DUNFORD–PETTIS AND STRONGLY-DUNFORD–PETTIS OPERATORS ON $L^1(\mu)$

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1. Introduction. Motivated by a problem in mathematical economics [**4**] Gretsky and Ostroy have shown [**5**] that every positive operator $T: L^1[0, 1] \to c_0$ is a Dunford–Pettis operator (i.e. T maps weakly convergent sequences to norm convergent ones), and hence that the same is true for every regular operator from $L^1[0, 1]$ to c_0. In a recent paper [**6**] we showed the converse also holds, thereby characterizing the D–P operators by this condition. In each case the proof depends (as do so many concerning D–P operators on $L^1[0, 1]$) on the following well-known result (see, e.g., [**2**]): If μ is a finite measure, an operator $T: L^1(\mu) \to E$ is a D–P operator $\Leftrightarrow T \cdot i: L^\infty(\mu) \xrightarrow{i} L^1(\mu) \xrightarrow{T} E$ is compact, where $i: L^\infty(\mu) \to L^1(\mu)$ is the canonical injection of $L^\infty(\mu)$ into $L^1(\mu)$. If μ is not a finite measure this characterization of D–P operators is no longer available, and hence results based on its use (e.g. [**5**], [**6**]) do not always have straightforward extensions to the case of operators on more general $L^1(\mu)$ spaces.

The purpose of this paper is two-fold. First, we show in §2 how arguments concerning D–P operators on a space $L^1(\mu)$ for μ a σ-finite measure can be reduced in a natural way to analogous arguments concerning operators on associated spaces $L^1(\mu_n)$, where each μ_n is a finite measure and hence where the above mentioned result may be applied to good effect. In particular, the results of Gretsky–Ostroy [**5**] and of the author [**6**] proved for $L^1[0, 1]$ are shown to be valid in this more general setting. Second, we explore in more detail the important distinction between the cases of a finite and of a σ-finite measure by introducing in §3 the concept of a strongly-Dunford–Pettis operator from a space $L^1(\mu)$ to some Banach space E. The strongly D–P operators turn out to be precisely the D–P operators when μ is a finite measure, but are a strictly smaller set otherwise. The interest in these strongly D–P operators lies in the fact that certain well-known results concerning D–P operators on the space $L^1(\mu)$ for μ finite, which are either meaningless or false when μ is infinite, find their natural statement and meaning in the context of strongly-D–P operators. Thus the strongly-D–P operators seem to play a unifying role in the study of operators on $L^1(\mu)$.

2. Throughout this paper (X, Σ, μ) will denote a positive measure space, E will denote some Banach space, and $T: L^1(\mu) \to E$ a bounded linear operator. In the case where μ is a σ-finite measure we will write $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \subset X_{n+1}$ and $\mu(X_n) < +\infty$ for all n. In this case we denote by $L^1(X_n)$ the subspace of $L^1(\mu)$ defined by $L^1(X_n) = \{f \in L^1(\mu) |\ \text{support } f \subset X_n\}$. Clearly $L^1(X_n)$ is isometrically isomorphic to $L^1(\mu_n)$, where μ_n is the measure induced on X_n by restricting μ to the measurable sets in $\Sigma \cap X_n$, under the mapping $Q_n: L^1(\mu_n) \to L^1(X_n)$ defined by $Q_n = f$ for $f(t)$ the function in X which equals $g(t)$ when $t \in X_n$ and is zero otherwise.

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The principal tool for extending results concerning D–P operators on the space $L^1(\mu)$ for μ a finite measure to the case where μ is σ-finite is the following:

THEOREM 2.1. *Suppose μ is a σ-finite measure. The operator $T: L^1(\mu) \to E$ is a* D–P *operator $\Leftrightarrow$ for every $n = 1, 2, \ldots$ the operator $T_n: L^1(\mu_n) \to E$ defined by $T_n(f) = T(Q_n f)$ is a* D–P *operator (where $Q_n f$ is the extension of f from X_n to X defined above).*

Proof. ($\Rightarrow$): If T is a D–P operator and $\{f_i\}_{i=1}^{\infty}$ is a sequence in $L^1(\mu_n)$ which converges weakly to 0 then clearly $\{Q_n f_i\}_{i=1}^{\infty}$ converges weakly to zero in $L^1(\mu)$. Therefore $\{T(Q_n f_i)\}_{i=1}^{\infty}$ converges weakly to zero in E, so by definition $\{T_n f_i\}_{i=1}^{\infty}$ converges to zero in E and T_n is a D–P operator.

($\Leftarrow$): Suppose T_n is a D–P operator for every n and $\{f_i\}_{i=1}^{\infty}$ is a sequence in $L^1(\mu)$ which is weakly convergent to zero. Define the operator

$$U: L^1(\mu) \to l^1 \quad \text{by} \quad U(f) = \left\{ \int_{X_{n+1}-X_n} f(t)\, d\mu(t) \right\}_{n=1}^{\infty}.$$

Clearly U is continuous, hence weakly continuous, and so maps weakly compact sets in $L^1(\mu)$ to weakly compact, hence compact [**3**, p. 296], sets in l^1. Since the set $\{f_i\}_{i=1}^{\infty}$ is weakly compact in $L^1(\mu)$ the set $\{|f_i|\}_{i=1}^{\infty}$ has the same property [**3**, p. 293] so by the above the set $\{U\,|f_i|\}_{i=1}^{\infty}$ is compact in l^1. Therefore given $\epsilon > 0$ there is an $N \geq 1$ so that

$$\begin{aligned} \int_{X-X_N} |f_i(t)|\, d\mu(t) &= \sum_{n=N}^{\infty} \int_{X_{n+1}-X_n} |f_i(t)|\, d\mu(t) \\ &= \left\| \sum_{n=N}^{\infty} \langle Uf, e_n \rangle e_n \right\| < \frac{\epsilon}{2\,\|T\|} \end{aligned}$$

for all i [**3**, p. 260].

If for each n we define a projection P_n on $L^1(\mu)$ by $P_n f = f \cdot \chi_{X_n}$, then P_n projects $L^1(\mu)$ onto $L^1(X_n)$ and for each i we have $f = P_n f_i + (f_i - P_n f_i)$. Hence when $n = N$ and $i = 1, 2, 3, \ldots$ we have

$$\|Tf_i\| \leq \|TP_N f_i\| + \|T(f_i - P_N f_i\|,$$

where

$$\|T(f_i - P_N f_i)\| \leq \|T\|\, \|f_i - P_N f_i\| = \|T\| \int_{X-X_N} |f_i(t)|\, d\mu(t) < \frac{\epsilon}{2}.$$

Since $\{P_N f_i\}_{i=1}^{\infty}$ converges weakly to 0 (and is in $L^1(X_n)$)$\{TP_N f_i\}_{i=1}^{\infty} = \{T_N g_i\}_{i=1}^{\infty}$, where g_i is f_i restricted to X_N for all i, and hence where $\{g_i\}_{i=1}^{\infty}$ converges weakly to zero in $L^1(\mu_N)$. By assumption T_N is a D–P operator so if $i \geq i_0$ then

$$\|TP_N f_i\| = \|T_N g_i\| < \frac{\epsilon}{2}.$$

It follows that $\{f_i\}_{i=1}^{\infty}$ converges to 0 in E, hence that T is a D–P operator, and the proof is complete.

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It is now a simple matter to combine this result with others concerning D–P operators on a space $L^1(\mu)$ for μ finite to get analogous results in the case where μ is σ-finite. In particular we derive the following extensions of the results of Gretsky–Ostroy [5] and of the author [6] proved for the space $L^1[0, 1]$.

COROLLARY 2.2. *Let E be a Banach space with an order compatible basis, μ a σ-finite measure, and $T: L^1(\mu) \to E$ a positive operator. Then T is a* D–P *operator.*

Proof. If $T: L^1(\mu) \to E$ is a positive operator then it is clear from the definition of the isometry $Q_n: L^1(\mu_n) \to L^1(X_n)$ that $T_n = TQ_n: L^1(\mu_n) \to E$ is also positive for all n, hence a D–P operator [5], and it follows from Theorem 2.1 that T is also a D–P operator.

COROLLARY 2.3. *If μ is any σ-finite measure and $T: L^1(\mu) \to c_0$ is a bounded linear operator, then the following are equivalent:*

1) *T is regular*
2) *T is a* D–P *operator*
3) $\{|T^*e_i|\}_{i=1}^{\infty}$ *is w^*-convergent to* 0 *in* $L^\infty(\mu)$.

Proof. (1)⇒(2): If T is regular (i.e. a difference of positive operators) it follows from Corollary 2.2 that T is a D–P operator, since the unit vector basis $\{e_i\}_{i=1}^{\infty}$ for c_0 is certainly order compatible.

(2)⇒(3): If $T: L^1(\mu) \to c_0$ is a D–P operator then according to Theorem 2.1 we know that each operator $T_n: L^1(\mu_n) \to c_0$ is a D–P operator. We also know that T has the representation $T = \sum_{i=1}^{\infty} T^*e_i \otimes e_i$, where $\{T^*e_i\}$ is w^*-convergent to 0 in $L^\infty(\mu)$ (see, e.g., [6]). Let $T^*e_i = h_i$ for all i, so $T = \sum_{i=1}^{\infty} h_i \otimes e_i$. Given any $n = 1, 2, \ldots$ we see that $T_n: L^1(\mu_n) \to c_0$ then has the representation $T_n = \sum_{i=1}^{\infty} \bar{h}_i \otimes e$: where $\bar{h}_i$ is the restriction of h_i to X_n, a function in $L^\infty(\mu_n)$ for all i. Since T_n is a D–P operator and μ_n is a finite measure we know from our earlier work [6] that $\{|\bar{h}_i|\}_{i=1}^{\infty}$ is w^*-convergent to 0 in $L^\infty(\mu_n)$ for all n. Given any $f \in L^1(\mu)$ and any $\epsilon > 0$ choose n so that

$$\int_{X-X_n} |f(t)|\, d\mu(t) < \frac{\epsilon}{2 \sup \|h_i\|_\infty}$$

and write $f = P_n f + (f - P_n f)$ where $P_n f = f\chi_{X_n}$ (as in the proof of Theorem 2.1). Then

$$\begin{aligned} \langle |h_i|, f\rangle| &\le |\langle |h_i|, P_n f\rangle| + |\langle |h_i|, f - P_n f\rangle| \\ &\le \left|\int_X |h_i(t)|\, P_n f(t)\, d\mu(t) + \sup_i \|h_i\| \cdot \frac{\epsilon}{2 \sup \|h_i\|_\infty}\right. \\ &= \int_{X_n} |\bar{h}_i(t)|\, g(t)\, d\mu(t) + \frac{\epsilon}{2}, \end{aligned}$$

where

$$g = f\,|_{X_n} \in L^1(\mu_n).$$

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We noted above that $\{|\bar{h}_i|\}_{i=1}^{\infty}$ is w^*-convergent to 0 in $L^{\infty}(\mu_n)$, so there is an integer i_0 such that if $i \geq i_0$ then

$$\left|\int_{X_m} |\bar{h}_i(t)|\, g(t)\, d\mu(t)\right| < \frac{\epsilon}{2},$$

and hence so that $|\langle |h_i|, f\rangle < \varepsilon$ if $i \geq i_0$. Since $f \in L^1(\mu)$ is arbitrary it follows that $\{|T^*e_i|\}_{i=1}^{\infty}$ is w^*-convergent to 0 in $L^{\infty}(\mu)$.

(3)$\Rightarrow$(1): *If* $\{|T^*e_i|\}_{i=1}^{\infty}$ is w^*-convergent to 0 in $L^1(\mu)$ then the operator $|T|: L^1(\mu) \to c_0$ with representation $|T| = \sum_{i=1}^{\infty} |T^*e_i| \otimes e_i$ is well-defined. Hence, since $|T| + T$ and $|T| - T$ are both positive and $T = \frac{1}{2}(|T| + T) - \frac{1}{2}(|T| - T)$, it follows that T is regular.

3. We have just shown that if μ is a σ-finite measure and T is an operator from $L^1(\mu)$ to c_0, then T is a D–P operator $\Leftrightarrow \{|T^*e_i|\}_{i=1}^{\infty}$ is w^*-convergent to 0 in $L^{\infty}(\mu)$. In the case where μ is actually a finite measure the statement that $\{|T^*e_i|\}_{i=1}^{\infty}$ is w^*-convergent to 0 in $L^{\infty}(\mu)$ is easily seen to be equivalent to the condition that $\{T^*e_i\}_{i=1}^{\infty}$ converges to zero in $L^1(\mu)$ or that $\{T^*e_i\}_{i=1}^{\infty}$ converges to 0 in measure on X, since $\{T^*e_i\}_{i=1}^{\infty}$ is bounded in $L^{\infty}(\mu)$. However in the case where $\mu(X) = +\infty$ this latter condition is a strictly stronger one than that of the Dunford–Pettis property. In fact, we have:

PROPOSITION 3.1. *Let μ be a σ-finite, non-finite, measure on X and $T: L^1(\mu) \to c_0$ a bounded linear operator. If the sequence $\{T^*e_i\}_{i=1}^{\infty}$ in $L^{\infty}(\mu)$ converges in measure to* 0 *on X then T is a* D–P *operator. However there exists a* D–P *operator T for which $\{T^*e_i\}_{i=1}^{\infty}$ does not converge in measure to* 0.

Proof. Suppose $\{T^*e_i\}_{i=1}^{\infty} = \{h_i\}_{i=1}^{\infty}$, a bounded sequence in $L^{\infty}(\mu)$ which converges in measure to 0. If, as in §2, $X = \bigcup_{n=1}^{\infty} X_n$ for $X_n \subset X_{n+1}$ and $\mu(X_n) < +\infty$ for all n then certainly for any fixed n the sequence $\{\bar{h}_i\}_{i=1}^{\infty} = \{h_i \mid_{X_n}\}_{i=1}^{\infty}$ is a sequence in $L^{\infty}(\mu_n)$ (where we recall μ_n is the restriction of μ to X_n) which converges in μ_n-measure to zero on X_n. Hence, as noted above, the operator $T_n: L^1(\mu_n) \to c_0$ defined by $T_n = \sum_{i=1}^{\infty} \bar{h}_i \otimes e_i$ is a D–P operator, and it follows from Theorem 2.1 that T is a D–P operator.

On the other hand, since μ is not a finite measure there exist disjoint measurable sets $\{E_n\}_{n=1}^{\infty}$ in X with

$$0 < \inf_n \mu(E_n) \leq \sup_n \mu(E_n) < +\infty.$$

Let $h_n = \chi_{E_n}$ for $n = 1, 2, \ldots$, so $\{h_n\}_{n=1}^{\infty}$ is a bounded sequence of non-negative functions which is w^*-convergent to 0 in $L^{\infty}(\mu)$ but which does not converge to 0 in measure. Setting $T = \sum_{n=1}^{\infty} h_n \otimes e_n$ we get a positive operator from $L^1(\mu)$ to c_0 for which $\{T^*e_n\}_{n=1}^{\infty} = \{h_n\}_{n=1}^{\infty}$ does not converge in measure to zero. Since by Corollary 2.2 T is a D–P operator, we have produced the desired example.

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Thus we see that for an arbitrary σ-finite measure μ the set of operators $T:L^1(\mu)\to c_0$ for which $\{T^*e_i\}_{i=1}^{\infty}$ converges in measure to 0 is a subset of the D–P operators, with equality holding if and only if μ is finite. In fact, rather than being peculiar to the study of operators from $L^1(\mu)$ to c_0, these operators are a special case of what we call *strongly*-D–P *operators* on an L^1-space.

Definition. An operator $T:L^1(\mu)\to E$ is called a strongly-Dunford–Pettis operator if it maps bounded, uniformly integrable subsets of $L^1(\mu)$ into compact subsets of E.

(Recall that a subset $A\subset L^1(\mu)$ is uniformly integrable if for every $\epsilon>0$ there is a $\delta>0$ so that $|\int_E f(t)\,d\mu(t)|<\epsilon$ whenever $f\in A$ and $\mu(E)<\delta$; see e.g., [**7**, p. 134]).

It is well known that for any measure μ every weakly compact subset of $L^1(\mu)$ is both bounded and uniformly integrable and that the converse is true if μ is finite [**3**, p. 294]. Thus we see from the definition that every strongly-D–P operator is a D–P operator and that the two sets of operators coincide when μ is finite. In the case where $\mu(X)=+\infty$ it is the strongly-D–P operators, rather than the D–P operators themselves, which have an analogous characterization to that of [**2**].

Theorem 3.2. *An operator $T:L^1(\mu)\to E$ is a strongly*-D–P *operator* $\Leftrightarrow$ *the operator $T.\,i:L^\infty(\mu)\cap L^1(\mu)\xrightarrow{i}L^1(\mu)\xrightarrow{T}E$ is compact (where the norm on the space $L^\infty(\mu)\cap L^1(\mu)$ is defined by $\|f\|=\max\{\|f\|_\infty,\|f\|_1\}$).*

Proof. ($\Rightarrow$): *Suppose* T is a strongly D–P operator. If u denotes the unit ball of $L^\infty(\mu)\cap L^1(\mu)$ then $i(U)\subset L^1(\mu)$ and is clearly a bounded, uniformly integrable subset of $L^1(\mu)$ (since by definition U is in the unit ball of $L^\infty(\mu)$). Hence $T(i(U))$ is a compact set in E, implying $T.\,i$ is compact.

($\Leftarrow$): Suppose $T.\,i$ is compact and that A is any bounded, uniformly integrable subset of $L^1(\mu)$. Clearly the set $|A|=\{|f|\mid f\in A\}$ has the same properties. Therefore, given any $\epsilon>0$ choose $\delta>0$ so that if $\mu(E)<\delta$ then

$$\int_E |f(t)|\,d\mu(t)<\frac{\epsilon}{4\,\|T\|}$$

for all $f\in A$. Since A is bounded in $L^1(\mu)$ there is an M so that if $f\in A$ then $|f(t)|\le M$ for all $t\in X-E_f$ where E_f is a measurable subset of X for which $\mu(E_f)<\delta$. It follows that if $f\in A$ we have $f=g+r$, where $g=f\chi_{X-E_f}$ is in the M-ball of $L^\infty(\mu)\cap L^1(\mu)$ and where

$$r=f\chi_{E_f},\quad \text{with}\quad \|r\|_1=\int|r(t)|\,d\mu(t)<\frac{\epsilon}{4\,\|T\|}.$$

Hence if $\{f_n\}$ is any sequence then $f_n=g_n+r_n$ as above for all n, and for all m and n $\|Tf_n-Tf_m\|\le\|Tg_n-Tg_m\|+\|Tr_n-Tr_m\|<\|Tg_n-Tg_n\|+\dfrac{\epsilon}{2}$ (since $\|Tr_i\|<\epsilon/4$ for all i). But by assumption $\{Tg_n\}_{n=1}^{\infty}$ is compact in E so there is a subsequence $\{Tg_{n_i}\}_{i=1}^{\infty}$ and an integer N so that if $i,j\ge N$ then $\|Tg_{n_i}-TG_{n_j}\|<\dfrac{\epsilon}{2}$. Hence, by the above, if $i,j\ge N$ then

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$\|Tf_{n_i} - Tf_{n_j}\| < \epsilon$, so the subsequence $\{Tf_{n_i}\}_{i=1}^{\infty}$ of $\{Tf_n\}_{n=1}^{\infty}$ converges in E. Thus the set $T(A)$ is compact in E, and T is a strongly-D–P operator.

Now let us return to the study of operators $T: L^1(\mu) \to c_0$. We have seen that for any σ-finite measure μ, T is a D–P operator $\Leftrightarrow T$ is regular $\Leftrightarrow \{|T^*e_i|\}_{i=1}^{\infty}$ is w^*-convergent to 0 in $L^{\infty}(\mu)$, and the condition that $\{T^*e_i\}_{i=1}^{\infty}$ converges in measure to 0 is sufficient for T to be a D–P operator, but not necessary unless $\mu(X) < +\infty$. As promised earlier, we now show that the operators for which $\{T^*e_i\}$ converges in measure to 0 are precisely the strongly-D–P operators, thereby emphasizing the distinction which we find in general between the D–P and the strongly-D–P operators in the case where $\mu(X) = +\infty$.

THEOREM 3.3. *Let μ be a σ-finite measure and $T: L^1(\mu) \to c_0$ a bounded linear operator. Then T is a strongly*-D–P *operator* $\Leftrightarrow \{T^*e_i\}_{i=1}^{\infty}$ *converges in measure to* 0 *in* $L^{\infty}(\mu)$.

Proof. ($\Rightarrow$): Let $T: L^1(\mu) \to c_0$ be a strongly-D–P operator having the representation $T = \sum_{i=1}^{\infty} h_i \otimes e_i$, where $\{h_i\}_{i=1}^{\infty} = \{T^*e_i\}_{i=1}^{\infty}$ is w^*-convergent to 0 in $L^{\infty}(\mu)$. If $\{h_i\}_{i=1}^{\infty}$ does not converge in measure to 0 then there exist $\epsilon > 0$, $\delta > 0$, and a subsequence $\{h_{i_n}\}_{n=1}^{\infty}$ of $\{h_i\}_{i=1}^{\infty}$ for which $\mu\{t \in X \mid |h_{i_n}(t)| \geq \epsilon\} > \delta$, $n = 1, 2, \ldots$. Since, for each n, either $\mu\{t \mid h_{i_n}(t) \geq \epsilon\}$ or $\mu\{t \mid h_{i_n}(t) \leq -\epsilon\}$ must be greater than $\delta/2$ we may assume without loss of generality that $\mu\{t \mid h_{i_n}(t) \geq \epsilon\} > \delta$ for all $n = 1, 2, 3, \ldots$, and hence for each n there is a set $E_n \subset X$ so that

$$\frac{\delta}{2} \leq \inf_n \mu(E_n) \leq \sup_n \mu(E_n) < +\infty$$

and $h_{i_n}(t) \geq \epsilon$ for all $t \in E_n$. If $f_k = \chi_{e_k}$ for $k = 1, 2, 3, \ldots$, then the set $\{f_k\}_{k=1}^{\infty}$ is clearly a uniformly integrable subset of $L^1(\mu)$ (being a bounded subset of $L^{\infty}(\mu)$). Moreover,

$$Tf_k = \sum_{i=1}^{\infty} \langle h_i, f_k \rangle e_i,$$

so for any k and any

$$N \left\| \sum_{i=N}^{\infty} \langle h_i, f_k \rangle e_i \right\| = \sup_{i \geq N} |\langle h_i, f_k \rangle| = \sup_{i \geq N} \left| \int_{E_k} h_i(t)\, d\mu(t) \right|$$

$$\geq \sup_{n \geq N} \left| \int_{E_k} h_{i_n}(t)\, d\mu(t) \right|.$$

If we choose $k = i_n$ (for any $n \geq N$) we then get this last is

$$\geq \left| \int_{E_{i_n}} \epsilon\, d\mu(t) \right| \geq \epsilon \cdot \frac{\delta}{2},$$

by definition of $\{h_{i_n}\}_{n=1}^{\infty}$ and E_{i_n}. Therefore

$$Tf_k = \sum_{i=1}^{\infty} \langle h_i, f_k \rangle e_i,$$

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where this series converges, but not uniformly, over the set $\{f_k\}_{k=1}^{\infty}$. It follows that the set $\{Tf_k\}_{k=1}^{\infty}$ is not compact in c_0 [**3**, p. 260] and hence that T is not strongly-D–P, a contradiction. Consequently it must be that $\{T^*e_i\}_{i=1}^{\infty}$ converges in μ-measure to zero.

($\Leftarrow$): On the other hand, assume $\{T^*e_i\}_{i=1}^{\infty}$ converges to 0 in μ-measure and write

$$T=\sum_{i=1}^{\infty} T^*e_i \otimes e_i = \sum_{i=1}^{\infty} h_i \otimes e_i.$$

Let A be any bounded, uniformly integrable subset of $L^1(\mu)$ and let $\epsilon>0$ be given. By assumption, if $E_i=\{t \mid |h_i(t)|\geq \epsilon\}$ then $\mu(E_i)\to 0$. Therefore if $f\in A$ and $i=1,2,\ldots$ we have

$$|\langle h_i, f\rangle| = \left|\int_{X-E_i} |h_i(t)|\,|f(t)|\,d\mu(t)\right| \leq \sup_i \|h_i\|_\infty \int_{E_i} |f(t)|\,d\mu(t) + \epsilon\cdot \int_X |f(t)|\,d\mu(t).$$

Since A is uniformly integrable there is a $\delta>0$ so that if $\mu(E)<\delta$ then $\int_E |f(t)|\,d\mu(t)<\epsilon$ for all $f\in A$, so since $\mu(E_i)\to 0$ there is a N for which $\mu(E_i)<\delta$ whenever $i\geq N$ and hence for which $\int_{E_i} |f(t)|\,d\mu(t)<\epsilon$ for all $f\in A$. Therefore for any $f\in A$

$$\left\|\sum_{i=N}^{\infty} \langle h_i, f\rangle e_i\right\|_{c_0} = \sup_{i\geq N} |\langle h_i, f\rangle| < \sup_i \|h_i\|_\infty \cdot \epsilon + \sup_{f\in A} \|f\|_1 \cdot \epsilon \quad \text{(by the above)},$$

so $Tf=\sum_{i=1}^{\infty} \langle T^*e_i, f\rangle e_i$ converges in c_0 uniformly over $f\in A$, implying that the set $T(A)$ is compact in c_0 and hence that T is a strongly-D–P-operator.

More generally, we have the following characterization of strongly-D–P operators from $L^1(\mu)$ to any separable Banach space:

THEOREM 3.4. *Let μ be a σ-finite measure and E a separable Banach space. An operator $T: L^1(\mu)\to E$ is a strongly-D–P operator $\Leftrightarrow$ whenever $\{w_n^*\}_{n=1}^{\infty}$ is a sequence in E^* which is w^*-convergent to 0, then the sequence $\{T^*w_n^*\}_{n=1}^{\infty}$ in $L^\infty(\mu)$ converges in measure to 0.*

Proof. ($\Rightarrow$): Suppose $T: L^1(\mu)\to E$ is a strongly-D–P operator. If $\{w_n^*\}_{n=1}^{\infty}$ is w^*-convergent to 0 in E^* then $V=\sum_{n=1}^{\infty} w_n^* \otimes e_n$ is a well-defined, bounded linear operator from E to c_0 and hence $V\,.\,T: L^1(\mu)\xrightarrow{T} E \xrightarrow{V} c_0$ is also a strongly-D–P operator. According to Theorem 3.3 above the sequence $\{(V\,.\,T)^*e_n\}_{n=1}^{\infty}$ in $L^\infty(\mu)$ converges to 0 in measure. But $(V\,.\,T)^*e_n = T^*w_n^*$ for all n, so $\{T^*w_n^*\}_{n=1}^{\infty}$ converges to 0 in measure.

($\Leftarrow$): Conversely, suppose whenever $\{w_n^*\}_{n=1}^{\infty}$ is w^*-convergent to 0 in E^* then $\{T^*w_n^*\}_{n=1}^{\infty}$ converges in measure to 0 in $L^\infty(\mu)$. Since E is separable there is an isometric isomorphism Q of E into $C[0,1]$ [**1**, Chap. XI, Theorem 10], where $C[0,1]$ has a Schauder basis $\{\Phi_i\}_{n=1}^{\infty}$ with coefficient functionals $\{\Phi_i^*\}_{i=1}^{\infty}$ in $C[0,1]^*$ (see e.g. [**8**, p. 11]). If A is a bounded, uniformly integrable set in $L^1(\mu)$ for which $T(A)$ is *not* compact in E, then $Q\,.\,T(A)$ is not compact in $C[0,1]$, so $\sum_{i=1}^{\infty} \langle \Phi_i^*, QT(f)\rangle \Phi_i$ is not uniformly

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convergent in $C[0, 1]$ over $f \in A$. Therefore there is an $\varepsilon > 0$, a sequence $\{f_n\}_{n=1}^{\infty} \subset A$, and an increasing sequence of integers $\{p_m\}_{n=1}^{\infty}$ for which

$$\left\| \sum_{i=p_n+1}^{p_{n+1}} \langle \Phi_i^*, QT(f_n) \rangle \Phi_i \right\|_{\infty} > \epsilon \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Correspondingly, for each n there is a functional $\alpha_n \in C[0, 1]^*$ with $\|\alpha_n\| = 1$ for which

$$\left| \sum_{i=p_n+1}^{p_{n+1}} \langle \Phi_i^*, QT(f_n) \rangle \langle \Phi_i, \alpha_n \rangle \right| > \epsilon,$$

and hence for which

$$\left| \sum_{i=p_n+1}^{p_{n+1}} \langle \Phi_i, \alpha_n \rangle \langle \Phi_i^*, QT(f_n) \rangle \right| > \epsilon.$$

If we set

$$q_n = \sum_{i=p_n+1}^{p_{n+1}} \langle \Phi_i, \alpha_n \rangle \Phi_i^* \quad \text{for} \quad n = 1, 2, \ldots,$$

then $\{q_n\}_{n=1}^{\infty}$ is a bounded sequence in $C[0, 1]^*$, and hence clearly converges to 0 in the w^*-topology on $C[0, 1]^*$ (since $\{\Phi_i, \Phi_i^*\}_{i=1}^{\infty}$ is a basis for $C[0, 1]$). But then $\{w_n\}_{n=1}^{\infty} = \{Q^*(q_n)\}_{n=1}^{\infty}$ is w^*-convergent to 0 in E^*, so by assumption $\{T^*(Q^*q_n)\}_{n=1}^{\infty}$ converges in measure to 0 in $L^{\infty}(\mu)$.

As the second part of the proof of Theorem 3.3 shows, it follows that $\{\langle T^*Q^*q_n, f \rangle\}_{n=1}^{\infty}$ converges to 0 uniformly over f in any bounded, uniformly integrable subset of $L^1(\mu)$, in particular over the subset $\{f_n\}_{n=1}^{\infty} \subset A$. Thus there is an N so that if $n \geq N$ then $|\langle T^*Q^*q_n, f_n \rangle| < \epsilon$. That is,

$$|\langle q_n, QTf_n \rangle = \left| \left\langle \sum_{i=p_n+1}^{p_{n+1}} \langle \Phi_i, \alpha_n \rangle \Phi_i^*, QTFf_n \right\rangle \right| < \epsilon \quad \text{for} \quad n \geq N.$$

But this is just

$$\left| \left\langle \sum_{i=p_n+1}^{p_{n+1}} \langle \Phi_i^*, QTf_n \rangle \left\langle \Phi_i, \alpha_n \right\rangle \right| ,$$

which is $>\epsilon$ for all n, a contradiction. Hence it must be that T maps bounded, uniformly integrable subsets of $L^1(\mu)$ to compact subsets of E, and it follows that T is a strongly-D–P operator.

Finally, we note that in the case where $\mu(X) < +\infty$ Theorem 3.4 yields the following characterization of D–P operators.

Corollary 3.5. *If μ is a finite measure and E a separable Banach space, then an operator $T: L^1(\mu) \to E$ is a* D–P *operator* $\Leftrightarrow$ *T^* maps w^*-convergent sequences in E^* to $L^1(\mu)$-convergent sequences in $L^{\infty}(\mu)$.*

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