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A Coincidence Theorem for Holomorphic Maps to G/P

Dedicated to Professor Peter Zvengrowski on the occasion of his sixty-first birthday

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Abstract. The purpose of this note is to extend to an arbitrary generalized Hopf and Calabi-Eckmann manifold the following result of Kalyan Mukherjea: Let $V_n = \mathbb{S}^{2n+1} \times \mathbb{S}^{2n+1}$ denote a Calabi-Eckmann manifold. If $f, g\colon V_n \longrightarrow \mathbb{P}^n$ are any two holomorphic maps, at least one of them being non-constant, then there exists a coincidence: $f(x) = g(x)$ for some $x \in V_n$. Our proof involves a coincidence theorem for holomorphic maps to complex projective varieties of the form G/P where G is complex simple algebraic group and $P \subset G$ is a maximal parabolic subgroup, where one of the maps is dominant.

1 Introduction

Let G be a simply connected simple algebraic group over $\mathbb{C}$ and $P \subset G$ a *maximal* parabolic subgroup. Let $\mathcal{L}$ be the ample generator of $\mathrm{Pic}(G/P) \cong \mathbb{Z}$. Let E denote the total space of the principal $\mathbb{C}^*$ bundle associated to $\mathcal{L}^{-1}$. Let λ be any complex number with $|\lambda| > 1$ and let $\varphi\colon E \longrightarrow E$ denote the bundle map $e \mapsto \lambda \cdot e$, $e \in E$. The quotient space, denoted V_λ, is a compact complex homogeneous non-Kähler manifold. The manifold V_λ (or simply V) is called a generalized Hopf manifold [14]. (See also [10, §2].) One has an elliptic curve bundle $q\colon V \longrightarrow G/P$ with fibre and structure group the elliptic curve $\mathbb{T} = \mathbb{C}^*/\langle\lambda\rangle$ with periods $\{1, \tau\}$ where $\exp(2\pi\sqrt{-1}\tau) = \lambda$. (Note that $\mathrm{Im}\,(\tau) \neq 0$ as $|\lambda| > 1$.) One has a diffeomorphism $V \cong \mathbb{S}^1 \times K/L$ where K is a maximal compact subgroup of G and L the semisimple part of the centralizer of a subgroup of K isomorphic to the circle $\mathbb{S}^1$. If we take G/P to be the complex projective space $\mathbb{P}^n$, then the above construction yields the usual Hopf manifold $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$.

Suppose that both G, G' are simply connected simple algebraic groups over $\mathbb{C}$ and P, P', maximal parabolic subgroups of G, G' respectively. Let $E \longrightarrow G/P$, $E' \longrightarrow G'/P'$ denote the principal $\mathbb{C}^*$-bundles associated to the negative ample generators of the Picard groups of G/P, G'/P' respectively. The product bundle $E \times E' \longrightarrow G/P \times G'/P'$ is a principal $\mathbb{C}^* \times \mathbb{C}^*$ bundle. Let τ be any complex number with $\mathrm{Im}(\tau) \neq 0$. One has a complex analytic monomorphism $\mathbb{C} \longrightarrow \mathbb{C}^* \times \mathbb{C}^*$ defined by $z \mapsto \big(\exp(2\pi\sqrt{-1}\tau z), \exp(2\pi\sqrt{-1}z)\big)$. Denote the image of this group by $\mathbb{C}_\tau$. The group $\mathbb{C}_\tau \subset (\mathbb{C}^*)^2$ acts on $E \times E'$ as bundle automorphisms and the quotient $U = E \times E'/\mathbb{C}_\tau$ is the total space of an elliptic curve bundle with fibre and structure group the elliptic curve $(\mathbb{C}^*)^2/\mathbb{C}_\tau = \mathbb{T}$ with periods $\{1, \tau\}$. Up to a diffeomorphism, U can be identified with the space $K/L \times K'/L'$ where $K \subset G$,

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$K' \subset G'$ are maximal compact subgroups and L and L' are semisimple parts of centralizers of certain subgroups of K and K' isomorphic to $\mathbb{S}^1$. The compact complex manifold U is homogeneous and non-Kähler, which we call a generalized Calabi-Eckmann manifold. When $G/P = \mathbb{P}^m$, and $G'/P' = \mathbb{P}^n$, U is a Calabi-Eckmann manifold $\mathbb{S}^{2m+1} \times \mathbb{S}^{2n+1}$ [1]. We shall denote by p the bundle projection $U \longrightarrow G/P \times G'/P'$.

The manifold U is an example of a simply connected compact complex homogeneous manifold. Such manifolds have been completely classified by H.-C. Wang [18].

Theorem 1 *We keep the above notations.*

(i) *Let $\varphi, \psi \colon V \longrightarrow G/P$ be any two holomorphic maps with φ non-constant. Then there exists an $x \in V$ such that $\varphi(x) = \psi(x)$.*
(ii) *Assume that $\dim(G/P) \leq \dim(G'/P')$ and let $\varphi, \psi \colon U \longrightarrow G/P$ be a holomorphic map with φ non-constant. Then there exists an $x \in U$ such that $\varphi(x) = \psi(x)$.*

The above theorem will be derived from the following:

Theorem 2

(i) *Let M be any connected compact complex analytic manifold and let $\varphi, \psi \colon M \longrightarrow G/P$ be holomorphic where $P \subset G$ is a maximal parabolic subgroup. Assume that at least one of the maps φ, ψ is surjective. Then there exists an $x \in M$ such that $\varphi(x) = \psi(x)$.*
(ii) *Furthermore, if M is a projective variety, or if Kähler manifold with $\dim(M) = \dim(G/P)$, and $f, g \colon M \longrightarrow G/P$ are continuous maps homotopic to φ, ψ respectively, then there exists an $x \in M$ such that $f(x) = g(x)$.*

The special case of Theorem 1 when $U = \mathbb{S}^{2n+1} \times \mathbb{S}^{2n+1}$ is a Calabi-Eckmann manifold is due to K. Mukherjea [12]. He uses D. Toledo's approach [17] to analytic fixed point theory. In particular he uses Borel's computation of Dolbeault cohomology of Calabi-Eckmann manifolds and a certain 'analytic Thom class' $\xi^0_\Delta \in H^n(\mathbb{P}^n \times \mathbb{P}^n ; \Omega^n) \cong H^{2n}(\mathbb{P}^n \times \mathbb{P}^n ; \mathbb{C})$ supported on the diagonal to detect coincidences. Our proof is based on the observation that any holomorphic map from U (resp. V) to a complex projective variety factors through $G/P \times G'/P'$ (resp. G/P) (see Lemma 8, Section 3). Any non-constant holomorphic map of G/P or of $G/P \times G'/P'$ into G/P is shown to be dominant (Lemma 3). Theorem 1 is then deduced from Theorem 2. Theorem 2 is proved using positivity of cup products in cohomology of G/P and the fact that any effective cycle is rationally equivalent to a *positive* linear combination of Schubert cycles (*cf.* [8], [9].)

2 Holomorphic Maps to G/P

We keep the notations of Section 1. In particular, G is a simply connected simple complex algebraic group. Let $Q \subset G$ be any parabolic subgroup (not necessarily maximal). Fix a maximal torus T and a Borel subgroup B containing T such that

$B \subset Q$. Let W be the Weyl group of G with respect to T and $W(Q) \subset W$ that of Q and let $S \subset W$ denote the set of simple reflections with respect to our choice of B. Recall that the T fixed points of G/Q are labelled by $W/W(Q)$. We shall identify $W/W(Q)$ with the set of coset representatives $W^Q \subset W$ having minimal length with respect to S. The B-orbits of these T fixed points give an algebraic cell decomposition for G/Q. Denote by $X(w)$ the Schubert variety $X(w) \subset G/Q$ which is the B-orbit closure of the T-fixed point corresponding to $w \in W^Q$. The class of the Schubert varieties $[X(w)]$, $w \in W^Q$, form a $\mathbb{Z}$-basis for the singular cohomology group $H^*(G/Q\,;\mathbb{Z})$ as well as the Chow cohomology group $A^*(G/Q)$. Recall that $\mathrm{Pic}(G/Q) \cong A^1(G/Q)$ which is infinite cyclic when Q is a maximal parabolic subgroup.

Lemma 3 *Let $P \subset G$ be a maximal parabolic subgroup and let X be an irreducible complex projective variety with* $\mathrm{Pic}(X) = \mathbb{Z}$. *Suppose that* $\dim(X) \geq \dim(G/P)$. *Then any non-constant algebraic map $\varphi\colon X \longrightarrow G/P$ is dominant with finite fibres; in particular* $\dim(G/P) = \dim(X)$.

Proof Let $\mathcal{L}$ be a very ample line bundle over G/P. Let $Z = \mathrm{Im}(\varphi)$. As φ is non-constant, Z is not a point variety and $\mathcal{L}|Z$ is very ample. Since the bundle $\mathcal{L}|Z$ is generated by global global sections, it follows that $\mathcal{L}' := \varphi^*(\mathcal{L})$ is generated by its sections over X. Also $\mathcal{L}'$ cannot be trivial since any non-zero section of a trivial bundle is nowhere zero, whereas sections of $\mathcal{L}'$ arising as pull-back of non-zero sections of $\mathcal{L}|Z$ are non-zero sections which vanish somewhere in X. Since $\mathrm{Pic}(X) \cong \mathbb{Z}$ is it follows that some *positive* multiple of $\mathcal{L}'$ must be very ample. Hence, restricted to any fibres of φ the bundle $\mathcal{L}' = \varphi^*(\mathcal{L})$ must be ample. This implies that the fibres of φ must be finite. Therefore $\dim(X) \leq \dim(G/P)$. Since $\dim(X) \geq \dim(G/P)$ by hypothesis, we must actually have equality and the map φ must be dominant. ■

Remark 4 (i) The assumption that $\mathrm{Pic}(X) = \mathbb{Z}$ is not superfluous. For example, take $X = \mathbb{P}^1 \times \mathbb{P}^n$, $n \geq 2$. Let $\varphi\colon X \longrightarrow \mathbb{P}^3$ be the composition $\varphi_1 \circ pr_1$, where pr_1 is the first projection map and $\varphi_1\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ is defined by $\varphi_1(z_0 : z_1) = (z_0 : z_1 : 0 : 0)$.

(ii) Suppose that $\varphi\colon Z \longrightarrow G/P$ is holomorphic and that $X \subset Z$ is a complex analytic subset which satisfies the hypothesis of the lemma above. If $\varphi\colon Z \longrightarrow G/P$ is holomorphic and $\varphi|X$ is non-constant, then φ must be dominant. On the other hand, suppose that $\pi\colon Z \longrightarrow M$ is a complex analytic fibre bundle with fibre X as in the lemma above. Suppose $\dim(X) > \dim(G/P)$, then any complex analytic map $\psi\colon Z \longrightarrow G/P$ factors through π, *i.e.*, $\psi = \theta \circ \pi$ for some complex analytic map $\theta\colon M \longrightarrow G/P$ since the lemma implies that ψ restricted to any fibre has to be constant.

(iii) A result of K. Paranjape and V. Srinivas [13] says that if G/P is not the projective space, any non-constant self morphism $\varphi\colon G/P \longrightarrow G/P$ is an automorphism of varieties. The full group of automorphisms of G/P has been determined by I. Kantor [7].

We shall now prove Theorem 2:

Proof of Theorem 2 (i) Suppose $\varphi\colon M \longrightarrow G/P$ is surjective. Set $d = \dim(G/P)$, $m = \dim(M)$. Let $\Delta \subset G/P \times G/P$ denote the diagonal of G/P. Let $\Gamma := \Gamma_\theta \subset G/P \times G/P$ denote the image of the map $\theta\colon M \longrightarrow G/P \times G/P$, $\theta(x) = \big(\varphi(x), \psi(x)\big)$, $x \in M$. We need only show that $\Gamma \cap \Delta \neq \varnothing$. Note that Γ is a complex analytic subspace of the projective variety G/P and hence algebraic by GAGA [15]. Also $k := \dim(\Gamma) \geq \dim(G/P)$ since φ is surjective. Now, as for any effective cycle in $G/P \times G/P$, the class $[\Gamma] \in A_k(G/P \times G/P)$ is a *positive* linear combination of Schubert cycles in $G/P \times G/P$. (*Cf.* [7]. See also [9].) Thus, $[\Gamma] = \sum_i a_i[X(w_i)] \times [X(w_i')]$ where a_i are positive integers, $w_i, w_i' \in W^P$ are suitable elements such that $\dim\big(X(w_i)\big) + \dim\big(X(w_i')\big) = \dim(\Gamma)$. Since φ is surjective, in the above expression for Γ, the term $[G/P] \times [X(w)]$ must occur with positive coefficient for some $w \in W^P$. The same arguments can be applied to the class of the diagonal Δ in $G/P \times G/P$ as well. In fact, it is known that $[\Delta] = \sum_{v \in W^P} [X(v)] \times [X(v'')]$ where $X(v'')$ is the Schubert variety 'dual' to $X(v)$, *i.e.*, $v'' = w_0.v$ where $w_0 \in W^P$ represents the longest element of W/W_P. (*Cf.* [11, Theorem 11.11].) Hence $[\Gamma].[\Delta] = \big(a[G/P] \times [X(w)] + \text{other terms}\big) \cdot (1 \times [G/P] + \text{other terms}) = a\big(1 \times [X(w)]\big) +$ other terms, where $a > 0$ and the coefficients of the remaining terms (with respect to the basis consisting of Schubert cocycles) in the rhs of the last equality are *non-negative integers*. Hence $[\Gamma][\Delta] \neq 0$ in $A_{k-d}(G/P \times G/P)$ and so $\Gamma \cap \Delta \neq \varnothing$.

(ii) Let $h\colon M \longrightarrow G/P \times G/P$ be the map $x \mapsto \big(f(x), g(x)\big)$ for $x \in M$. It suffices to show that $h^*([\Delta]) \in H^{2d}(M ; \mathbb{Z})$ is non-zero, where we regard $[\Delta]$ as an element of the singular cohomology group $H^{2d}(G/P \times G/P ; \mathbb{Z})$. Note that h is homotopic to $\theta = (\varphi, \psi)$ and so we have $h^*([\Delta]) = \theta^*([\Delta])$. To complete the proof, it suffices to show that $\theta^*([\Delta]) \neq 0$ in $H^*(M ; \mathbb{Z})$.

Suppose that M is Kähler and $\dim(M) = d = \dim(G/P)$. By de Rham and Hodge theory [6, §15.7], we have $H^r(M ; \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}_{\bar\partial}(M)$ and $\theta^*([\Delta])$ can be thought of as an element of $H^{d,d}_{\bar\partial}(M ; \mathbb{C})$. Since φ is dominant, $\dim(\Gamma) = d = \dim(M)$ and the fundamental class $\mu_M \in H_{2d}(M ; \mathbb{Z})$ maps to $n[\Gamma] \in H_{2d}(G/P \times G/P ; \mathbb{Z}) \cong A_d(G/P \times G/P)$ for some $n \geq 1$. From what has been shown already the intersection product $[\Delta].[\Gamma_\theta] \neq 0$ in the Chow ring of $G/P \times G/P$. This implies that $[\Delta] \cap [\Gamma_\theta] \neq 0$ in $H_0(G/P \times G/P)$. Now we have (see [16, ch. 5, §6])

$$\theta_*\big(\theta^*([\Delta]) \cap \mu_M\big) = n[\Delta] \cap \theta_*(\mu_M) = n[\Delta] \cap [\Gamma] \neq 0.$$

It follows that $\theta^*([\Delta]) \neq 0$ in $H^*(M ; \mathbb{Z})$.

If M is a complex projective variety then, one can always find an irreducible subvariety $Z \subset M$ with $\dim(Z) = \dim(\Gamma_\theta)$ which maps onto Γ. It follows that $\theta_*([Z]) = n[\Gamma_\theta] \in H_*(G/P \times G/P)$ for some $n > 0$. Using the fact that $H_*(G/P \times G/P)$ has no torsion, proceeding just as before we conclude that $\theta^*(\Delta) \neq 0$. ■

Remark 5 (i) In the statement of Theorem 2, the hypothesis that M be nonsingular is not necessary. Indeed, H. Hironaka [5] has shown that any irreducible complex analytic space which is countable at infinity can be desingularized. So replacing M by $\widetilde{M}$ and the maps φ, ψ by $\widetilde{\varphi} := \varphi \circ \pi$, $\widetilde{\psi} := \psi \circ \pi$ respectively where $\pi\colon \widetilde{M} \longrightarrow M$ is a desingularization map, we see that $\widetilde{\varphi}$, $\widetilde{\psi}$ must have a coincidence. This immediately implies that φ and ψ must have a coincidence.

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(ii) In case M is not Kähler, it is not true in general that $\theta^*([\Delta]) \neq 0$ in $H^*(M)$ although φ and ψ must have a coincidence as our theorem shows. Such coincidences have been described as "homologically invisible" by Kalyan Mukherjee [12]. He observed that when M is the Calabi-Eckmann manifold $\mathbb{S}^{2n+1} \times \mathbb{S}^{2n+1}$ with $n \geq 1$, for any two maps $\varphi, \psi \colon M \longrightarrow \mathbb{P}^n$ the homomorphism $\theta^* \colon H^*(\mathbb{P}^n \times \mathbb{P}^n) \longrightarrow H^*(M)$ is zero in positive dimensions where $\theta = (\varphi, \psi)$. In particular, if f, g are *continuous* maps homotopic to holomorphic maps φ, ψ respectively with φ dominant, we do not know if f and g must have a coincidence.

(iii) When M is Kähler and $\dim(M) > \dim(G/P)$, the conclusion of the theorem is still valid provided $[\Gamma_\theta] \in H_*(G/P\ ;\ \mathbb{Q})$ is in the image of $\theta_* \colon H_*(M\ ;\ \mathbb{Q}) \longrightarrow H_*(G/P\ ;\mathbb{Q})$.

Corollary 6

(i) *Let $P \subset G$ be a maximal parabolic subgroup and let $f, g \colon G/P \longrightarrow G/P$ be any two continuous maps homotopic to holomorphic maps φ, ψ respectively where φ is non-constant. Then $f(x) = g(x)$ for some $x \in G/P$.*

(ii) *Let $f, g \colon G/P \times G'/P' \longrightarrow G/P$ be any two continuous maps which are homotopic to holomorphic maps φ, ψ respectively with φ being non-constant. Assume that $\dim(G/P) \leq \dim(G'/P')$. Then there exists an $x \in G/P \times G'/P'$ such that $f(x) = g(x)$.*

Proof Part (i) follows immediately from Lemma 3 and Theorem 2 (ii). To prove (ii), suppose $\varphi|Z$ is constant for every fibre $Z \cong G'/P'$ of the first projection $pr_1 \colon G/P \times G'/P' \longrightarrow G/P$ map, then φ can be factored as $\varphi_1 \circ pr_1$ where $\varphi_1 \colon G/P \times G'/P' \longrightarrow G/P$ defined by φ. It follows from 3 that φ_1 is dominant. Hence φ is also dominant. Otherwise for some fibre $Z \cong G'/P'$, $\varphi|Z$ is non-constant. By Lemma 3 it follows that $\varphi|Z$ is dominant. It follows from Lemma 3 again that $\varphi|Z$—and hence φ—must be dominant and the corollary follows from Theorem 2. ■

Remark 7 It follows from the above corollary that any continuous map homotopic to a holomorphic map has a fixed point. However, in general, the spaces G/P do not have fixed point property. For example, the Grassmannian $G_k(\mathbb{C}^n) = \mathrm{SL}(n, \mathbb{C})/P_k$ admits a continuous fixed point free involution whenever n is even and k odd or if $n = 2k$. As another example, the complex quadric $\mathrm{SO}(n)/P_1$ is diffeomorphic to the oriented real Grassmann manifold $\widetilde{G}_2(\mathbb{R}^n)$ of oriented 2-planes in $\mathbb{R}^n$. The involution that reverses the orientation on each element of $\widetilde{G}_2(\mathbb{R}^n)$ is obviously fixed point free. However, it is known that $G_k(\mathbb{C}^n)$ has fixed point property (for continuous maps) when n is large compared to k and at most one of $n-k, k$ is odd. See [2], [3].

3 Proof of Main Theorem

We now prove the main result of the paper, namely, Theorem 1. We keep the notations of Section 1.

Lemma 8 *Let U, V be generalized Calabi-Eckmann and generalized Hopf manifolds. (See Section 1.) Let Z be a complex projective variety. Any holomorphic maps*

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$\varphi\colon U \longrightarrow Z$, $\psi\colon V \longrightarrow Z$ can be factored as $\varphi = \varphi_1 \circ p$, $\psi = \psi_1 \circ q$, where $p\colon U \longrightarrow G/P \times G'/P'$ and $q\colon V \longrightarrow G/P$ are projections of the principal $\mathbb{T}$-bundles.

Proof It was shown in the proof of [14, Theorem 3], that $H^2(V\,;\mathbb{Z}) = H^2(K/L\,;\mathbb{Z}) = 0$ where $K \subset G$ is a maximal compact subgroup of G and L is the semisimple part of the centralizer in K of a subgroup of K isomorphic to $\mathbb{S}^1$. The same argument shows that $H^2(U\,;\mathbb{Z}) = H^2(K/L \times K'/L'\,;\mathbb{Z}) = 0$. In particular the manifolds U, V are not Kähler.

A theorem of Grauert and Remmert [4] says that for a compact complex homogeneous manifold M of dimension n, the transcendence degree over $\mathbb{C}$ of the field $\mathcal{M}(M)$ of meromorphic functions on M is equal to n if and only if it a projective algebraic variety. In our case U, V fibre over projective varieties of dimension one less. It follows that $\operatorname{tr.deg}_{\mathbb{C}}\big((V)\big) \geq \operatorname{tr.deg}_{\mathbb{C}}\big(\mathcal{M}(X)\big) = \dim(G/P)$. Suppose ψ is not constant along a fibre. Then there exists an open set (in the analytic topology) $N \subset G/P$ such that ψ is non-constant on $q^{-1}(x)$, for any $x \in N$. Let $x \in N$. Composing with a suitable meromorphic function on Z which is non-constant on $\psi\big(q^{-1}(x)\big)$, we get a meromorphic function θ on V. We claim that θ is transcendental over $\mathcal{M}(G/P) \subset \mathcal{M}(V)$. Assume, if possible, that $\theta^k + a_1\theta^{k-1} + \cdots + a_k = 0$, $a_i \in \mathcal{M}(G/P)$. By changing the $x \in N$ if necessary, we may assume that x is not on the polar divisor for any a_i. Restricting this equation to the fibre over x, we see that $\theta|q^{-1}(x)$ is algebraic over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, we must have $\theta|p^{-1}(x) \in \mathbb{C}$. This is absurd since θ is non-constant on $q^{-1}(x)$. Hence we conclude that θ is constant along the fibres of q. Proof that φ is constant along the fibres of p is entirely similar. ■

Proof of Theorem 1 (i) By Lemma 8, the maps φ, ψ factor through the projection of the elliptic curve bundle $p\colon V \longrightarrow G/P$. Write $\varphi = \varphi_1 \circ p$, $\psi = \psi_1 \circ p$. Now, it suffices to show that the holomorphic maps φ_1 and ψ_1 have a coincidence. Since φ_1 is non-constant, this is now immediate from Corollary 6 (i).

(ii) Proceeding exactly as in (i), we write $\varphi = \varphi_1 \circ q$, $\psi = \psi_2 \circ q$, where $\varphi_1, \psi_1\colon G/P \times G'/P' \longrightarrow G/P$ are holomorphic. Note that since φ_1 is non-constant. By Corollary 6 φ_1 and ψ_1 must have a coincidence. Hence φ and ψ must also have a coincidence. ■

We conclude with the following observation.

Lemma 9 *Let $\pi\colon W \longrightarrow M$ be a holomorphic fibre bundle with M compact connected, and fibre a complex torus $\mathbb{T}$. Suppose that $H_2(F\,;\mathbb{Q}) \longrightarrow H_2(W\,;\mathbb{Q})$ is zero. Then any holomorphic map $\varphi\colon W \longrightarrow G/Q$ is constant on the fibres of π where $Q \subset G$ is any parabolic subgroup.*

Proof Assume that $\varphi\colon W \longrightarrow G/Q$ is a holomorphic map such that $\varphi|F$ is not constant for some fibre F of the T-bundle $\pi\colon W \longrightarrow M$. Let $\iota\colon F \subset W$ denote the inclusion map. Let $C \subset G/Q$ be the image of F. Note that $\dim(C) = 1 = \dim(F)$. Since φ is holomorphic, C is an algebraic subvariety of G/Q. In particular, it represents a non-zero element of $H_2(G/Q\,;\mathbb{Q})$. In fact C is rationally equivalent to a positive linear combination of certain 1-dimensional Schubert subvarieties in G/Q [8]. It follows that $(\varphi|F)_*\colon H_2(F\,;\mathbb{Q}) \longrightarrow H_2(G/Q\,;\mathbb{Q})$ maps the fundamental class of F to a nonzero element of $H_2(G/Q\,;\mathbb{Q})$. On the other hand, $(\varphi|F)_* = \varphi_* \circ \iota_* = 0$ in dimension 2, since $\iota_*\colon H_2(T\,;\mathbb{Q}) \longrightarrow H_2(W\,;\mathbb{Q})$ is zero by hypothesis. We conclude that φ must be constant on the fibres of π. ∎

Remark 10 (i) Let $\pi\colon W \longrightarrow M$ be as in the above lemma. Let $\varphi, \psi\colon W \longrightarrow G/P$ be any two holomorphic maps where φ is non-constant and $P \subset G$ a maximal parabolic. Let $\varphi_1\colon M \longrightarrow G/P$ be such that $\varphi = \varphi_1 \circ \pi$. Suppose $X \subset M$ is irreducible and has the structure of a complex projective variety with $\operatorname{Pic}(X) = \mathbb{Z}$ and $\dim(X) = \dim(G/P)$. If $\varphi_1|X$ is non-constant, then, in view of the above lemma and Remark 4(ii), φ_1 must be dominant. It follows that φ itself must be dominant. By Theorem 2 it follows that φ and ψ must have a coincidence.

(ii) I do not know if Theorem 1 still holds if one merely assumes that φ, ψ are continuous maps homotopic to holomorphic maps one of which is dominant.

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