Second Meeting, December 12th, 1884.

A. J. G. Barclay, Esq., M.A., President, in the Chair.

Failing Cases of Fourier's Double-Integral Theorem.

By Peter Alexander, M.A.

Fourier's Theorem, as usually stated, is

$$\phi(x)=\frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{+\infty}\phi(v)\cos w(x-v)\,dv\right\}dw.$$

Most writers give this without limitation, but De Morgan (*Diff. and Int. Calc. pp.* 618 *&c.*) directs attention to what he calls the *apparent* neglect by previous writers of the limitation of the theorem to functions which satisfy the condition

$$\int_{-\infty}^{+\infty}\phi(v)dv=\text{a finite quantity.}$$

After showing that this limitation is neglected in each of three methods by which the theorem had been verified, he proceeds to show that the limitation may be removed, and that the theorem is universal. He says—

"Returning to the expression

$$\phi(x)=\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv+\frac{1}{\pi}\Sigma\left\{\int_{-\infty}^{+\infty}\cos w(x-v)\phi(v)dv.\Delta w\right\}\qquad\text{(A.)}$$

"first observe that

$$\int(A_1x_1+A_2x_2+\&c.)da\text{ or}\int\left[\Sigma\,(Ax)\right]da$$

"is identical with

$$\int A_1da.x_1+\int A_2da.x_2+\&c.\text{ or }\Sigma\left(\int A\;da.x\right)$$

"provided only that x_1, x_2, &c., are independent of a.

"Write the expression (A) in the form

$$\frac{1}{\pi}\int_{-l}^{+l}\left\{\tfrac{1}{2}\Delta w\cos 0\,(x-v)+\Delta w\cos\Delta w\,(x-v)\right.$$

$$\left.+\Delta w\cos 2\Delta w\,(x-v)+\&c.\right\}\phi(v)\;dv=\phi(x).$$

"This expression is absolutely true for $-l(x)l$, whatever the values of "l may be, and the series it contains is the limit of a set of convergent "series made by diminishing κ without limit in

$$\tfrac{1}{2}\,\Delta w \cos 0\,(x-v).\epsilon^{-0.\kappa\Delta w} + \Delta w \cos \Delta w\,(x-v).\epsilon^{-\kappa\Delta w}$$
$$+ \Delta w \cos 2\Delta w\,(x-v).\epsilon^{-2\kappa\Delta w} + \ldots\ldots$$

"Let κ have any positive value, however small, and let the preceding "be multiplied by $\phi(v)$ and integrated with respect to v from $v = -l$ "to $v = +l$; that is from $v = -(\pi \div \Delta w)$ to $v = +(\pi \div \Delta w)$; and if x "lie between these limits the result will be as near as we please to "$\phi(x)$ if κ be taken small enough. Since the series is convergent this "might be verified by actual arithmetical operation.

"*Now, since the individual terms of the preceding diminish without "limit with* Δw, *any one or more of them, in fact any finite and fixed "number of them, might be erased or altered in any finite ratio without "affecting the result.*

"If then in the first term we change $\frac{1}{2}$ into 1, (or if we erased the "first term altogether,) the limit of the result, when Δw is dimished "without limit, is strictly

$$\frac{1}{\pi}\int_{-\pi\div 0}^{+\pi\div 0}\int_{0}^{\infty} \cos w\,(x-v).\epsilon^{-\kappa w}\,\phi\,(v)\,dv.dw = \phi\,(x) + a$$

"where a and κ are comminuent.

"Diminish κ without limit and we have Fourier's theorem as given."

This reasoning seems plausible, but is fallacious. The fallacy is contained in the words "Now, since......the result," which I have italicised. He takes it for granted that, because the term

$$\tfrac{1}{2}\,\Delta w \cos 0\,(x-v).\epsilon^{-0.\kappa\Delta w}.\phi(v) \quad \text{or} \quad \tfrac{1}{2}.\Delta w\,\phi(v)$$

diminishes without limit with Δw, its integral

$$\int_{\frac{-\pi}{\Delta w}}^{\frac{\pi}{\Delta w}} \tfrac{1}{2}\,\Delta w.\phi(v)\,dv$$

also dininishes without limit with Δw, which does not of necessity follow. In fact he admits that it does not in a former part of his discussion of this theorem, where he says :—"$\int \phi(v)dv \div 2l$ may in such case (*i.e.*, if $\int \phi(v)dv$ increase without limit with l) either increase without limit, have a finite limit, or diminish without limit." [Here $l = \dfrac{\pi}{\Delta w}$ and π is dropped.]

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To prove that $\int_{-\frac{\pi}{\Delta w}}^{\frac{\pi}{\Delta w}} \frac{1}{2}\,\Delta w.\phi(v)dv$ is not necessarily zero,

let $\phi(v) = \frac{d}{dv}.\psi(v)$. Then

$$\int_{-\frac{\pi}{\Delta w}}^{\frac{\pi}{\Delta w}} \frac{1}{2}\,\Delta w.\phi(v)dv = \frac{1}{2}\,\Delta w\left\{\psi\left(\frac{\pi}{\Delta w}\right) - \psi\left(-\frac{\pi}{\Delta w}\right)\right\}$$

or $$\frac{1}{2}\,\Delta w\left\{\left[\psi\left(\frac{\pi}{\Delta w}\right) - \psi(v)\right] + \left[\psi(v) - \psi\left(-\frac{\pi}{\Delta w}\right)\right]\right\}$$

Now if $\psi\left(\frac{\pi}{\Delta w}\right) - \psi\left(-\frac{\pi}{\Delta w}\right)$ become zero or remain finite when Δw is dimished without limit the integral will become zero. But if $\psi\left(\frac{\pi}{\Delta w}\right) - \psi\left(-\frac{\pi}{\Delta w}\right)$ increase without limit when Δw is diminished without limit the integral will not necessarily become zero.

For example, suppose

$\phi(v) = p$ for positive values of v

and $= q$ „ negative „ „

Then $\psi(v) = pv$ for positive values of v

and $= qv$ „ negative „ „

Then $$\int_{-\frac{\pi}{\Delta w}}^{\frac{\pi}{\Delta w}} \frac{1}{2}\,\Delta w\phi(v)\,dv = \frac{1}{2}\,\Delta w\left\{\psi\left(\frac{\pi}{\Delta w}\right) - \psi\left(-\frac{\pi}{\Delta w}\right)\right\}$$

$$= \frac{1}{2}\,\Delta w\left\{p\,\frac{\pi}{\Delta w} - q\left(-\frac{\pi}{\Delta w}\right)\right\}$$

$$= \frac{p+q}{2}\pi$$

which is not zero unless $q = -p$.

In cases such as this Fourier's Theorem fails, and instead of it we should use

$$\phi(x) = \mathop{\mathrm{L}}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv\right] + \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{+\infty}\phi(v)\cos w(x-v)dvdw. \tag{B}$$

which is universal.

This follows at once from (A) which is true for all periodic functions of period $-l(x)l$.

Now a non-periodic function may be looked upon as a periodic

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function of infinite period $-\infty(x)\infty$. But (A) is true however great the period $-l(x)l$ be, hence its limiting form, (B), will be true for a function of infinite range, that is for a non-periodic function.

I shall now give one or two illustrations of the failure of Fourier's theorem, and show that the modified theorem (B) gives the true expression.

First example—

Let $\phi(x)=p$ for all positive values of x.

and $\quad =q$,, negative ,, ,, .

Then
$$\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{+\infty}\phi(v).\cos w(x-v)\,dv.dw$$
$$=\frac{1}{\pi}\int_0^\infty\left\{p\int_0^\infty\cos w(x-v)\,dv+q\int_{-\infty}^0\cos w(\chi-v)\,dv\right\}$$
$$=\frac{1}{\pi}\int_0^\infty\left\{p\left(\frac{\sin wx}{w}\right)+q\left(-\frac{\sin wx}{w}\right)\right\}dw$$
$$=\frac{p-q}{\pi}\int_0\frac{\sin w\,x}{w}dw$$
$$=\frac{p-q}{\pi}\left(\pm\frac{\pi}{2}\right)=\pm\frac{p-q}{2}$$
according as x is positive or negative.

This is certainly not $=\phi(x)$. Hence the ordinary form of Fourier's theorem fails in this case.

But let us see what (B) gives.
$$\mathrm{L}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv\right]=\mathrm{L}_{l=\infty}\left[\frac{1}{2l}\left\{\int_0^l p\,dv+\int_{-l}^0 q\,dv\right\}\right]$$
$$=\mathrm{L}_{l=\infty}\left[\frac{1}{2l}\left\{p\,l+q\,l\right\}\right]$$
$$=\frac{p+q}{2};$$
consequently
$$\mathrm{L}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv\right]+\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{+\infty}\phi(v).\cos w(x-v)\,dv.dw)$$
$$=\frac{p+q}{2}\pm\frac{p-q}{2}=p\text{ or }q,$$
according as x is positive or negative.

Therefore (B) holds in this case.

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Second example—

Let $\phi(x) = a(1+\epsilon^{-mx})$ for positive values of x.

and $ = b(1+\epsilon^{mx})$,, negative ,, ,, .

Then

$$\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{+\infty}\phi(v)\cos w(x-v)\,dv.dw$$

$$=\frac{1}{\pi}\int_0^\infty\left\{\int_0^\infty a(1+\epsilon^{-mv})\cos w(x-v)\,dv\right.$$

$$\left.+\int_{-\infty}^0 b(1+\epsilon^{mv})\cos w(x-v)\,dv\right\}dw$$

$$=\frac{1}{\pi}\int_0^\infty\left\{a\left[\int_0^\infty\cos w(x-v)\,dv+\int_0^\infty\epsilon^{-mv}\cos w(x-v)\,dv\right]\right.$$

$$\left.+b\left[\int_{-\infty}^0\cos w(x-v)\,dv+\int_{-\infty}^0\epsilon^{mv}\cos w(x-v)\,dv\right]\right\}dw$$

$$=\frac{1}{\pi}\int_0^\infty\left\{a\left[\cos wx\left(\int_0^\infty\cos wv\,dv+\int_0^\infty\epsilon^{-mv}\cos wv\,dv\right)\right.\right.$$

$$\left.+\sin wx\left(\int_0^\infty\sin wv\,dv+\int_0^\infty\epsilon^{-mv}\sin wv\,dv\right)\right]$$

$$+b\left[\cos wx\left(\int_{-\infty}^0\cos wv\,dv+\int_{-\infty}^0\epsilon^{mv}\cos wv\,dv\right)\right.$$

$$\left.\left.+\sin wx\left(\int_{-\infty}^0\sin wv\,dv+\int_{-\infty}^0\epsilon^{mv}\sin wv\,dv\right)\right]\right\}dw$$

$$=\frac{1}{\pi}\int_0^\infty\left\{a\left[\cos wx\left(0+\frac{m}{m^2+w^2}\right)+\sin wx\left(\frac{1}{w}+\frac{w}{m^2+w^2}\right)\right]\right.$$

$$\left.+b\left[\cos wx\left[0+\frac{m}{m^2+w^2}\right)+\sin wx\left(-\frac{1}{w}-\frac{w}{m^2+w^2}\right)\right]\right\}dw$$

$$=\frac{1}{\pi}\left\{(a+b)\,m\int_0^\infty\frac{\cos wx}{m^2+w^2}\,dw+(a-b)\int_0^\infty\frac{\sin wx}{w}dw\right.$$

$$\left.+(a-b)\int_0^\infty\frac{w\sin wx}{m^2+w^2}dw\right\}$$

$$=\frac{1}{\pi}\left\{(a+b)\;m.\frac{\pi}{2m}\epsilon^{\pm mx}+(a-b).\left(\pm\frac{\pi}{2}\right)\right.$$

$$\left.+(a-b)\left(\pm\frac{\pi}{2}\epsilon^{\mp mx}\right)\right\}$$

$$=\pm\frac{a-b}{2}+\left(\frac{a+b}{2}\pm\frac{a-b}{2}\right)\epsilon^{\mp mx}$$

the upper or lower sign according as x is positive or negative.

This becomes

$$\frac{a-b}{2}+a\,\epsilon^{-mx} \text{ when } x \text{ is positive,}$$

and $$-\frac{a-b}{2}+b\,\epsilon^{-mx} \text{ when } x \text{ is negative,}$$

and is therefore not $=\phi(x)$. Hence Fourier's theorem again fails.

Let us now examine what (B) gives in this case.

$$\mathop{\mathrm{L}}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv\right]=\mathop{\mathrm{L}}_{l=\infty}\left[\frac{1}{2l}\left\{\int_0^l a\,(1+\epsilon^{-mv})\,dv + \int_{-l}^{0} b\,(1+\epsilon^{mv})\,dv\right\}\right]$$
$$=\mathop{\mathrm{L}}_{l=\infty}\left[\frac{1}{2l}\left\{a\left(l+\frac{1-\epsilon^{-ml}}{m}\right) + b\left(l+\frac{1-\epsilon^{-ml}}{m}\right)\right\}\right]$$
$$=\frac{a+b}{2}.$$

Hence

$$\mathop{\mathrm{L}}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv\right]+\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{+\infty}\phi(v)\cos w\,(x-v)\,dv.dw$$
$$=\frac{a+b}{2}\pm\left\{\frac{a-b}{2}\pm\left(\frac{a+b}{2}\pm\frac{a-b}{2}\right)\epsilon^{\pm mx}\right\}$$
$$=\left(\frac{a+b}{2}\pm\frac{a-b}{2}\right)\left(1+\epsilon^{\mp mx}\right).$$

according as x is positive or negative.

This gives $a\,(1+\epsilon^{-mx})$ when x is positive.

and $b\,(1+\epsilon^{mx})$ „ „ negative.

that is, it gives $\phi(x)$.

Hence (B) holds for this case.

Some writers, as Duhamel and Price, to avoid difficulty, assume that $\phi\,(x)$ never $=\infty$, and $\phi\,(x)=0$ when $x=\pm\infty$, thus ignoring all such functions as the two dealt with above.

Similarly, Todhunter annexes to the theorem the condition that

$$\frac{1}{2l}\int_{-l}^{l}\phi\,(v)\,dv \text{ vanishes with } \frac{1}{l}.$$

Freeman, Fourier's translator, (*Theory of Heat*, p. 351) says that Poisson first gave a direct proof that

$$\phi\,(x)=\left\{\frac{1}{\pi}\int_0^\infty dq\int_{-\infty}^{+\infty}d\alpha.\epsilon^{-\kappa q}\cos\,(qx-q\alpha)\;\phi\,(\alpha)\right\}_{\kappa=0}$$

κ being put $= 0$ after the integrations, and that Boole regarded Fourier's theorem as unproved, unless equivalent to this. But this is just what De Morgan tried to prove, and failed in doing, as I have shown. Moreover, if we test this on the first of the two examples I have chosen, we have

$$\frac{1}{\pi}\int_0^\infty dw\int_{-\infty}^{+\infty} dv.\epsilon^{-\kappa w}\cos w\,(x-v)\,.\,\phi\,(v)$$

$$=\frac{1}{\pi}\int_0^\infty dw.\epsilon^{-\kappa w}\left\{p\int_0^\infty \cos w\,(x-v)\,dv+q\int_{-\infty}^0 \cos w\,(x-v)\,dv\right\}$$

$$=\frac{1}{\pi}\int_0^\infty dw.\epsilon^{-\kappa w}\left\{p\left(\frac{\sin wx}{w}\right)+q\left(-\frac{\sin wx}{w}\right)\right\}$$

$$=\frac{p-q}{\pi}\int_0^\infty \frac{\epsilon^{-\kappa w}\sin wx}{w}\,dw$$

$$=\frac{p-q}{\pi}\tan^{-1}\frac{x}{\kappa}.$$

If now we put $\kappa=0$ this becomes

$$\frac{p-q}{\pi}\left(\pm\frac{\pi}{2}\right)=\pm\frac{p-q}{2}$$

which is not $=\phi(x)$.

Therefore Poisson's formula gives the same result as Fourier's and seems subject to the same limitations.

Note.—In the discussion which followed, Professor Chrystal and Dr Muir mentioned one or two papers* worth reading on this subject; Dr Muir also stated that he had some little hesitation in accepting the remarks on Poisson's form, and hoped Mr Alexander would examine further into the matter.

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* *G(regory?).* On Fourier's Theorem. *Cambridge Mathematical Journal*, III., pp. 288-290.

Walton, W. A demonstration of Fourier's Theorem. *Quarterly Journal of Mathematics*, VIII., pp. 136-138.

Glaisher, J. W. L. On Fourier's double-integral Theorem. *Messenger of Mathematics*, II., pp. 20-24.

Du Bois Reymond, P. Sur les formules de représentation des fonctions. *Comptes rendus*...Paris, XCII., pp. 915-918, 962-964.

These are in addition to the list given by Freeman at the place above cited.