $\mathcal{O}$-algebra A and a homomorphism ϕ from G into the automorphism group $\operatorname{Aut}(A)$ of A. We write ${}^g a$ instead of $(\phi(g))(a)$ for $g \in G$ and $a \in A$ and define

$$A^H := \{a \in A : {}^h a = a \text{ for } h \in H\},$$

for every subgroup H of G. If K is a subgroup of H and $b \in A^K$ then ${}^{hk}b = {}^h b$ for $h \in H$ and $k \in K$. We therefore write ${}^{hK}b$ instead of ${}^h b$. The *transfer map* $\operatorname{Tr}_K^H : A^K \to A^H$ is then defined by $\operatorname{Tr}_K^H(b) := \sum_{hK \in H/K} {}^{hK}b$ for $b \in A^K$; here H/K denotes the set of cosets hK ($h \in H$). We set $A_K^H := \operatorname{Tr}_K^H(A^K)$. Then A_K^H is an ideal in A^H.

The group algebra $\mathcal{O}G$ will be considered as a G-algebra in such a way that ${}^g a = gag^{-1}$ for $g \in G$ and $a \in \mathcal{O}G$. In this case the map

$$\mathcal{O}G \longrightarrow \mathbb{F}\mathrm{C}_G(Q), \quad \sum_{g \in G} \alpha_g g \longmapsto \sum_{g \in \mathrm{C}_G(Q)} \bar{\alpha}_g g,$$

restricts to a homomorphism $\mathrm{Br}_Q : (\mathcal{O}G)^Q \to \mathbb{F}\mathrm{C}_G(Q)$ which is called the *Brauer homomorphism* with respect to Q, for any p-subgroup Q of G.

We will prove Brauer's Second Main Theorem on Blocks in the following form.

THEOREM 1. *Let G be a finite group, u a p-element in G, s a p-regular element in $\mathrm{C}_G(u)$ and e an idempotent in $\mathrm{Z}\mathcal{O}G$. We denote by e_u the unique idempotent in $\mathrm{Z}\mathcal{O}\mathrm{C}_G(u)$ such that $\mathrm{Br}_{\langle u \rangle}(e) = \mathrm{Br}_{\langle u \rangle}(e_u)$. Then $eus \equiv e_u us$ (mod $[\mathcal{O}G, \mathcal{O}G]$).*

We note that the existence and uniqueness of e_u follow from lifting theorems for idempotents.

In order to get from Theorem 1 the Second Main Theorem in its usual form (see 5.4.1 in [16], for example) we just apply an irreducible character χ to the congruence above (using the fact that χ vanishes on $[\mathcal{O}G, \mathcal{O}G]$).

The reader may wish to consult the references for other proofs of the Second Main Theorem.

PROOF. We may assume that $u \neq 1$ and wish to show that $(e - e_u)su \in [\mathcal{O}G, \mathcal{O}G]$. Since s is a linear combination of idempotents in $\mathcal{O}\langle s \rangle$ it suffices to show that $(e - e_u)fu \in [\mathcal{O}G, \mathcal{O}G]$ for any idempotent f in $\mathcal{O}\mathrm{C}_G(u)$. By the

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definition of e_u, the idempotent $(e - e_u)f \in (\mathcal{O}G)^{\langle u\rangle}$ is contained in the kernel of $\mathrm{Br}_{\langle u\rangle}$. But it is well-known (and easy to see) that $\mathrm{Ker}(\mathrm{Br}_{\langle u\rangle})$ is the sum of the two ideals $(J\mathcal{O})(\mathcal{O}G)^{\langle u\rangle}$ and $(\mathcal{O}G)^{\langle u\rangle}_{\langle v\rangle}$ (where $v := u^p$) of $(\mathcal{O}G)^{\langle u\rangle}$. Since 0 is the only idempotent contained in $(J\mathcal{O})(\mathcal{O}G)^{\langle u\rangle}$ it therefore follows from Rosenberg's lemma (see 5.1 in [13], for example) that $(e - e_u)f \in (\mathcal{O}G)^{\langle u\rangle}_{\langle v\rangle}$. Hence, by Puig's version of Green's indecomposability theorem (see [18] or Theorem 9 below), there is an idempotent j in $(\mathcal{O}G)^{\langle v\rangle}$ orthogonal to ${}^g j$ for $g \in \langle u\rangle \setminus \langle v\rangle$ such that $(e - e_u)f = \mathrm{Tr}^{\langle u\rangle}_{\langle v\rangle}(j)$. Thus

$$ju = ju - j({}^u j)u = j(ju) - (ju)j \in [\mathcal{O}G, \mathcal{O}G]$$

and

$$(e - e_u)fu = \mathrm{Tr}^{\langle u\rangle}_{\langle v\rangle}(ju) \in \mathrm{Tr}^{\langle u\rangle}_{\langle v\rangle}([\mathcal{O}G, \mathcal{O}G] \cap (\mathcal{O}G)^{\langle v\rangle}) \subseteq [\mathcal{O}G, \mathcal{O}G],$$

as we wanted to show.

We would like to add some related results on idempotents. We start with a simple lemma due to Cliff [3]. Similar results can also be found in Oliver [17] and Taylor [20]. The analogous fact in prime characteristic goes back to Brauer. For a recent account, see Külshammer [12].

LEMMA 2. *Let A be the free $\mathcal{O}$-algebra in generators $x_1, \dots, x_k$, and let m and n be non-negative integers such that $m \leq n$. Then $(x_1 + \cdots + x_k)^{p^n} = a + b + c$ where $a \in p^m A$, b is a sum of p^{n-m+1}-th powers of monomials in $x_1, \dots, x_k$, and $c \in [A, A]$.*

PROOF. We write $(x_1 + \cdots + x_k)^{p^n}$ as the sum of the k^{p^n} different terms $y_1, \dots, y_{p^n}$ with $y_1, \dots, y_{p^n} \in \{x_1, \dots, x_k\}$. The cyclic group $Z = \langle z\rangle$ of order p^n acts on the set of these terms in such a way that ${}^z(y_1 y_2 \dots y_{p^n}) = y_2 \dots y_{p^n} y_1$. It is obvious that terms in the same Z-orbit lie in the same coset modulo $[A, A]$. Hence, if B is a Z-orbit containing at least p^m elements then $\sum_{b \in B} b$ is contained in $p^m A + [A, A]$. On the other hand, if B is a Z-orbit containing less than p^m elements then, for $b \in B$, the stabilizer of b in Z has order at least p^{n-m+1}. This means that b is of the form $b = (y_1 \dots y_{p^{m-1}})^{p^{n-m+1}}$, and the result follows.

We wish to apply Lemma 2 to group algebras. Thus let G be a finite group, and let K be a conjugacy class of G. We call K *p-regular* if it consists of p-regular elements, and *p-singular* otherwise. For a subset X of G, we set $X^+ := \sum_{g \in X} g \in \mathcal{O}G$.

The following result is due to Cliff [3].

PROPOSITION 3. *Let G be a finite group, and let e be an idempotent in $\mathcal{O}G$. We write $e = \sum_{g \in G} \epsilon_g g$ with $\epsilon_g \in \mathcal{O}$ for $g \in G$. Then $\sum_{g \in L} \epsilon_g = 0$ for every p-singular conjugacy class L of G.*

PROOF. Let L be a p-singular conjugacy class of G. It suffices to show that $\sum_{g \in L} \epsilon_g \in p^m \mathcal{O}$ for every positive integer m. We therefore fix a positive integer m and choose a positive integer $n \geq m$ such that $g^{p^{n-m+1}} = 1$ for every p-element $g \in G$.

Let A be the free $\mathcal{O}$-algebra in $|G|$ generators x_g ($g \in G$). There is a unique homomorphism of algebras $\phi : A \to \mathcal{O}G$ satisfying $\phi(x_g) = \epsilon_g g$ for $g \in G$. Thus

$$\phi\Big(\Big(\sum_{g \in G} x_g\Big)^{p^n}\Big) = \Big(\sum_{g \in G} \phi(x_g)\Big)^{p^n} = e^{p^n} = e.$$

We write $(\sum_{g \in G} x_g)^{p^n} = a + b + c$ with a, b, c as in Lemma 2. Then $e = \phi(a) + \phi(b) + \phi(c)$ where $\phi(a) \in p^m \mathcal{O}G$, $\phi(b)$ is a linear combination of p-regular elements in G, and $\phi(c) \in [\mathcal{O}G, \mathcal{O}G]$. Hence

$$\sum_{g \in L} \epsilon_g = \lambda(e(L^{-1})^+) = \lambda(\phi(a)(L^{-1})^+) + \lambda(\phi(b)(L^{-1})^+) + \lambda(\phi(c)(L^{-1})^+)$$

where $\lambda(\phi(a)(L^{-1})^+) \in p^m \mathcal{O}$, $\lambda(\phi(b)(L^{-1})^+) = 0$ since L is p-singular, and

$$\lambda(\phi(c)(L^{-1})^+) = 0$$

since $\phi(c)(L^{-1})^+ \in [\mathcal{O}G, \mathcal{O}G]$. Thus $\sum_{g \in L} \epsilon_g \in p^m \mathcal{O}$ as we wished to show.

An immediate consequence of Proposition 3 is the following result.

COROLLARY 4. *Let G be a finite group, s a p-regular element in G and e an idempotent in $Z\mathcal{O}G$. We write $es = \sum_{g \in G} \alpha_g g$ with $\alpha_g \in \mathcal{O}$ for $g \in G$. Then $\sum_{g \in L} \alpha_g = 0$ for every p-singular conjugacy class L of G.*

PROOF. The p-regular element $s \in G$ is a linear combination of idempotents in $\mathcal{O}\langle s \rangle$, so es is a linear combination of idempotents in $\mathcal{O}G$, and the result follows from Proposition 3.

We recall that every element $g \in G$ can be written uniquely in the form $g = us$ where u is a p-element in G and s is a p-regular element in G such that $us = su$. Then u is called the *p-factor* of g, and s is called the *p-regular factor* of g. Two elements in G are said to be contained in the same *p-section* of G if their p-factors are conjugate in G.

The following result is known to be a consequence of the Second Main Theorem (see [11], for example). We present a proof using the ideas above.

PROPOSITION 5. *Let K be a conjugacy class of G, and let e be an idempotent in $\mathrm{Z}\mathcal{O}G$. Then K^+e is a linear combination of elements contained in the same p-section as K. In particular, e is a linear combination of p-regular elements in G.*

PROOF. We write $K^+e = \sum_{g\in G} \alpha_g g$ with $\alpha_g \in \mathcal{O}$ for $g \in G$. Then $\alpha_g = \lambda(K^+eg^{-1})$ for $g \in G$, so it suffices to show that $\lambda(K^+eg^{-1}) = 0$ whenever g is not contained in the same p-section as K. We fix such an element g and denote by u the p-factor and by s the p-regular factor of g^{-1}, so that $g^{-1} = us = su$. Moreover, we denote by e_u the unique idempotent in $\mathrm{Z}\mathcal{O}\mathrm{C}_G(u)$ such that $\mathrm{Br}_{\langle u\rangle}(e) = \mathrm{Br}_{\langle u\rangle}(e_u)$. Then, by Theorem 1, $eus \equiv e_u us \pmod{[\mathcal{O}G, \mathcal{O}G]}$, so $K^+eg^{-1} \equiv K^+e_u g^{-1} \pmod{[\mathcal{O}G, \mathcal{O}G]}$, and therefore

$$\lambda(K^+eg^{-1}) = \lambda(K^+e_u g^{-1}) = \lambda((K \cap \mathrm{C}_G(u))^+ e_u g^{-1})$$

since $e_u g^{-1} \in \mathcal{O}\mathrm{C}_G(u)$. We write $e_u s = \sum_{h\in \mathrm{C}_G(u)} \beta_h h$ with $\beta_h \in \mathcal{O}$ for $h \in \mathrm{C}_G(u)$. Then, by Corollary 4, $\sum_{h\in L} \beta_h = 0$ for every p-singular conjugacy class L of $\mathrm{C}_G(u)$. But, since g and K are contained in different p-sections, $(K \cap \mathrm{C}_G(u))u$ is a union of p-singular conjugacy classes of $\mathrm{C}_G(u)$. Thus we have

$$\lambda((K \cap \mathrm{C}_G(u))^+ e_u g^{-1}) = \sum_{h\in (K\cap \mathrm{C}_G(u))u} \beta_{h^{-1}} = 0$$

as we wanted to show.

Let u be a p-element in G, and let K be a conjugacy class of G contained in the same p-section of G as u. Then the elements in K with p-factor u form a conjugacy class K_u of $\mathrm{C}_G(u)$, and the map $K \mapsto K_u$ is a bijection between the set of conjugacy classes of G contained in the same p-section of G as u and the set of conjugacy classes of $\mathrm{C}_G(u)$ contained in the same p-section of $\mathrm{C}_G(u)$ as u.

Let e be an idempotent in $\mathrm{Z}\mathcal{O}G$, and let e_u be the unique idempotent in $\mathrm{Z}\mathcal{O}\mathrm{C}_G(u)$ such that $\mathrm{Br}_{\langle u\rangle}(e) = \mathrm{Br}_{\langle u\rangle}(e_u)$. We wish to compare K^+e and $K_u^+e_u$.

The following result can be found in Iizuka [11].

THEOREM 6. *Let G be a finite group, u a p-element in G, K a conjugacy class of G contained in the p-section of u in G, and* $K_u := \{g \in K : g \text{ has } p\text{-factor } u\}$. *Let e be an idempotent in $Z\mathcal{O}G$, and let e_u be the unique idempotent in $Z\mathcal{O}C_G(u)$ such that $\mathrm{Br}_{\langle u\rangle}(e) = \mathrm{Br}_{\langle u\rangle}(e_u)$. We write $K^+e = \sum_{g\in G}\alpha_g g$ and $K_u^+e_u = \sum_{h\in C_G(u)}\beta_h h$ with $\alpha_g, \beta_h \in \mathcal{O}$ for $g \in G$ and $h \in C_G(u)$. Then $\alpha_g = 0$ if g is not contained in the p-section of u, and $\alpha_{us} = \beta_{us}$ for any p-regular element s in $C_G(u)$.*

PROOF. The first assertion follows from Proposition 5. Thus let s be a p-regular element in $C_G(u)$. Then $\alpha_{us} = \lambda(K^+eu^{-1}s^{-1})$ and $\beta_{us} = \lambda(K_u^+e_uu^{-1}s^{-1})$. As in the proof of Proposition 5, we have

$$\lambda(K^+eu^{-1}s^{-1}) = \lambda(K^+e_uu^{-1}s^{-1}) = \lambda((K \cap C_G(u))^+u^{-1}e_us^{-1}).$$

If L is a conjugacy class of $C_G(u)$ contained in $(K \cap C_G(u)) \setminus K_u$ then Lu^{-1} is a p-singular conjugacy class of $C_G(u)$, so $\lambda(L^+u^{-1}e_us^{-1}) = 0$ by Corollary 4. Thus

$$\lambda((K \cap C_G(u))^+u^{-1}e_us^{-1}) = \lambda(K_u^+u^{-1}e_us^{-1}) = \beta_{us}$$

as we wanted to show.

The theorem implies that one can compute K^+e from $K_u^+e_u$ and vice versa. As an application, we mention the following result taken from Broué [2].

PROPOSITION 7. *Let G be a finite group, u a p-element in G and U the p-section of G containing u. Let B be a block of $\mathcal{O}G$ with block idempotent e and defect group D. If u is not conjugate in G to an element in D then $K^+e = 0$ for every conjugacy class K of G contained in U.*

PROOF. If u is not conjugate to an element in D then $\mathrm{Br}_{\langle u\rangle}(e) = 0$. But then $e_u = 0$, in the notation of Theorem 6. Thus $K_u^+e_u = 0$ for every conjugacy class K of G contained in U, and therefore $K^+e = 0$ by Theorem 6.

The following result also appears in Broué [2].

PROPOSITION 8. *Let G be a finite group, u a p-element in G and U the p-section of G containing u. Moreover, let B be a block of $\mathcal{O}G$ with block idempotent e and defect group D. Then the following statements are equivalent:*

(1) *$K^+e \in JZ\mathcal{O}G$ for every conjugacy class K of G contained in U;*

(2) *u is not conjugate in G to an element in $Z(D)$.*

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PROOF. Suppose first that $u \in \mathrm{Z}(D)$. By Brauer's First Main Theorem on Blocks, $\mathrm{Br}_D(e)$ is a block idempotent in $\mathbb{F}\mathrm{N}_G(D)$ with defect group D. Since $\mathrm{Br}_D(e)^2 = \mathrm{Br}_D(e)$, Proposition 5 implies that there is a p-regular conjugacy class S of $\mathrm{N}_G(D)$ with defect group D such that $S^+\mathrm{Br}_D(e) \notin \mathrm{JZ}\mathbb{F}\mathrm{N}_G(D)$. We choose $s \in S$ and note that $S \subseteq \mathrm{C}_G(D)$. Moreover, we denote by L the conjugacy class of $\mathrm{N}_G(D)$ containing $us = su$, and by $\nu : \mathbb{F}\mathrm{N}_G(D) \to \mathbb{F}[\mathrm{N}_G(D)/D]$ the natural epimorphism. It is easy to see that

$$\nu(L^+) = |\mathrm{N}_G(D) \cap \mathrm{C}_G(s) : \mathrm{N}_G(D) \cap \mathrm{C}_G(us)|\nu(S^+) \neq 0.$$

Since the kernel of ν is nilpotent this means that

$$L^+ - |\mathrm{N}_G(D) \cap \mathrm{C}_G(s) : \mathrm{N}_G(D) \cap \mathrm{C}_G(us)|S^+ \in \mathrm{JZ}\mathbb{F}\mathrm{N}_G(D),$$

so $\mathrm{Br}_D(e)L^+ \notin \mathrm{JZ}\mathbb{F}\mathrm{N}_G(D)$. If K denotes the conjugacy class of G containing us then K has defect group D, and $K \cap \mathrm{C}_G(D) = L$. Thus

$$\mathrm{Br}_D(K^+e) = \mathrm{Br}_D(K^+)\mathrm{Br}_D(e) = L^+\mathrm{Br}_D(e) \notin \mathrm{JZ}\mathbb{F}\mathrm{N}_G(D),$$

so $K^+e \notin \mathrm{JZ}\mathcal{O}G$.

Now suppose conversely that u is not conjugate to an element in $\mathrm{Z}(D)$, and let K be a conjugacy class of G contained in U. If Q denotes a defect group of K then

$$K^+e \in (\mathcal{O}G)_Q^G \cap (\mathcal{O}G)_D^G \subseteq \sum_{R<D}(\mathcal{O}G)_R^G + (\mathrm{J}\mathcal{O})\mathrm{Z}\mathcal{O}G,$$

so $K^+e \in \left(\sum_{R<D}(\mathcal{O}G)_R^G + (\mathrm{J}\mathcal{O})\mathrm{Z}\mathcal{O}G\right) \cap \mathrm{Z}\mathcal{O}Ge \subseteq \mathrm{JZ}\mathcal{O}G$.

Appendix: Green's theorem à la Puig

The theorem we need is the following one.

THEOREM 9. *Let P be a finite p-group, let A be a P-algebra, and let i be an idempotent in A_1^P. Then there is an idempotent j in A orthogonal to ${}^g j$ for $g \in P \setminus \{1\}$ such that $i = Tr_1^P(j)$.*

The theorem proved by Puig in [18] is more general, but this version suffices for our purposes.

In the proof of Theorem 1, the theorem is applied with $P = \langle u\rangle/\langle v\rangle$, $A = (\mathcal{O}G)^{\langle v\rangle}$ and $i = (e - e_u)f$.

PROOF. Since $iA_1^Pi = (iAi)_1^P$ we may replace A by iAi and therefore assume that $i = 1_A$. We denote by $\mathrm{M}_P(A)$ the $\mathscr{O}$-algebra consisting of all matrices of degree $|P|$ with coefficients in A. It will be convenient to index rows and columns of elements in $\mathrm{M}_P(A)$ by elements in P. Then $\mathrm{M}_P(A)$ becomes a P-algebra over $\mathscr{O}$ in such a way that $({}^xm)_{y,z} = {}^x(m_{x^{-1}y,x^{-1}z})$ for $x, y, z \in P$ and $m \in \mathrm{M}_P(A)$.

For $a \in A$, we define $\delta(a) \in \mathrm{M}_P(A)$ by $\delta(a)_{x,y} := a$ if $x = y = 1$, and $\delta(a)_{x,y} = 0$ otherwise. Then $\delta : A \to \mathrm{M}_P(A)$ is a (non-unitary) monomorphism of algebras.

For $a \in A$, we define $\theta(a) := \mathrm{Tr}_1^P(\delta(a)) \in \mathrm{M}_P(A)^P$. Since

$$\theta(a)_{x,y} = \sum_{z\in P}({}^z\delta(a))_{x,y} = \sum_{z\in P}{}^z(\delta(a)_{z^{-1}x,z^{-1}y})$$

for $a \in A$ and $x, y \in P$ we have $\theta(a)_{x,y} = {}^xa$ if $x = y$, and $\theta(a)_{x,y} = 0$ otherwise. Thus $\theta : A \to \mathrm{M}_P(A)^P$ is a unitary homomorphism of algebras.

We write $1_A = \mathrm{Tr}_1^P(c)$ with $c \in A$ and define $\alpha(a) \in \mathrm{M}_P(A)$ for $a \in A$ by $\alpha(a)_{x,y} := ({}^xc)a$ for $x, y \in P$. Then $\alpha : A \to \mathrm{M}_P(A)$ is a homomorphism of P-algebras since

$$(\alpha(a)\alpha(b))_{x,y} = \sum_{z\in P}\alpha(a)_{x,z}\alpha(b)_{z,y} = \sum_{z\in P}({}^xc)a({}^zc)b = ({}^xc)ab = \alpha(ab)_{x,y}$$

and

$$({}^x\alpha(a))_{y,z} = {}^x(\alpha(a)_{x^{-1}y,x^{-1}z}) = {}^x(({}^{x^{-1}y}c)a) = ({}^yc)({}^xa) = \alpha({}^xa)_{y,z}$$

for $x, y, z \in P$ and $a \in A$. Moreover, α is injective since

$$\sum_{x\in P}\alpha(a)_{x,1} = \sum_{x\in P}({}^xc)a = a$$

for $a \in A$. Finally, $\alpha(A) = \alpha(1)\mathrm{M}_P(A)\alpha(1)$ since

$$\begin{aligned}(\alpha(1)m\alpha(1))_{x,y} &= \sum_{u,v\in P}\alpha(1)_{x,u}m_{u,v}\alpha(1)_{v,y} \\ &= \sum_{u,v\in P}({}^xc)m_{u,v}({}^vc) = \alpha\left(\sum_{u,v\in P}m_{u,v}({}^vc)\right)_{x,y}\end{aligned}$$

for $m \in \mathrm{M}_P(A)$ and $x, y \in P$.

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For $x \in P$, we define $\gamma(x) \in \mathrm{M}_P(A)$ by $\gamma(x)_{y,z} = 1$ if $z = yx$, and $\gamma(x)_{y,z} = 0$ otherwise. Then

$$(\gamma(u)\gamma(v))_{x,y} = \sum_{z\in P} \gamma(u)_{x,z}\gamma(v)_{z,y} = \gamma(v)_{xu,y} = \gamma(uv)_{x,y}$$

and

$$({}^u\gamma(v))_{x,y} = {}^u(\gamma(v)_{u^{-1}x,u^{-1}y}) = \gamma(v)_{u^{-1}x,u^{-1}y} = \gamma(v)_{x,y}$$

for $u, v, x, y \in P$. Since $\gamma(1) = 1$, $\gamma : P \to \mathrm{UM}_P(A)^P$ is a homomorphism of groups.

If $m \in \mathrm{M}_P(A)^P$ then $m_{x,y} = ({}^z m)_{x,y} = {}^z(m_{z^{-1}x,z^{-1}y})$ for $x, y, z \in P$. Thus

$$\begin{aligned}\left(\sum_{z\in P}\theta(m_{1,z})\gamma(z)\right)_{x,y} &= \sum_{u,z\in P}\theta(m_{1,z})_{x,u}\gamma(z)_{u,y}\\ &= \sum_{u\in P}\theta(m_{1,u^{-1}y})_{x,u}\\ &= {}^x(m_{1,x^{-1}y}) = m_{x,y}\end{aligned}$$

for $x, y \in P$, so $m = \sum_{z\in P}\theta(m_{1,z})\gamma(z)$ and $\mathrm{M}_P(A)^P = \sum_{z\in P}\theta(A)\gamma(z)$. Moreover,

$$(\gamma(x)\theta(a)\gamma(x^{-1}))_{y,z} = \sum_{u,v\in P}\gamma(x)_{y,u}\theta(a)_{u,v}\gamma(x^{-1})_{v,z} = \theta(a)_{yx,zx} = \theta({}^x a)_{y,z}$$

for $x, y, z \in P$ and $a \in A$, so $\gamma(x)\theta(a) = \theta({}^x a)\gamma(x)$ for $a \in A$ and $x \in P$. Suppose that $\sum_{x\in P}\theta(a_x)\gamma(x) = 0$ where $a_x \in A$ for $x \in P$. Then

$$0 = \sum_{x\in P}(\theta(a_x)\gamma(x))_{1,y} = \sum_{x,z\in P}\theta(a_x)_{1,z}\gamma(x)_{z,y} = \sum_{x\in P}\theta(a_x)_{1,yx^{-1}} = a_y$$

for $y \in P$, so $\mathrm{M}_P(A)^P = \bigoplus_{z\in P}\theta(A)\gamma(z)$ is isomorphic to the skew group algebra AP of P over A. The usual proof of Green's indecomposability theorem (see [10] or the proposition below) shows that any primitive idempotent in A stays primitive in AP. Let us therefore write $1_A = e_1 + \cdots + e_r$ with pairwise orthogonal primitive idempotents $e_1, \ldots, e_r$ in A. Then we have $1_{\mathrm{M}_P(A)} = \theta(1) = \theta(e_1) + \cdots + \theta(e_r)$ with pairwise orthogonal primitive idempotents $\theta(e_1), \ldots, \theta(e_r)$ in $\mathrm{M}_P(A)^P$. Since $\alpha(1)$ is an idempotent in $\mathrm{M}_P(A)^P$ there are a subset J of $\{1, \ldots, r\}$ and a unit w in $\mathrm{M}_P(A)^P$ such that $\alpha(1) = {}^w(\sum_{i\in J}\theta(e_i))$ (see 2.10 in [13], for example). Then $j' := {}^w(\sum_{i\in J}\delta(e_i))$ is an idempotent in $\mathrm{M}_P(A)$ orthogonal to ${}^g j'$ for $g \in P\setminus\{1\}$ such that $\alpha(1) = \mathrm{Tr}_1^P(j')$; in particular,

$j' \in \alpha(1)\mathrm{M}_P(A)\alpha(1) = \alpha(A)$, so $j' = \alpha(j)$ for an idempotent j in A satisfying the required properties.

The seemingly technical calculations of the proof are by now standard tools in ring theory (see Cohen and Montgomery [4], for example).

It remains to prove Green's indecomposability theorem in the following version.

PROPOSITION 10. *Let P be a finite p-group, A a P-algebra over $\mathcal{O}$ and AP the corresponding skew group algebra of P over A. Then every primitive idempotent in A remains primitive in AP.*

PROOF. Let i be a primitive idempotent in A. Then $i + \mathrm{J}A$ is a primitive idempotent in $A/\mathrm{J}A$. Since $\mathrm{J}A$ is a P-invariant ideal of A, it generates a nilpotent ideal $(\mathrm{J}A)(AP) = (AP)(\mathrm{J}A)$ of AP such that $AP/(\mathrm{J}A)(AP)$ is isomorphic to $(A/\mathrm{J}A)P$, the skew group algebra of P over the P-algebra $A/\mathrm{J}A$. Since it suffices to prove that $i + \mathrm{J}A$ is primitive in $(A/\mathrm{J}A)P$, we may assume that $\mathrm{J}A = 0$.

In this case A is a direct product of complete matrix algebras over $\mathbb{F}$ permuted by P. If A is isomorphic to $A_1 \times A_2$ with P-algebras A_1, A_2 then AP is isomorphic to $A_1P \times A_2P$. Thus we may assume that $A = B_1 \times \cdots \times B_q$ with complete matrix algebras $B_1, \ldots, B_q$ over $\mathbb{F}$ transitively permuted by P. We denote by Q the stabilizer of B_1 in P and by $g_1, \ldots, g_q$ a set of represesentatives for the cosets gQ in P. Then the map

$$\mathrm{Mat}(B_1Q) \longrightarrow AP, \quad [b_{ij}] \longmapsto \sum_{i,j=1}^{q} g_i b_{ij} g_j^{-1},$$

is easily seen to be an isomorphism of algebras. In particular, any primitive idempotent in B_1Q remains primitive in AP. Thus we may assume that A itself is a complete matrix algebra over $\mathbb{F}$.

For $g \in G$, there is an element $u_g \in \mathrm{U}A$ such that ${}^g a = u_g a u_g^{-1}$ for $a \in A$, by the Skolem-Noether theorem. Moreover, $u_g u_h u_{gh}^{-1} \in \mathrm{UZ}A = \mathrm{U}\mathbb{F}1_A$ for $g, h \in P$, and the map $(g,h) \mapsto u_g u_h u_{gh}^{-1}$ is a 2-cocycle of P with values in $\mathrm{UZ}A = \mathrm{U}\mathbb{F}1_A$. Since P is a p-group we have $\mathrm{H}^2(P, \mathrm{U}\mathbb{F}) = 1$, so we may assume that $u_g u_h = u_{gh}$ for $g, h \in P$. But then the map

$$A \otimes_{\mathbb{F}} \mathbb{F}P \longrightarrow AP, \quad a \otimes g \longmapsto a u_g^{-1} g,$$

is an isomorphism of algebras; in particular, $AP/J(AP)$ is isomorphic to A, and the result follows.

The result is known to hold, more generally, for crossed products instead of skew group algebras. Essentially the same proof works in this more general situation.

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