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Chern Characters of Fourier Modules

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Abstract. Let A_θ denote the rotation algebra—the universal C^*-algebra generated by unitaries U, V satisfying $VU = e^{2\pi i\theta}UV$, where θ is a fixed real number. Let σ denote the Fourier automorphism of A_θ defined by $U \mapsto V$, $V \mapsto U^{-1}$, and let $B_\theta = A_\theta \rtimes_\sigma \mathbb{Z}/4\mathbb{Z}$ denote the associated C^*-crossed product. It is shown that there is a canonical inclusion $\mathbb{Z}^9 \hookrightarrow K_0(B_\theta)$ for each θ given by nine canonical modules. The unbounded trace functionals of B_θ (yielding the Chern characters here) are calculated to obtain the cyclic cohomology group of order zero $\mathrm{HC}^0(B_\theta)$ when θ is irrational. The Chern characters of the nine modules—and more importantly, the Fourier module—are computed and shown to involve techniques from the theory of Jacobi's theta functions. Also derived are explicit equations connecting unbounded traces across strong Morita equivalence, which turn out to be non-commutative extensions of certain theta function equations. These results provide the basis for showing that for a dense G_δ set of values of θ one has $K_0(B_\theta) \cong \mathbb{Z}^9$ and is generated by the nine classes constructed here.

1 Introduction

Let θ be a real number and let A_θ denote the rotation C^*-algebra generated (universally) by unitaries U and V such that $VU = \lambda UV$, where $\lambda = e^{2\pi i\theta}$. Throughout, we shall write $A := A_\theta$ and identify it with its canonical smooth dense $*$-subalgebra A_θ^∞ coming from the canonical action of the 2-torus. It is closed under the holomorphic functional calculus so that modules over A are in one-to-one correspondence with those over A_θ^∞ [6].

The objective of this paper is to study the *Fourier automorphism* σ of A_θ characterized by the conditions

$$\sigma(U) = V, \quad \sigma(V) = U^{-1}$$

and its associated crossed product C^*-algebra $B = A \rtimes_\sigma \mathbb{Z}_4$ (where $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$). When θ is irrational, it is a simple C^*-algebra with a unique normalized trace and can be characterized as the unique C^*-algebra generated by unitaries U, V, W satisfying the commutation relations

$$VU = \lambda UV, \quad WUW^{-1} = V, \quad WVW^{-1} = U^{-1}, \quad W^4 = I. \tag{1.1}$$

In fact, B has the cancellation property (θ irrational), which follows from [10, Proposition 6.2]. When θ is rational, B is the universal C^*-algebra generated by unitaries satisfying (1.1). In this case, B has a canonical trace τ defined (relative to the generators U, V) by the conditions

$$\tau(1) = 1, \quad \tau(U^m V^n) = 0,$$

for $(m, n) \neq (0, 0)$.

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The Fourier automorphism is next in order of difficulty to the flip automorphism ($U \mapsto U^{-1}$, $V \mapsto V^{-1}$), which is the square of σ. The flip was the object of serious study in [1], [3], [4], [5], [14], and [15]. As was demonstrated in [3, Section 4], [4, Section 5], [14], and [15], techniques using unbounded traces of modules proved to be useful in studying the structure of automorphisms and their associated crossed products.

In Section 2 the unbounded traces on B (the Chern characters) are computed and it is shown that they form a 7-dimensional vector space. Together with the (unique) bounded trace, one obtains, when θ is irrational, the zeroth cyclic cohomology group (defined in [7, Chapter 3]):

$$\mathrm{HC}^0(A_\theta \rtimes_\sigma \mathbb{Z}_4) \cong \mathbb{C}^8.$$

(See Theorem 2.3.) Six basic projections and three modules are constructed and their Chern characters computed, via the natural pairing

$$K_0(B) \times \mathrm{HC}^0(B) \to \mathbb{C}.$$

The three modules arise from the Fourier module $\mathcal{F}$ constructed in Section 3. (The author is indebted to George Elliott for suggesting the use of the Fourier transform to construct this module from the Heisenberg modules of Connes [6] and Rieffel [12].) The computation of the Chern characters for the Fourier module is long and takes up most of the paper (see Sections 4 and 5). One of the goals of this paper lies in showing that this can be done in the module picture by classical means as well as showing how Jacobi's theta functions emerge naturally in such a connection and help in the computation of these characters. The results of this computation are summarized in the Chern character table at the end of Section 2 (see Theorem 2.4). Further, obtained are explicit equations linking the unbounded traces themselves across strong Morita equivalence (see Theorems 4.2 and 5.2); such equations generalize some of the many classical Jacobi equations relating theta functions. More specifically, one uses Rieffel's equation relating traces across strong Morita equivalence given by

$$T'(\langle f, g\rangle_{B'}) = T(\langle g, f\rangle_B), \tag{1.2}$$

(see [11, Proposition 2.2]) where T and T' are traces on the strongly Morita equivalent algebras B and $B' = \mathrm{End}_B(\mathcal{F})$, respectively, given by the equivalence B'-B-bimodule structure of the Fourier module $\mathcal{F}$. The equation (1.2) is worked out explicitly in our case and, in fact, when looked at under a microscope (using probability density functions), it becomes one of the many Jacobi identities relating theta functions. For instance, one such equation is

$$\frac{1}{4}\big(\vartheta_3(s,u) + \vartheta_4(s,u)\big) \cdot \big(\vartheta_3(t,u) + \vartheta_4(t,u)\big) \\ = \vartheta_3(2t-2s, 8u) \cdot \vartheta_3(2t+2s, 8u) + \vartheta_2(2t-2s, 8u) \cdot \vartheta_2(2t+2s, 8u)$$

(see the equation (5.6) of Section 5), where the theta functions $\vartheta_2, \vartheta_3, \vartheta_4$ are recalled briefly in the Appendix (Section 7) *. Incidently, Boca [2] has recently been able to use theta

*For a good classic treatment of theta functions see [18, Chapter XXI].

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functions to construct projections of trace θ in the fixed point subalgebra of A_θ under the Fourier automorphism.

The Chern character table (Section 2) shows that for any θ the nine classes constructed here are independent in $K_0(B)$ and so yield an injection

$$\mathbb{Z}^9 \hookrightarrow K_0(B_\theta).$$

The Fourier automorphism was studied by Farsi and Watling in [9] in the rational case $\theta = p/q$ and its associated crossed product $B_{p/q} = A_{p/q} \rtimes \mathbb{Z}_4$ (as well as the fixed point algebra) explicitly described as an algebra of continuous matrix-valued functions on the 2-sphere with 3 singular points. (That is, at such points the functions commute with certain projections.) In the same paper they showed that $K_0(B_{p/q}) \cong \mathbb{Z}^9$ and $K_1(B_{p/q}) = 0$. Using this it will also follow that the (numerical) Connes Chern character

$$\mathbf{T}\colon K_0(B_{p/q}) \to \mathbb{R} \times \mathbb{Z} \times \mathbb{C}^2 \times \mathbb{R}^3,$$

which essentially consists of all the traces put together, is injective. (See Theorem 2.4.) Using the results of the current paper, together with [16], it will turn out that the above nine modules form a concrete basis for $K_0(B_{p/q})$, and in fact one which is continuously dependent on θ (from the point of view of the continuous field of C^*-algebras $\{B_\theta\}$). In [16], the author uses the results of the current paper and [9] to show that for a dense G_δ set of the parameter θ (containing the rationals) one has $K_0(B_\theta) \cong \mathbb{Z}^9$, and is generated by the nine canonical modules given here (so, in particular, it holds for many irrationals). It is also shown in [16] that one has $K_1(B_\theta) = 0$ for a dense G_δ.

2 Chern Characters and HC^0

In this section we shall determine a basis for the vector space of all unbounded traces on the crossed product $B = A_\theta \rtimes_\sigma \mathbb{Z}_4$, and show that it is 7-dimensional. The domains of these traces will be the dense $*$-subalgebra $A_\theta^\infty \rtimes_\sigma \mathbb{Z}_4$. (Throughout, B will be identified with this smooth subalgebra.) We shall also summarize one of the main results of the paper (Theorem 2.4), the proofs of which are given in later sections.

Definition If α is an automorphism of a C^*-algebra A, then a (not necessarily continuous) linear functional ϕ defined on a dense α-invariant *-subalgebra A' of A is said to be α-*tracial* (or is an α-*trace*) iff

$$\phi(xy) = \phi\big(\alpha(y)x\big),$$

$\forall x, y \in A'$. We will simply refer to ϕ as α^*-*tracial* if it is α^k-tracial for some integer k.

Clearly, an α-tracial map is α-invariant (when A is unital). Therefore, this relation is equivalent to $\phi(xy) = \phi\big(y\alpha^{-1}(x)\big)$. (In our case, A' will be the canonical smooth dense subalgebra of the irrational rotation algebra with respect to the canonical generators U, V.)

Given a C^*-dynamical system (A, Γ, α), where Γ is a discrete commutative group (written additively) acting on A, the crossed product $A \rtimes_\alpha \Gamma$ contains canonical unitaries W_g (for $g \in \Gamma$) defined by $W_g(h) = \delta_{g,h}$ and satisfy

$$\alpha_g(a) = W_g a W_g^*, \quad W_g W_h = W_{g+h}, \quad W_g^* = W_{-g}.$$

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In the present paper $\Gamma = \mathbb{Z}_4$.

Proposition 2.1 *Let T be a trace functional defined on $\ell^1(\Gamma, A)$. For each $t \in \Gamma$ define the map φ_t on A by*

$$\varphi_t(a) = T(aW_{-t}).$$

Then each φ_t is α-invariant and satisfies the equation

$$(*) \qquad \varphi_t(ab) = \varphi_t\big(\alpha_t(b)a\big)$$

$\forall a, b \in A$. (That is, φ_t is α_t-tracial).

Conversely, if $\{\varphi_t : t \in \Gamma\}$ is a family of α-invariant maps on A such that φ_t satisfies (*), then the functional T defined on $\ell^1(\Gamma, A)$ by

$$T\Big(\sum_{t\in\Gamma} a_t W_t\Big) = \sum_{t\in\Gamma} \varphi_t(a_{-t})$$

is a trace.

Proof ($\Rightarrow$) If T is tracial, then $T(aW_{-g}bW_{-h}) = T(bW_{-h}aW_{-g})$ becomes

$$\varphi_{g+h}\big(a\alpha_{-g}(b)\big) = \varphi_{g+h}\big(b\alpha_{-h}(a)\big)$$

since Γ is commutative. Taking $a = 1$ and $h = t - g$ in this equation shows that φ_t is α-invariant for all t. So this equation can be written as

$$\varphi_{g+h}\big(\alpha_g(a)b\big) = \varphi_{g+h}\big(\alpha_h(b)a\big),$$

and taking $g = 0$ shows that φ_h is an α_h-trace.

($\Leftarrow$) Conversely, suppose $\{\varphi_t : t \in \Gamma\}$ are given with the desired property. From the definition of T, one has $T(aW_{-g}) = \varphi_g(a)$. Thus,

$$\begin{aligned} T(aW_{-g}bW_{-h}) &= T(a\alpha_{-g}(b)W_{-g-h}) \\ &= \varphi_{g+h}\big(a\alpha_{-g}(b)\big) \\ &= \varphi_{g+h}\big(\alpha_{-h}(a)\alpha_{-g-h}(b)\big) \\ &= \varphi_{g+h}\big(b\alpha_{-h}(a)\big) \\ &= T(bW_{-h}aW_{-g}) \end{aligned}$$

which completes the proof. ■

Consequently, if $t_0 \in \Gamma$ is fixed and ϕ is α_{t_0}-tracial on A, then by letting

$$\varphi_t = \begin{cases} \phi & \text{if } t = -t_0 \\ 0 & \text{otherwise} \end{cases}$$

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then the family $\{\varphi_t : t \in \Gamma\}$ satisfies the conditions of Proposition 2.1 so that ϕ induces the trace functional T on $\ell^1(\Gamma, A)$ defined by

$$T\Big(\sum_{t\in\Gamma} a_t W_t\Big) = \phi(a_{-t_0}).$$

For convenience, write $\Lambda(n) = \lambda^n$.

Now fix a matrix $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z})$ which implements a natural automorphism of $A = A_\theta^\infty$. Thus, $\alpha(U) = U^a V^b$, $\alpha(V) = U^c V^d$. Let ϕ be α-tracial on A. Then

$$\phi(U^m V^n U^p V^q) = \phi\big(\alpha(U^p V^q) U^m V^n\big), \tag{2.1}$$

and using

$$\alpha(U^p V^q) = \Lambda\big(abp(p-1)/2 + cdq(q-1)/2 + bcpq\big) U^{ap+cq} V^{bp+dq}$$

equation (2.1) becomes

$$\begin{aligned}\Lambda(np)\phi(U^{m+p}V^{n+q}) &= \Lambda\big(abp(p-1)/2 + cdq(q-1)/2 + bcpq\big) \cdot \phi(U^{ap+cq}V^{bp+dq}U^m V^n) \\ &= \Lambda\big(abp(p-1)/2 + cdq(q-1)/2 + bcpq + mbp + mdq\big) \\ &\quad \cdot \phi(U^{ap+cq+m}V^{bp+dq+n})\end{aligned}$$

and so we have

$$\begin{aligned}\phi(U^{m+p}V^{n+q}) &= \Lambda\big(abp(p-1)/2 + cdq(q-1)/2 + bcpq + mbp + mdq - np\big) \\ &\quad \cdot \phi(U^{ap+cq+m}V^{bp+dq+n}).\end{aligned} \tag{2.2}$$

Upon replacing p by $p-m$ and q by $q-n$ equation (2.2) becomes

$$\begin{aligned}\phi(U^p V^q) = \Lambda\big(&ab(p-m)(p-m-1)/2 + cd(q-n)(q-n-1)/2 \\ &+ bc(p-m)(q-n) + mb(p-m) + md(q-n) - n(p-m)\big) \\ &\cdot \phi(U^{a(p-m)+c(q-n)+m}V^{b(p-m)+d(q-n)+n}).\end{aligned} \tag{2.3}$$

Now using the notation

$$G(m,n) = U^m V^n$$

which satisfy the relations

$$G(m,n)G(p,q) = \Lambda(np)G(m+p, n+q),$$

equation (2.3) can be re-written as

(2.4)

$$\begin{aligned}\phi\big(G(p,q)\big) = \Lambda\big(&ab(p-m)(p-m-1)/2 + cd(q-n)(q-n-1)/2 + bc(p-m)(q-n) \\ &+ mb(p-m) + md(q-n) - n(p-m)\big) \cdot \phi\Big(G\big((p,q)\alpha + (m,n)(I-\alpha)\big)\Big)\end{aligned}$$

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Hence for a given pair $(p,q) \in \mathbb{Z}^2$, the other pairs that depend on it are exactly $(p,q)\alpha + (m,n)(I-\alpha)$ for variable integers m, n. In fact, the following defines an equivalence relation on $\mathbb{Z}^2$: $(p,q) \sim (p',q')$ iff

$$(p',q') = (p,q)\alpha + (m,n)(I-\alpha),$$

for some $(m,n) \in \mathbb{Z}^2$. In fact, this is equivalent to saying that $(p',q') - (p,q) \in \mathrm{Range}(I-\alpha)$.

The σ-Traces

Now let us apply this to the Fourier automorphism $\alpha = \sigma = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ so that $I - \sigma = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$. Thus a given (p,q) is equivalent to $(p,q)\sigma + (m,n)(I-\sigma) = (-q+m+n, p-m+n)$. Now we take $n = q - m$ and $m = [(p+q)/2]$ (the greatest integer function) so that $p + q - 2m = m' = 0, 1$. Hence (p,q) is equivalent to $(0,0)$ or $(0,1)$, depending on whether the parity of $p+q$ is even or odd, respectively. This implies that a $\phi\big(G(p,q)\big)$ depends on $\phi\big(G(0,m')\big) = \phi(V^{m'})$ and that equation (2.4) becomes

$$\begin{aligned} \phi(U^pV^q) &= \Lambda\big(-(p-m)(q-n) + m(p-m) - n(p-m)\big) \cdot \phi(V^{m'}) \\ &= \Lambda\big(-(p-m)m + (m-n)(p-m)\big) \cdot \phi(V^{m'}) \\ &= \Lambda\big(-(q-m)(p-m)\big) \cdot \phi\big(V^{p+q-2[(p+q)/2]}\big) \end{aligned} \tag{2.5}$$

where $m = [(p+q)/2]$. Thus we have two independent σ-trace functionals, ψ_{10} and ψ_{11}, normalized as follows

$$\psi_{10}(1) = 1, \quad \psi_{10}(V) = 0, \quad \psi_{11}(1) = 0, \quad \psi_{11}(V) = 1.$$

Using equation (2.5) one can obtain full equations for these functionals. So if $x = \sum x_{pq}U^pV^q$ is a smooth vector, then

$$\psi_{10}(x) = \sum_{p,q} x_{pq}\Lambda\big(-(q-m)(p-m)\big) \cdot \psi_{10}\big(V^{p+q-2[(p+q)/2]}\big)$$

(here, $m = [(p+q)/2]$), which becomes

$$\psi_{10}(x) = \sum_{p,q} \lambda^{(p-q)^2}(x_{2p,2q} + x_{2p-1,2q-1}). \tag{2.6}$$

Proof Since $\psi_{10}(V) = 0$, (2.5) will be non-zero only when $p+q$ is even, in which case $m = [(p+q)/2] = (p+q)/2$ so (2.5) becomes

$$\psi_{10}(U^pV^q) = \Lambda\big(-(q-m)(p-m)\big) = \lambda^{(p-q)^2/4}.$$

Thus if $x = \sum x_{pq}U^pV^q$ is a smooth vector, then

$$\psi_{10}(x) = \sum_{p+q \text{ even}} x_{pq}\lambda^{(p-q)^2/4}$$

and since $p + q$ is even we split the sum according to when p, q are both even or both odd

$$= \sum_{p,q} x_{2p,2q} \lambda^{(2q-2p)^2/4} + \sum_{p,q} x_{2p-1,2q-1} \lambda^{(2q-2p)^2/4}$$
$$= \sum_{p,q} \lambda^{(q-p)^2} (x_{2p,2q} + x_{2p-1,2q-1})$$

which gives (2.6). ■

Similarly, for ψ_{11} one gets

(2.7) $$\psi_{11}(x) = \sum_{p,q} \lambda^{(q-p)(q-p+1)} (x_{2p-1,2q} + x_{2p-2,2q-1}).$$

At this point observe the following connection between ψ_{10} and ψ_{11}. Let γ denote the automorphism of A given by

$$\gamma(U) = \lambda^{1/2} U, \quad \gamma(V) = \lambda^{-1/2} V.$$

One can easily check the following equation

$$\psi_{11}(x) = \psi_{10}\big(\gamma(xV)\big),$$

using the relations

$$\psi_{10}(U^m V^n) = \lambda^{(m-n)^2/4} \delta_{\overline{m},\overline{n}}, \quad \psi_{11}(U^m V^n) = \lambda^{[(m-n)^2-1]/4} \delta_{\overline{m},\overline{n+1}},$$

where $\overline{k}$ is k reduced modulo 2 (so $\overline{k} = 0, 1$). One also has

$$\gamma \circ \sigma = \mathrm{Ad}(U) \circ \sigma \circ \gamma.$$

The σ^2-Traces

Since σ^2 is the flip automorphism, the σ^2-traces are four in number and have already been found in [3] and [14]. However, two of them are σ-invariant and the other two can be added together to get one σ-invariant trace. So we shall end up with three σ-invariant σ^2-traces. From [14], they are given by

$$\phi_{ij}(x) = \sum_{m,n} \lambda^{-(2m-i)(2n-j)/2} x_{2m-i,2n-j},$$

where $x = \sum x_{pq} U^p V^q$ and $i, j = 0, 1$. Or, for the generic vectors $U^m V^n$, one has

$$\phi_{ij}(U^m V^n) = \lambda^{-mn/2} \delta_{i,\overline{m}} \delta_{j,\overline{n}}.$$

It is easy to verify that ϕ_{00}, ϕ_{11}, and $\phi_{01} + \phi_{10}$ are σ-invariant σ^2-traces on A. So, adopt the following notation for consistency:

$$\psi_{20} = \phi_{00}, \quad \psi_{21} = \phi_{11}, \quad \psi_{22} = \phi_{01} + \phi_{10}.$$

The σ^3-Traces

The σ^3-traces turn out not to yield new K-theoretical data which the σ-traces don't already provide. This is because there is a one-to-one correspondence between the two given as follows. Given a σ-trace ϕ, the map

$$\tilde{\phi}(x) = \overline{\phi(x^*)}$$

defines a σ^3-trace. (And conversely.) To see this,

$$\tilde{\phi}(xy) = \overline{\phi(y^*x^*)} = \overline{\phi\big(\sigma(x^*)y^*\big)} = \overline{\phi\big(x^*\sigma^3(y^*)\big)} = \overline{\phi\Big(\big(\sigma^3(y)x\big)^*\Big)} = \tilde{\phi}\big(\sigma^3(y)x\big).$$

Therefore, using the equations for the σ-traces, one easily obtains equations for the σ^3-traces and can summarize the results as follows. In fact, using the definitions of ψ_{30} and ψ_{31} given in the following proposition, one easily verifies that $\psi_{30}(x) = \overline{\psi_{10}(x^*)}$ and $\psi_{31}(x) = \overline{\psi_{11}(x^*)}$.

Proposition 2.2 *One has the following 7-dimensional basis of the vector space of all unbounded traces on the fixed point subalgebra A^σ. More specifically, $\{\psi_{10}, \psi_{11}\}$ is a basis of σ-traces on A, $\{\psi_{20}, \psi_{21}, \psi_{22}\}$ a basis of σ^2-traces, and $\{\psi_{30}, \psi_{31}\}$ a basis of σ^3-traces, and are given by*

$$\begin{cases}
\psi_{10}(x) = \sum_{p,q} \lambda^{(p-q)^2}(x_{2p,2q} + x_{2p-1,2q-1}) \\
\psi_{11}(x) = \sum_{p,q} \lambda^{(q-p)(q-p+1)}(x_{2p-1,2q} + x_{2p-2,2q-1}) \\
\psi_{20}(x) = \sum_{m,n} \lambda^{-2mn} x_{2m,2n} \\
\psi_{21}(x) = \sum_{m,n} \lambda^{-(2m-1)(2n-1)/2} x_{2m-1,2n-1} \\
\psi_{22}(x) = \sum_{m,n} \lambda^{-m(2n-1)} x_{2m,2n-1} + \lambda^{-(2m-1)n} x_{2m-1,2n} \\
\psi_{30}(x) = \sum_{p,q} \lambda^{-(p+q)^2}(x_{2p,2q} + x_{2p+1,2q-1}) \\
\psi_{31}(x) = \sum_{p,q} \lambda^{-(p+q)(p+q-1)}(x_{2p-1,2q} + x_{2p,2q-1})
\end{cases}$$

where $x = \sum x_{pq} U^p V^q$ is in A.

One also has the relations

$$\begin{gathered}
\psi_{10}(U^mV^n) = \lambda^{(m-n)^2/4}\delta_{\overline{m},\overline{n}} \quad \psi_{11}(U^mV^n) = \lambda^{[(m-n)^2-1]/4}\delta_{\overline{m},\overline{n+1}} \\
\psi_{20}(U^mV^n) = \lambda^{-mn/2}\delta_{\overline{m},0}\delta_{\overline{n},0} \quad \psi_{21}(U^mV^n) = \lambda^{-mn/2}\delta_{\overline{m},1}\delta_{\overline{n},1} \\
\psi_{22}(U^mV^n) = \lambda^{-mn/2}\delta_{\overline{m},\overline{n+1}} \\
\psi_{30}(U^mV^n) = \lambda^{-(m+n)^2/4}\delta_{\overline{m},\overline{n}} \quad \psi_{31}(U^mV^n) = \lambda^{-[(m+n)^2-1]/4}\delta_{\overline{m},\overline{n+1}}.
\end{gathered} \tag{2.8}$$

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The unbounded traces T_{ij} on $B = A \rtimes_\sigma \mathbb{Z}_4$ are now given by

$$T_{ij}(a_0 + a_1 W + a_2 W^2 + a_3 W^3) = \psi_{ij}(a_{4-i}), \tag{2.9}$$

for $i = 1, 2, 3$ and j goes from 0 to n_i, where $n_1 = 1$, $n_2 = 2$, $n_3 = 1$. Therefore, using Proposition 2.2, one obtains all the traces on B giving its cyclic cohomology group of order zero.

Theorem 2.3 *For any irrational θ, one has the cyclic cohomology group of order zero*

$$\mathrm{HC}^0(A_\theta \rtimes_\sigma \mathbb{Z}_4) \cong \mathrm{HC}^0(A_\theta^\sigma) \cong \mathbb{C}^8.$$

The group on the left is generated by τ and T_{ij}, while the middle group is generated by τ and ψ_{ij} (restricted to A_θ^σ), where τ is the canonical bounded trace in each case.

Bracket Notation

It will ease notation and computation to write the ψ_{jk}'s in terms of what we shall call "bracket" functionals defined as follows. For fixed arbitrary integers ν, r, t, and real a, c, let $[\nu, a, c; r, t]$ denote the linear functional defined by

$$[\nu, a, c; r, t](x) = \sum_{p,q} \lambda^{\nu(p^2+q^2)-2pq+ap+cq} x_{2p-r,2q-t}$$

where $x = \sum_{p,q} x_{p,q} U^p V^q$ is a smooth vector. Here we adopt the convention that for real s

$$\lambda^s := \mathrm{e}^{2\pi i \theta s}.$$

The seven σ^*-traces in Proposition 2.2 can be written in terms of these functionals as follows:

$$\psi_{10} = [1, 0, 0; 0, 0] + [1, 0, 0; 1, 1]$$
$$\psi_{11} = [1, -1, 1; 1, 0] + [1, -1, 1; 2, 1]$$
$$\psi_{20} = [0, 0, 0; 0, 0]$$
$$\psi_{21} = \lambda^{-1/2}[0, 1, 1; 1, 1]$$
$$\psi_{22} = [0, 1, 0; 0, 1] + [0, 0, 1; 1, 0]$$
$$\psi_{30} = [-1, 0, 0; 0, 0] + [-1, 0, 0; -1, 1]$$
$$\psi_{31} = [-1, 1, 1; 1, 0] + [-1, 1, 1; 0, 1].$$

Basic Projections

If S is any unitary of order four, consider its associated projections

$$p_1(S) = \frac{1}{2}(1 + S^2)$$
$$p_2(S) = \frac{1}{2} + \Big(\frac{1+i}{4}\Big)S + \Big(\frac{1-i}{4}\Big)S^3$$
$$p_3(S) = \frac{1}{4}(1 + S + S^2 + S^3).$$

In our case, it is easy to see that we have the order four unitaries

$$S_{mn} = \lambda^{(m+n)^2/4}U^m V^n W.$$

However, most of these are unitarily equivalent to each other (by a unitary of the form $U^p V^q W^r$). In fact one can show that S_{mn} is unitarily equivalent to either $S_{00} = W$ or to $S_{10} = \lambda^{1/4}UW$. Thus, one gets the following six basic projections

$$p_1(W) = \frac{1}{2}(1 + W^2)$$
$$p_2(W) = \frac{1}{2} + \Big(\frac{1+i}{4}\Big)W + \Big(\frac{1-i}{4}\Big)W^3$$
$$p_3(W) = \frac{1}{4}(1 + W + W^2 + W^3)$$
$$p_1(\lambda^{1/4}UW) = \frac{1}{2}(1 + \lambda^{1/2}UVW^2)$$
$$p_2(\lambda^{1/4}UW) = \frac{1}{2} + \Big(\frac{1+i}{4}\Big)\lambda^{1/4}UW + \Big(\frac{1-i}{4}\Big)\lambda^{-1/4}VW^3$$
$$p_3(\lambda^{1/4}UW) = \frac{1}{4}(1 + \lambda^{1/4}UW + \lambda^{1/2}UVW^2 + \lambda^{-1/4}VW^3).$$

By computing their Chern characters it will follow (see table below) that these projections yield six independent classes in $K_0(B)$.

Second Order Chern Character

Now consider the unital *-embedding

$$\Psi\colon B_\theta \to M_4(A_\theta)$$

given by

$$\Psi(a_0 + a_1W + a_2W^2 + a_3W^3) = [\sigma^{-i}(a_{i-j})]_{i,j=0}^3 = \begin{bmatrix} a_0 & a_3 & a_2 & a_1 \\ \sigma^3(a_1) & \sigma^3(a_0) & \sigma^3(a_3) & \sigma^3(a_2) \\ \sigma^2(a_2) & \sigma^2(a_1) & \sigma^2(a_0) & \sigma^2(a_3) \\ \sigma(a_3) & \sigma(a_2) & \sigma(a_1) & \sigma(a_0) \end{bmatrix}$$

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where $i - j$ is reduced mod 4 and where $a_j \in A_\theta$. This shows that the range of the canonical (normalized) trace on $K_0(B_\theta)$ is contained in $\frac{1}{4}(\mathbb{Z} + \mathbb{Z}\theta)$. The Fourier module $\mathcal{F}_\theta$ has trace $\frac{\theta}{4}$, by Proposition 3.3 below, and since $p_3(W)$ has trace $\frac{1}{4}$ one obtains the equality

$$\tau_*\big(K_0(B_\theta)\big) = \tfrac{1}{4}(\mathbb{Z} + \mathbb{Z}\theta).$$

The embedding Ψ induces the map

$$\Psi_* \colon K_0(B_\theta) \to K_0(A_\theta)$$

such that if e is a projection in a matrix algebra over B_θ, then one takes the class of $\Psi(e)$ in $K_0(A_\theta)$. So, for example, the identity 1 of B_θ goes to the 4×4 identity matrix in $M_4(A_\theta)$, so that $\Psi_*[1] = 4[1]'$ in $K_0(A_\theta)$. (For the sake of clarity, we shall write $[e]'$ for classes in $K_0(A_\theta)$ and unprimed brackets $[e]$ for classes in $K_0(B_\theta)$.) Thus, it is not hard to obtain

$$(\dagger) \qquad \begin{aligned} \Psi_*[p_1(W)] &= \Psi_*[p_1(\lambda^{1/4}UW)] = 2[1]' \\ \Psi_*[p_2(W)] &= \Psi_*[p_2(\lambda^{1/4}UW)] = 2[1]' \\ \Psi_*[p_3(W)] &= \Psi_*[p_3(\lambda^{1/4}UW)] = [1]'. \end{aligned}$$

(These are clear for $\Psi\big(p_j(W)\big)$ since it is a scalar 4×4 matrix and one just looks at its rank; for the three other projections one observes that the order four unitary $W' = \lambda^{1/4}UW$ together with $U' = \lambda^{1/2}U$, $V' = \lambda^{-1/2}V$ satisfy the same commutation relations (1.1) so, under a suitable automorphism of B_θ, one gets the same result.)

Now recall Connes' canonical cyclic 2-cocycle φ which gives a non zero class in the cyclic cohomology group $HC^2(A_\theta)$ (for the smooth rotation algebra)

$$\varphi(x^0, x^1, x^2) = \frac{1}{2\pi i}\tau\Big(x^0\big(\delta_1(x^1)\delta_2(x^2) - \delta_2(x^1)\delta_1(x^2)\big)\Big)$$

(see [7, III.2.β]) where δ_i are the canonical derivations of A_θ under the canonical action of $\mathbb{T}^2$. It induces the canonical (second order) Chern character map

$$c_1 \colon K_0(A_\theta) \to \mathbb{Z}$$

given, using the cup product, as follows: if E is a projection in $M_n(A_\theta)$ then

$$c_1(E) = (\varphi\sharp\,\mathrm{Tr})(E, E, E)$$

where Tr is the usual (nonnormalized) trace on $M_n(\mathbb{C})$ and $\varphi\sharp\,\mathrm{Tr}$ is the unique cyclic cocycle such that

$$(\varphi\sharp\,\mathrm{Tr})(x^0 \otimes a^0, x^1 \otimes a^1, x^2 \otimes a^2) = \varphi(x^0, x^1, x^2) \cdot \mathrm{Tr}(a^0a^1a^2)$$

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where $a^j \in M_n(\mathbb{C})$ and $x^j \in A_\theta$. If e is a projection in A_θ, then

$$c_1[e] = \varphi(e,e,e) = \tau\big(e[\delta_1(e), \delta_2(e)]\big).$$

As Connes showed in [6, p. 601], c_1 has the property that if $[e]' = m[1]' + n[e_\theta]'$ (in $K_0(A_\theta)$) for some integers m, n, then $c_1[e] = -n$. (The power of this group homomorphism is that it picks out the "label" of a projection in an invariant way.) Although Connes assumed that $\theta \in (0,1)$ is irrational, this is not necessary so long as one uses the canonical trace on A_θ. (Note that the computation in [6, p. 601] for $c_1(e_\theta)$ is off by a negative sign, a fact pointed out by Elliott and which is used here.) The invariant that is of interest for our purposes is the composition

$$C_1 := c_1 \circ \Psi_* \colon K_0(B_\theta) \to \mathbb{Z}.$$

For θ in $(0,1)$, the map C_1 has the property that if $[e] \in K_0(B_\theta)$ is such that $\Psi_*[e] = m[1]' + n[e_\theta]'$ (in $K_0(A_\theta)$), so that its trace in B_θ is $\frac{1}{4}(m + n\theta)$, then $C_1[e] = -n$. This follows immediately from the above since

$$C_1[e] = c_1(\Psi_*[e]) = c_1(m[1]' + n[e_\theta]') = -n.$$

(In other words, one can write the trace of a projection as $\tau(e) = \frac{1}{4}(m - C_1[e]\theta)$.)

Therefore, the values of C_1 on the nine classes are as follows. For $j = 1, 2, 3$ and $S = W$, $\lambda^{1/4}UW$ one has

$$C_1[p_j(S)] = 0,$$

which follows from (†). For the Fourier module $\mathcal{F}_\theta$ constructed in section 3 (and whose trace is $\frac{\theta}{4}$ by Proposition 3.3 below) one has

$$C_1[\mathcal{F}_\theta] = -1.$$

This clearly follows for θ irrational by the above property. For the rational case it can be shown to follow from the construction of the Fourier module $\mathcal{F}_\theta$ since, considered as an A_θ-module, it is a Heisenberg module whose c_1-character value is -1, as can be seen from Connes' computation [6; Theorem 7 and the following sentence].

All the traces and Chern character invariants can be put together to form the Connes Chern character

$$\mathbf{T} = (\tau, C_1; T_{10}, T_{11}; T_{20}, T_{21}, T_{22})$$

where τ is the canonical (bounded) trace, C_1 the canonical second order Chern character, and T_{jk} the unbounded traces. It defines a group homomorphism $K_0(B) \to \mathbb{R} \times \mathbb{Z} \times \mathbb{C}^2 \times \mathbb{R}^3$. (Note that unlike T_{1k}, the traces T_{2k} are real on self-adjoint elements.)

The results for the Chern character values are as follows:

Chern Character Table

Projection	τ	C_1	T_{10}	$\lambda^{1/4}T_{11}$	T_{20}	T_{21}	T_{22}
$p_1(W)$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	0
$p_2(W)$	$\frac{1}{2}$	0	$\frac{1-i}{4}$	0	0	0	0
$p_3(W)$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0
$p_1(\lambda^{1/4}UW)$	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$	0
$p_2(\lambda^{1/4}UW)$	$\frac{1}{2}$	0	0	$\frac{1-i}{4}$	0	0	0
$p_3(\lambda^{1/4}UW)$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0
$\mathcal{F}$	$\frac{\theta}{4}$	-1	$\frac{1+i}{8}$	$\frac{1+i}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
$\mathcal{F}(i)$	$\frac{\theta}{4}$	-1	$\frac{i-1}{8}$	$\frac{i-1}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$
$\mathcal{F}(-1)$	$\frac{\theta}{4}$	-1	$-\frac{1+i}{8}$	$-\frac{1+i}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

The values in this table for the first six projections are easily verified using (2.8) and (2.9). (Their C_1 values having just been found above.) The last three rows give the values for the Fourier module $\mathcal{F} := \mathcal{F}_\theta$ and its transforms, $\mathcal{F}(i)$ and $\mathcal{F}(-1)$, under the dual automorphism $\hat{\sigma}$ (which fixes U and V and sends W to iW); its unbounded traces are computed in Sections 4 and 5. The fact that C_1 of $\mathcal{F}, \mathcal{F}(i)$, and $\mathcal{F}(-1)$ are all equal holds since they are all related by the automorphism $\hat{\sigma}$ which is the identity on A_θ and hence induces the identity on $K_0(A_\theta)$.

Remark 1 Given a B-module E, the dual automorphism $\hat{\sigma}$ induces another B-module structure on E where the new action of W on a vector ξ is given by $i\xi W$. This new module is here denoted by $E(i)$ (as in the above table). It is not hard to see that

$$T_{1k}\big([E(i)]\big) = iT_{1k}([E]), \quad \text{and} \quad T_{2k}\big([E(i)]\big) = -T_{2k}([E])$$

which explains how the T_{ij} values of the modules $\mathcal{F}(i)$ and $\mathcal{F}(-1)$ in the table were obtained from those of $\mathcal{F}$.

It will be shown in this paper that this table is valid for all real $\theta \in (0, 1]$. It is now straightforward to check that the above nine vectors are independent over the integers (for any θ), thus showing that there is an injection

$$\mathbb{Z}^9 \to K_0(B_\theta).$$

In [16] it will be shown that the nine modules form a basis for $K_0(B_\theta)$ in the rational case, an important fact used in showing that they form a basis for $K_0(B_\theta)$ for a dense G_δ set of the parameter θ. Therefore, it will follow that the above injection is actually an isomorphism

$$K_0(B_\theta) \cong \mathbb{Z}^9$$

for at least many irrationals.

The main result can now be stated as follows.

Theorem 2.4 *For any θ in $(0,1]$, there exists an injection*

$$\mathbb{Z}^9 \to K_0(A_\theta \rtimes_\sigma \mathbb{Z}_4)$$

implemented by the nine classes given in the table. When $\theta = p/q$ is rational the Connes Chern character

$$\mathbf{T}\colon K_0(B_{p/q}) \to \mathbb{R} \times \mathbb{Z} \times \mathbb{C}^2 \times \mathbb{R}^3$$

is injective.

The injectivity in the rational case follows from the fact that the nine classes (and their $\mathbf{T}$-images) are independent and $K_0(B_{p/q}) \cong \mathbb{Z}^9$ (see [9]).

Remark 2 The first six projections in the table are clearly given by continuous projection-valued sections of the field of C^*-algebras $\{B_\theta : \theta \in [0,1]\}$. The Fourier modules $\mathcal{F}_\theta$ (and the other two), as a function of the parameter θ, can also be shown to be part of a continuous section of the field (or of a matrix algebra over the field) and this is shown in [16]. More specifically, if Γ is the C^*-algebra of continuous sections of the field, then there is positive class $\xi \in K_0(\Gamma)$ such that $\mathcal{F}_\theta$ is its image under the evaluation map

$$ev_{\theta*}\colon K_0(\Gamma) \to K_0(B_\theta).$$

3 The Fourier Module

The Heisenberg Module

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing smooth functions on the reals. The Fourier transform of a function $\xi \in \mathcal{S}(\mathbb{R})$ is defined by

$$\hat{\xi}(t) = \int_{-\infty}^{\infty} \xi(s)\mathrm{e}(-st)\,ds$$

where for convenience we shall write $\mathrm{e}(t) := \mathrm{e}^{2\pi i t}$. It maps $\mathcal{S}(\mathbb{R})$ bijectively onto itself [13, Theorem 7.7] and defines a unitary operator on $L^2(\mathbb{R})$. We will use the Fourier transform to turn the Heisenberg right A_θ-module $\mathcal{S}(\mathbb{R})$ (constructed by Connes [6] and Rieffel [12]) into a right B-module, where $B = A_\theta \rtimes_\sigma \mathbb{Z}_4$. The Heisenberg module is defined for any $\theta \in (0,1]$ and is in fact an equivalence bimodule under suitable inner products [12, Section 1]. This will be recalled below in detail.

First, recall that the right actions of the unitaries U, V (generating A_θ) can be represented by

$$(\xi V)(t) = \mathrm{e}(t)\xi(t), \quad (\xi U)(t) = \xi(t+\theta),$$

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where $\xi \in \mathcal{S}(\mathbb{R})$. They satisfy $VU = \lambda UV$, where $\lambda = \mathrm{e}(\theta)$. From [12, p. 289], recall that $C_A := \mathrm{End}_A\big(\mathcal{S}(\mathbb{R})\big) \cong A_\beta$, where $\beta = \theta^{-1}$. The algebra A_β is generated by unitaries U_1, V_1 such that $V_1U_1 = \mu U_1V_1$ where $\mu = \mathrm{e}(\beta)$, and can be realized as operators on $\mathcal{S}(\mathbb{R})$ by the left actions

$$V_1(\xi)(t) = \mathrm{e}(-t/\theta)\xi(t), \quad U_1(\xi)(t) = \xi(t+1).$$

The module $\mathcal{S}(\mathbb{R})$ becomes an equivalence C_A-A bimodule [12, Theorem 1.1] with C_A-valued and A-valued inner products given by

$$\langle \xi, \eta \rangle_{A} = \sum_{m,n} \langle \xi, \eta \rangle_{A}(m,n) \cdot U^m V^n,$$

where

$$\langle \xi, \eta \rangle_{A}(m,n) = \theta \int_{-\infty}^{\infty} \overline{\xi(t+m\theta)}\eta(t)\mathrm{e}(-nt)\,dt,$$

and

$$\langle \xi, \eta \rangle_{C_A} = \sum_{m,n} \langle \xi, \eta \rangle_{C_A}(m,n) \cdot U_1^m V_1^n,$$

where

$$\langle \xi, \eta \rangle_{C_A}(m,n) = \int_{-\infty}^{\infty} \xi(t-m)\overline{\eta(t)}\mathrm{e}(nt/\theta)\,dt.$$

The last of these follows from the equation $\tau(\langle \xi, \eta \rangle_{C_A}) = \langle \xi, \eta \rangle_{L^2(\mathbb{R})}$ [12, Lemma 1.6] where τ is the normalized trace on $C_A = A_\beta$, and the first inner product follows from Lemma 1.5 of [12]. Below, this bimodule is extended to an equivalence A_β^σ-B bimodule.

The Fourier Module

To extend the Heisenberg bimodule, define the right action of W on $\mathcal{S}(\mathbb{R})$ by

$$(\xi W)(t) = \frac{1}{\sqrt{\theta}}\hat{\xi}(-t/\theta),$$

where $\hat{\xi}$ is the Fourier transform of ξ. It is straightforward to verify that the relations in (1.1) are satisfied. It is easily checked that $(\xi W^2)(t) = \xi(-t)$, which corresponds to the flip automorphism, and that $(\xi W^*)(t) = \frac{1}{\sqrt{\theta}}\hat{\xi}(t/\theta)$. We shall denote this right B-module by $\mathcal{F} = \mathcal{F}_\theta$. In Proposition 3.2 it is verified that it is projective and yields an equivalence bimodule.

Let $C_B = \mathrm{End}_B\big(\mathcal{S}(\mathbb{R})\big)$. One clearly has an injection $C_B \hookrightarrow C_A \cong A_\beta$. It is natural to expect that C_B turns out to be the fixed point subalgebra of A_β under the Fourier automorphism on it. In fact, this can be realized as follows.

Let us also denote by σ the Fourier automorphism of A_β, so that

$$\sigma(U_1) = V_1, \quad \sigma(V_1) = U_1^{-1}.$$

Proposition 3.1 *We have a C^*-isomorphism $C_B \cong A_\beta^\sigma$.*

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Proof A simple computation shows that if $L \in C_B$, say

$$L = \sum_{m,n} c_{m,n} U_1^m V_1^n,$$

where $\{c_{m,n}\}$ is rapidly decreasing, then the condition $L(\xi W) = L(\xi)W$ implies that the coefficients $c_{m,n}$ satisfy $c_{m,n} = c_{n,-m}\mu^{mn}$, which exactly says that L is in the fixed point subalgebra of A_β under the Fourier automorphism σ. By symmetry, the converse follows. ■

Next, define a B-valued inner product $\langle\ ,\ \rangle_{B}$ by symmetrization with respect to W,

$$\langle \xi, \eta \rangle_{B} = \sum_{i=0}^{3} \langle \xi, \eta W^{-i} \rangle_{A} W^i.$$

Also, define the C_B-valued inner product as follows. For $\xi, \eta \in \mathcal{S}(\mathbb{R})$, let $\langle \xi, \eta \rangle_{C_B}$ denote the operator on $\mathcal{S}(\mathbb{R})$ given by

$$\langle \xi, \eta \rangle_{C_B}(\zeta) = \xi \langle \eta, \zeta \rangle_{B}.$$

It can be easily expressed in terms of the C_A-valued inner product by

$$\langle \xi, \eta \rangle_{C_B} = \sum_{i=0}^{3} \sigma^i(\langle \xi, \eta \rangle_{C_A}).$$

Proposition 3.2 *The space $\mathcal{S}(\mathbb{R})$ is a projective right Hilbert B-module under the B-valued inner product $\langle\ ,\ \rangle_{B}$ and a left Hilbert C_B-module under $\langle\ ,\ \rangle_{C_B}$. This turns the Fourier module $\mathcal{F}$ into an equivalence C_B-B bimodule.*

Proof Given the exact diagram of B-modules

$$\begin{array}{ccccc} & & N & & \\ & & \downarrow{\scriptstyle\varphi} & & \\ M & \xrightarrow{\pi} & \mathcal{S}(\mathbb{R}) & \longrightarrow & 0 \end{array}$$

then, considered as an A-module diagram, the map φ lifts to an A-linear map $\varphi' \colon N \to M$ such that $\pi\varphi' = \varphi$. Then the map

$$\varphi''(x) = \frac{1}{4} \sum_i \varphi'(xW^{-i})W^i,$$

is B-linear and satisfies $\pi\varphi'' = \varphi$, as required. The rest is straightforward. ■

As in [12, p. 291] choose vectors $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R})$ (as a C_A-A-bimodule, where $C_A \cong A_\beta$) such that

$$\sum_{k=1}^{n} \langle f_k, f_k \rangle_{C_A} = 1_{C_A}.$$

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Letting $g_k = \frac{1}{2} f_k$, one gets

$$\sum_{k=1}^{n} \langle g_k, g_k \rangle_{C_B} = 1_{C_B}.$$

To see this, apply the sum to any vector $\xi \in \mathcal{S}(\mathbb{R})$:

$$\begin{aligned}
\sum_k \langle g_k, g_k \rangle_{C_B}(\xi) &= \sum_k g_k \langle g_k, \xi \rangle_B \\
&= \sum_k g_k \Big(\sum_j \langle g_k, \xi W^{-j} \rangle_A W^j \Big) \\
&= \sum_{j,k} g_k \langle g_k, \xi W^{-j} \rangle_A W^j \\
&= \sum_{j,k} \langle g_k, g_k \rangle_{C_A} (\xi W^{-j}) W^j \\
&= \frac{1}{4} \sum_j \Big(\sum_k \langle f_k, f_k \rangle_{C_A} \Big) (\xi W^{-j}) W^j \\
&= \frac{1}{4} \sum_j (\xi W^{-j}) W^j \\
&= \xi.
\end{aligned}$$

This shows that the B-module $\mathcal{F}$ is represented by the $n \times n$ matrix projection

$$e = [\langle g_j, g_k \rangle_B] \in M_n(B).$$

Proposition 3.3 *For the Fourier module $\mathcal{F}$ one has*

$$(\tau_B)_*[\mathcal{F}] = \frac{1}{4}\theta,$$

where $[\mathcal{F}] \in K_0(B)$ and τ_B is the canonical trace of B.

Proof Follows easily as in the proof in [12, p. 291]. ■

As this takes care of the (bounded) trace of the Fourier module, we now turn our attention to finding its unbounded traces and prepare a program for doing so. This is done in the remainder of this section.

Remark A close examination of the proofs of Propositions 2.1 and 2.2 of [11] shows that they still hold if the C^*-algebras there are replaced by dense *-subalgebras closed under the holomorphic functional calculus (and containing the identities of the C^*-algebras). (All the unbounded traces here are finite on these dense subalgebras.) Modules over such smooth

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subalgebras are in one-to-one correspondence with the modules over the underlying C^*-algebra (in view of Lemma 1 of [6]). This smooth version of Rieffel's result (particularly his Proposition 2.2) will be used freely below. For convenience, we quote his result as follows.

Proposition 3.4 **([11, Proposition 2.2])** *Let C and D be C^*-algebras with identity elements, and let X be a C-D-equivalence bimodule. Then there is a bijection between the (non-normalized) finite traces on C and those on D, under which to a trace τ on C there is associated a trace τ' on D such that*

$$\tau'(\langle x, y\rangle_D) = \tau(\langle y, x\rangle_C)$$

for all $x, y \in X$.

In the next result we show that σ^*-traces on A are related to those on A_β in essentially the same way as ordinary traces are related by Proposition 3.4.

Proposition 3.5 *Fix r. For each σ^r-trace ϕ on A $(= A_\theta)$, there is a uniquely associated σ^{4-r}-trace ϕ' on C_A $(\cong A_\beta)$ such that*

$$\phi'(\langle f, g\rangle_{C_A}) = \phi(\langle g, fW^r\rangle_A).$$

Proof The map ϕ naturally defines a trace map $\tilde{\phi}$ on B by setting

$$\tilde{\phi}\Big(\sum a_j W^j\Big) = \phi(a_{4-r}).$$

By Proposition 3.4 there is a unique trace $\tilde{\phi}'$ on C_B such that

$$\tilde{\phi}'(\langle f, g\rangle_{C_B}) = \tilde{\phi}(\langle g, f\rangle_B) = \phi(\langle g, fW^{-(4-r)}\rangle_A).$$

Now we can extend $\tilde{\phi}'$ to a map ϕ' on A_β simply by symmetrizing,

$$\phi'(x) = \tilde{\phi}'\Big(\sum_{j=0}^{3} \sigma^j(x)\Big),$$

where here "σ" is the Fourier automorphism of A_β. It is clear that ϕ' is σ-invariant and

$$\phi'(\langle f, g\rangle_{C_A}) = \phi(\langle g, fW^{-(4-r)}\rangle_A) = \phi(\langle g, fW^r\rangle_A).$$

To see this, one has

$$\phi'(\langle f, g\rangle_{C_A}) = \tilde{\phi}'\Big(\sum_{j=0}^{3} \sigma^j(\langle f, g\rangle_{C_A})\Big) = \tilde{\phi}'(\langle f, g\rangle_{C_B}) = \tilde{\phi}(\langle g, f\rangle_B) = \phi(\langle g, fW^{-(4-r)}\rangle_A).$$

To show that ϕ' is σ^{4-r}-tracial, *i.e.*

$$\phi'(ab) = \phi'\big(\sigma^{4-r}(b)a\big),$$

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it suffices to assume that a and b have the form $a = \langle f, g\rangle_{C_A}$ and $b = \langle h, k\rangle_{C_A}$, since these elements span C_A. We have

$$\begin{aligned}
\phi'(\langle f,g\rangle_{C_A} \cdot \langle h,k\rangle_{C_A}) &= \phi'\big(\langle\langle f,g\rangle_{C_A}(h),k\rangle_{C_A}\big)\\
&= \phi'\big(\langle f\langle g,h\rangle_A,k\rangle_{C_A}\big)\\
&= \phi\big(\langle k, f\langle g,h\rangle_A W^r\rangle_A\big)\\
&= \phi\big(\langle k, fW^r\sigma^{-r}(\langle g,h\rangle_A)\rangle_A\big)\\
&= \phi\big(\langle k, fW^r\rangle_A \cdot \sigma^{-r}(\langle g,h\rangle_A)\big)\\
&= \phi\big(\sigma^r(\langle k, fW^r\rangle_A)\cdot\langle g,h\rangle_A\big)\\
&= \phi(\langle g,h\rangle_A \cdot \langle k, fW^r\rangle_A)\\
&= \phi\big(\langle g, h\langle k, fW^r\rangle_A\rangle_A\big)\\
&= \phi'\big(\langle h\langle k, fW^r\rangle_A W^{-r}, g\rangle_{C_A}\big)\\
&= \phi'\big(\langle hW^{-r}\sigma^r(\langle k, fW^r\rangle_A), g\rangle_{C_A}\big)\\
&= \phi'\big(\langle hW^{-r}\langle kW^{-r}, f\rangle_A, g\rangle_{C_A}\big)\\
&= \phi'\big(\langle\langle hW^{-r}, kW^{-r}\rangle_{C_A}(f), g\rangle_{C_A}\big)\\
&= \phi'(\langle hW^{-r}, kW^{-r}\rangle_{C_A}\cdot\langle f,g\rangle_{C_A})\\
&= \phi'\big(\sigma^{4-r}(\langle h,k\rangle_{C_A})\cdot\langle f,g\rangle_{C_A}\big) \qquad \blacksquare
\end{aligned}$$

Our next objective is to compute the unbounded trace functionals T_{ij} on $\mathcal{F}$. Recall that these functionals are defined on B by (2.9). Proposition 3.5 implements a unique σ^{4-i}-trace ψ'_{ij} on A_β such that

$$\psi_{ij}(\langle g, fW^i\rangle_A) = \psi'_{ij}(\langle f,g\rangle_{C_A}).$$

This gives

$$\begin{aligned}
(T_{ij})_*[\mathcal{F}] = (T_{ij})_*[e] &= \sum_k T_{ij}(\langle g_k, g_k\rangle_B)\\
&= \sum_k \psi_{ij}(\langle g_k, g_k W^i\rangle_A)\\
&= \sum_k \psi'_{ij}(\langle g_k, g_k\rangle_{C_A})\\
&= \frac{1}{4}\sum_k \psi'_{ij}(\langle f_k, f_k\rangle_{C_A})\\
&= \frac{1}{4}\psi'_{ij}(1),
\end{aligned}$$

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where $1 = 1_{C_B}$. We thus obtain the result:

Proposition 3.6 *The unbounded traces of the Fourier module are given by*

$$(T_{ij})_*[\mathcal{F}] = \frac{1}{4}\psi'_{ij}(1),$$

where ψ'_{ij} is the unique σ^{4-i}-trace on A_β associated with the σ^i-trace ψ_{ij} on A (according to Proposition 3.5).

Clearly, our next goal is to find $\psi'_{ij}(1)$. To do this, we shall find explicit equations linking the ψ'_{ij} with the analogues of ψ_{ij} on A_β. Such a result will contain the values of $\psi'_{ij}(1)$. (See Theorems 4.2 and 5.2 below.) First, we need some notation.

Notation

Let $\overline{\psi}_{jk}$ denote the σ^j-traces on A_β (relative to its generators U_1 and V_1 given in Section 3). They are given by the same formulas as in Proposition 2.2, except where λ is to be replaced by $\mu = \mathrm{e}(\beta)$. Similarly, denote the bracket functionals on A_β by $\overline{[\nu, a, c; r, t]}$, so that one can use these same formulas given for them in Section 2 (but with λ replaced by μ). Thus,

$$\overline{[\nu, a, c; r, t]}\,(x) = \sum_{p,q} \mu^{\nu(p^2+q^2)-2pq+ap+cq} x_{2p-r,2q-t},$$

where $x = \sum_{p,q} x_{p,q} U_1^p V_1^q$ is a smooth vector in A_β.

4 Computation of $T_{2k}([\mathcal{F}])$

In this section and the next we shall consider the Schwartz functions

$$f_{b,d}(x) = \mathrm{e}(bx)\exp\Big(-\frac{\pi}{\theta}(x-d)^2\Big)$$

of the real variable x, where $b, d \in \mathbb{C}$. Its Fourier transform is easily checked to be

$$\hat{f}_{b,d} = \sqrt{\theta}\mathrm{e}(bd)\cdot e(-dx)e^{-\pi\theta(x-b)^2}$$

where $\mathrm{e}(t) := \mathrm{e}^{2\pi i t}$. Hence

$$f_{b,d}W = \mathrm{e}(bd) f_{\frac{d}{\theta},-b\theta} \quad \text{and} \quad f_{b,d}W^2 = f_{-b,-d}. \tag{4.1}$$

Notation Henceforth we shall write $f = f_{b,d}$.

The objective of this section will be to calculate $\psi'_{2k}(1)$ by explicitly working out the equation

$$\psi'_{2k}(\langle f, f\rangle_{A_\beta}) = \psi_{2k}(\langle f, fW^2\rangle_A) \tag{4.2}$$

for $k = 0, 1, 2$. In doing so, observe that all the bracket functionals $[\nu, a, c; k, t]$ involved in ψ_{2k} (and their counterparts $\overline{\psi}_{2k}$ on A_β defined at the end of Section 3) are such that $\nu = 0$, $c = k$, $a = t$. Thus one need only find $[0, t, k; k, t]$ on the inner product $\langle f, fW^2\rangle_A$. This will facilitate the computation of both sides of (4.2).

Proposition 4.1 *With $f = f_{b,d}$ and letting*

$$v = \frac{\pi d}{\theta}, \quad w = \pi b, \quad u = \frac{i}{2\theta}$$

one has

$$\psi_{20}(\langle f, fW^2\rangle_A) = \frac{1}{2}\sqrt{\frac{\theta}{2}}\vartheta_3(w, u)\cdot\vartheta_3(v, u)$$

$$\psi_{21}(\langle f, fW^2\rangle_A) = \frac{1}{2}\sqrt{\frac{\theta}{2}}\vartheta_4(w, u)\cdot\vartheta_4(v, u)$$

$$\psi_{22}(\langle f, fW^2\rangle_A) = \frac{1}{2}\sqrt{\frac{\theta}{2}}\big(\vartheta_4(w, u)\cdot\vartheta_3(v, u) + \vartheta_3(w, u)\cdot\vartheta_4(v, u)\big)$$

and

$$\overline{\psi}_{20}(\langle f, f\rangle_{A_\beta}) = \sqrt{\frac{\theta}{2}}\vartheta_3(2w, 4u)\cdot\vartheta_3(2v, 4u)$$

$$\overline{\psi}_{21}(\langle f, f\rangle_{A_\beta}) = \sqrt{\frac{\theta}{2}}\vartheta_2(2w, 4u)\cdot\vartheta_2(2v, 4u)$$

$$\overline{\psi}_{22}(\langle f, f\rangle_{A_\beta}) = \sqrt{\frac{\theta}{2}}\big(\vartheta_3(2w, 4u)\cdot\vartheta_2(2v, 4u) + \vartheta_2(2w, 4u)\cdot\vartheta_3(2v, 4u)\big)$$

Proof The result basically follows from Lemma 6.2 (of Section 6). Applying this lemma with $b' = -b$, $d' = -d$, and with $\nu = 0$, $a = t$, $c = k$, one has (as in the notation of Lemma 6.2) $E = F = 0$, $K = 2b\theta$, $L = -2d$. Thus, since $fW^2 = f_{-b,-d}$ one gets

$$\begin{aligned}
[0, t, k; k, t](\langle f, fW^2\rangle_A) &= \frac{\theta^{3/2}}{\sqrt{2}}\lambda^{tk/2}e^{-\pi(4d^2+4b^2\theta^2)/(2\theta)}e^{-\pi\theta(k^2+t^2)/2}e^{\pi(-2dk+2b\theta t)}\\
&\quad\cdot\vartheta_3(-\pi i\theta k - 2\pi i d, 2i\theta)\vartheta_3(-\pi i\theta t + 2\pi i b\theta, 2i\theta)\\
&= \frac{\theta^{3/2}}{\sqrt{2}}\lambda^{tk/2}e^{-\pi(4d^2+4b^2\theta^2)/(2\theta)}e^{-\pi\theta(k^2+t^2)/2}e^{\pi(-2dk+2b\theta t)}\\
&\quad\cdot\frac{1}{2\theta}\exp\left(-\frac{(-\pi i\theta k - 2\pi i d)^2 + (-\pi i\theta t + 2\pi i b\theta)^2}{2\pi\theta}\right)\\
&\quad\cdot\vartheta_3\left(v + \frac{\pi}{2}k, u\right)\vartheta_3\left(w - \frac{\pi}{2}t, u\right)\\
&= \frac{1}{2}\sqrt{\frac{\theta}{2}}\lambda^{tk/2}\vartheta_3\left(v + \frac{\pi}{2}k, u\right)\cdot\vartheta_3\left(w - \frac{\pi}{2}t, u\right)
\end{aligned}$$

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where in the second equality we have used the double inversion formula (A7) (of the Appendix) for a product of two ϑ_3's, used that ϑ_3 is even in the first variable, and in the third equality noted that all the exponential factors cancel out. Now since

$$\vartheta_3\Big(x \pm \frac{\pi}{2}t, u\Big) = \vartheta_{3+t}(x, u)$$

by (A12), where $t = 0, 1$, one gets

$$[0, t, k; k, t](\langle f, fW^2\rangle_{A}) = \frac{1}{2}\sqrt{\frac{\theta}{2}}\lambda^{tk/2}\vartheta_{3+t}(w, u)\vartheta_{3+k}(v, u)$$

where $k, t = 0, 1$. This gives the first three equations of the proposition in view of the bracket notation of Section 2. The last three follow in a similar but simpler way from Lemma 6.2 which gives

$$\begin{aligned}\overline{[0, t, k; k, t]}\,(\langle f, f\rangle_{A_\beta}) &= \sqrt{\frac{\theta}{2}}\mu^{kt/2}\mathrm{e}(bk - dt/\theta)\mathrm{e}^{-\pi(k^2+t^2)/2\theta} \\ &\quad \cdot \vartheta_3\Big(2\pi b + i\frac{\pi k}{\theta}, \frac{2}{\theta}i\Big)\vartheta_3\Big(2\pi\frac{d}{\theta} - i\frac{\pi t}{\theta}, \frac{2}{\theta}i\Big) \\ &= \sqrt{\frac{\theta}{2}}\mu^{kt/2}\vartheta_{3-k}\Big(2\pi b, \frac{2}{\theta}i\Big)\vartheta_{3-t}\Big(2\pi\frac{d}{\theta}, \frac{2}{\theta}i\Big) \\ &= \sqrt{\frac{\theta}{2}}\mu^{kt/2}\vartheta_{3-k}(2w, 4u)\vartheta_{3-t}(2v, 4u)\end{aligned}$$

where the second equality follows from (A8) and (A9). ■

Derivation of $\psi'_{20}(1) = \frac{1}{2}$

Since ψ'_{2k} must be a linear combination of the $\overline{\psi}_{2s}$'s (by Proposition 2.2), say

$$\psi'_{2k} = C\overline{\psi}_{20} + D\overline{\psi}_{21} + E\overline{\psi}_{22} \tag{4.3}$$

for some constants C, D, E, we need to calculate $C = \psi'_{20}(1)$. So, equation (4.2) becomes, by Proposition 4.1 in the case $k = 0$,

$$\begin{aligned}\frac{1}{2}\vartheta_3(w, u)\vartheta_3(v, u) &= C\vartheta_3(2w, 4u)\vartheta_3(2v, 4u) + D\vartheta_2(2w, 4u)\vartheta_2(2v, 4u) \\ &\quad + E\{\vartheta_3(2w, 4u)\vartheta_2(2v, 4u) + \vartheta_2(2w, 4u)\vartheta_3(2v, 4u)\}\end{aligned}$$

where the constants C, D, E are independent of v, w. It is not hard to see that this equation holds for $C = D = E = \frac{1}{2}$. This is because in this case it factors out as a product of two equations of the form

$$\vartheta_3(w, u) = \vartheta_3(2w, 4u) + \vartheta_2(2w, 4u) \tag{4.4}$$

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which is known to hold (and not hard to verify). To see that $C = D = E = \frac{1}{2}$ is necessary, put $w = v$ so that

$$\frac{1}{2}\vartheta_3(v,u)^2 = C\vartheta_3(2v,4u)^2 + D\vartheta_2(2v,4u)^2 + 2E\vartheta_3(2v,4u)\vartheta_2(2v,4u). \tag{4.5}$$

Now take $2v$ to be a zero of $\vartheta_2(2v,4u)$. (One such zero can be obtained from any of the zeros of ϑ_3—given by (A5)—and using equation (A8) of the Appendix. For example, $2v = \frac{\pi}{2}$.) Thus (4.5) becomes

$$\frac{1}{2}\vartheta_3(v,u)^2 = C\vartheta_3(2v,4u)^2$$

and since (4.4) implies that $\vartheta_3(v,u) = \vartheta_3(2v,4u)$, one gets

$$\psi'_{20}(1) = C = \frac{1}{2}.$$

(Noting that $\vartheta_3(2v,4u) \neq 0$.) Similarly, one obtains $D = E = \frac{1}{2}$.

Derivation of $\psi'_{21}(1) = \frac{1}{2}$

In a similar fashion, for the case $k = 1$ equation (4.2) becomes

$$\begin{aligned}\frac{1}{2}\vartheta_4(w,u)\vartheta_4(v,u) = C\vartheta_3(2w,4u)\vartheta_3(2v,4u) + D\vartheta_2(2w,4u)\vartheta_2(2v,4u)\\ + E\{\vartheta_3(2w,4u)\vartheta_2(2v,4u) + \vartheta_2(2w,4u)\vartheta_3(2v,4u)\}\end{aligned}$$

where the constants C, D, E are independent of v, w. Taking $w = v$ it becomes

$$\frac{1}{2}\vartheta_4(v,u)^2 = C\vartheta_3(2v,4u)^2 + D\vartheta_2(2v,4u)^2 + 2E\vartheta_3(2v,4u)\vartheta_2(2v,4u). \tag{4.6}$$

This holds for $C = D = \frac{1}{2}$ and $E = -\frac{1}{2}$ since it just becomes a product of two equations of the form

$$\vartheta_4(v,u) = \vartheta_3(2v,4u) - \vartheta_2(2v,4u). \tag{4.7}$$

To see that $C = \frac{1}{2}$, choose v so that, as before, $\vartheta_2(2v,4u) = 0$. Then (4.7) gives $\vartheta_4(v,u) = \vartheta_3(2v,4u)$ and hence (4.6) yields

$$\psi'_{21}(1) = C = \frac{1}{2}.$$

Similarly, one obtains $D = \frac{1}{2}$, $E = -\frac{1}{2}$.

Derivation of $\psi_{22}'(1) = 1$

Here, $k = 2$ so equation (4.2) yields

$$\frac{1}{2}\{\vartheta_4(w,u)\vartheta_3(v,u) + \vartheta_3(w,u)\vartheta_4(v,u)\}$$
$$= C\vartheta_3(2w,4u)\vartheta_3(2v,4u)$$
$$+ D\vartheta_2(2w,4u)\vartheta_2(2v,4u) + E\{\vartheta_3(2w,4u)\vartheta_2(2v,4u) + \vartheta_2(2w,4u)\vartheta_3(2v,4u)\}$$

which with $w = v$ becomes

$$\vartheta_4(v,u)\cdot\vartheta_3(v,u) = C\vartheta_3(2v,4u)^2 + D\vartheta_2(2v,4u)^2 + 2E\vartheta_3(2v,4u)\vartheta_2(2v,4u). \tag{4.8}$$

This equation holds with $C = 1$, $D = -1$, $E = 0$ since then it is just the product of equations (4.4) and (4.7). To check that $C = 1$ is necessary, once again choose v such that $\vartheta_2(2v,4u) = 0$. Then, for such v, (4.4) and (4.7) give

$$\vartheta_4(v,u) = \vartheta_3(v,u) = \vartheta_3(2v,4u) \neq 0$$

so that (4.8) entails

$$\psi_{22}'(1) = C = 1,$$

as required. Similarly, one gets $D = -1$, $E = 0$.

One can now deduce explicit equations that connect the ψ_{2k}'s with the $\overline{\psi}_{2s}$'s, and which generalize some of the Jacobi equations relating theta functions and which arose above.

Theorem 4.2 *The Chern characters of the Fourier module $\mathcal{F}$ which arise from σ^2-traces have the values*

$$T_{20}[\mathcal{F}] = T_{21}[\mathcal{F}] = \frac{1}{8}, \quad T_{22}[\mathcal{F}] = \frac{1}{4}.$$

Furthermore, we have, for all $f, g \in S(\mathbb{R})$, the equations relating the σ^2-traces on A with those on A_β:

$$\psi_{20}(\langle g, fW^2\rangle_A) = \frac{1}{2}\overline{\psi}_{20}(\langle f,g\rangle_{A_\beta}) + \frac{1}{2}\overline{\psi}_{21}(\langle f,g\rangle_{A_\beta}) + \frac{1}{2}\overline{\psi}_{22}(\langle f,g\rangle_{A_\beta})$$
$$\psi_{21}(\langle g, fW^2\rangle_A) = \frac{1}{2}\overline{\psi}_{20}(\langle f,g\rangle_{A_\beta}) + \frac{1}{2}\overline{\psi}_{21}(\langle f,g\rangle_{A_\beta}) - \frac{1}{2}\overline{\psi}_{22}(\langle f,g\rangle_{A_\beta})$$
$$\psi_{22}(\langle g, fW^2\rangle_A) = \overline{\psi}_{20}(\langle f,g\rangle_{A_\beta}) - \overline{\psi}_{21}(\langle f,g\rangle_{A_\beta}).$$

5 Computation of $T_{1k}([\mathcal{F}])$

In this section we compute the traces $T_{10}[\mathcal{F}]$ and $T_{11}[\mathcal{F}]$ of the Fourier module $\mathcal{F}$. (The computation here is longer than those of the previous section.) To do this, we shall calculate $\psi_{1r}'(1)$, for $r = 0, 1$. Our goal will also be to find other related constants that would give explicit equations connecting the σ-traces on $A = A_\theta$ with the σ^3-traces on $C_A = A_\beta$ (Theorem 5.2).

Since ψ_{1r} is σ-tracial on A_θ, Proposition 3.5 yields a unique σ^3-trace ψ'_{1r} on A_β such that

$$\psi'_{1r}(\langle g, h\rangle_{A_\beta}) = \psi_{1r}(\langle h, gW\rangle_{A}).$$

Taking $g = h = f$ (where $f = f_{b,d}$ is as defined in Section 4), this becomes

$$\psi'_{1r}(\langle f, f\rangle_{A_\beta}) = \psi_{1r}(\langle f, fW\rangle_{A}). \tag{5.1}$$

Now ψ'_{1r}, being a σ^3-trace on A_β, is a linear combination of the two basic σ^3-trace functionals on A_β, namely $\overline{\psi}_{30}$ and $\overline{\psi}_{31}$ (given by Proposition 2.2 but with $\mu = \mathrm{e}(\beta)$ in place of λ):

$$\psi'_{1r} = C\overline{\psi}_{30} + D\overline{\psi}_{31},$$

where C and D are constants (depending only on r and θ) to be determined. Clearly, $C = \psi'_{1r}(1)$, which we need to find. So equation (5.1) becomes

$$C\overline{\psi}_{30}(\langle f, f\rangle_{A_\beta}) + D\overline{\psi}_{31}(\langle f, f\rangle_{A_\beta}) = \psi_{1r}(\langle f, fW\rangle_{A}). \tag{5.2}$$

We shall compute both sides of this equation quite explicitly using Lemma 6.2.

The Right Side of (5.2)

Using $f_{b,d}W = \mathrm{e}(bd) f_{\frac{d}{\theta},-b\theta}$ one gets

$$\begin{aligned}\psi_{1r}(\langle f, fW\rangle_{A}) &= [1,-r,r;r,0](\langle f, fW\rangle_{A}) + [1,-r,r;r+1,1](\langle f, fW\rangle_{A})\\ &= \mathrm{e}(bd)[1,-r,r;r,0](\langle f, f_{\frac{d}{\theta},-b\theta}\rangle_{A}) + \mathrm{e}(bd)[1,-r,r;r+1,1](\langle f, f_{\frac{d}{\theta},-b\theta}\rangle_{A})\end{aligned}$$

and now we shall compute each of these bracket expressions separately. Let us denote them by B_1 and B_2, respectively. For the first, using the notation of Lemma 6.2, with $b' = \frac{d}{\theta}$ and $d' = -b\theta$, one has

$$K = b\theta - d, \quad L = -d - b\theta, \quad E = \frac{1}{2}L, \quad F = \frac{1}{2}K,$$

and

$$\begin{aligned}B_1 &= [1,-r,r;r,0](\langle f, fW\rangle_{A})\\ &= \mathrm{e}(bd)\frac{\theta^{3/2}}{\sqrt{2}}\mathrm{e}(K^2/2\theta)\mathrm{e}(-rL/2)\exp\bigl(-\pi(L^2+K^2)/2\theta\bigr)\exp(\pi Lr)\exp(-\pi\theta r/2)\\ &\quad\cdot \vartheta_3\bigl(-\pi r\theta + \pi(-d-b\theta) + i\pi(-d-b\theta-r\theta), 2\theta(i+1)\bigr)\\ &\quad\cdot \vartheta_3\bigl(\pi(-d+b\theta) + i\pi(-d+b\theta), 2\theta(i+1)\bigr)\end{aligned}$$

Now let us consider the first of these ϑ_3 expressions and call it Θ_{11} (and the second call Θ_{12}). One has

$$\begin{aligned}\Theta_{11} &= \vartheta_3\bigl(-\pi r\theta + \pi(-d-b\theta) + i\pi(-d-b\theta-r\theta), 2\theta(i+1)\bigr)\\ &= \vartheta_3\bigl(-\pi d(i+1) - \pi b\theta(i+1) - r\pi\theta(i+1), 2\theta(i+1)\bigr)\end{aligned}$$

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and now let

$$v = -\frac{1}{2}\pi d(i+1), \quad w = \frac{1}{2}\pi b\theta(i+1), \quad u = \frac{1}{4}\theta(i+1)$$

so that by (A8)

$$\begin{aligned}\Theta_{11} &= \vartheta_3(2v - 2w - 4r\pi u, 8u) \\ &= e^{2\pi iru} e^{ir(2v-2w-r\pi\theta(i+1))}\vartheta_{3-r}(2v-2w, 8u) \\ &= e^{-2\pi iru} e^{ir(2v-2w)}\vartheta_{3-r}(2v-2w, 8u).\end{aligned}$$

(Note that this is trivial when $r = 0$, and when $r = 1$ it holds by (A8).) Similarly,

$$\Theta_{12} = \vartheta_3\big(\pi(-d+b\theta) + i\pi(-d+b\theta), 2\theta(i+1)\big) = \vartheta_3(2v+2w, 8u).$$

Thus,

$$\begin{aligned}B_1 = \mathrm{e}(bd)\frac{\theta^{3/2}}{\sqrt{2}}&\mathrm{e}(K^2/2\theta)\mathrm{e}(-rL/2)\exp\big(-\pi(L^2+K^2)/2\theta\big)\exp(\pi Lr)\exp(-\pi\theta r/2) \\ &\cdot e^{-2\pi iru}e^{ir(2v-2w)}\vartheta_{3-r}(2v-2w, 8u)\cdot\vartheta_3(2v+2w, 8u)\end{aligned}$$

which we shall write as

$$B_1 = \frac{\theta^{3/2}}{\sqrt{2}}M\vartheta_{3-r}(2v-2w, 8u)\vartheta_3(2v+2w, 8u)$$

where

$$\begin{aligned}M := \mathrm{e}(bd)&\mathrm{e}(K^2/2\theta)\mathrm{e}(-rL/2)\exp\big(-\pi(L^2+K^2)/2\theta\big)\exp(\pi Lr)\exp(-\pi\theta r/2) \\ &\cdot\exp\big(-2\pi iru + ir(2v-2w)\big).\end{aligned}$$

Now let us look at B_2, using Lemma 6.2 again

$$\begin{aligned}B_2 &= [1, -r, r; r+1, 1](\langle f, fW\rangle_A) \\ &= \mathrm{e}(bd)\frac{\theta^{3/2}}{\sqrt{2}}\lambda^{(r+1)/2}\mathrm{e}(K^2/2\theta)\mathrm{e}\Big(-\big(K+(r+1)L\big)/2\Big) \\ &\quad\cdot\exp\big(-\pi(L^2+K^2)/2\theta\big)\exp\big(\pi L(r+1)+\pi K\big)\exp\Big(-\pi\theta\big((r+1)^2+1\big)/2\Big) \\ &\quad\cdot\vartheta_3\Big(-\pi r\theta - \pi\theta + \pi L + i\pi\big(L-(r+1)\theta\big), 8u\Big) \\ &\quad\cdot\vartheta_3\big(r\pi\theta - \pi\theta(r+1) + \pi K + i\pi(K-\theta), 8u\big).\end{aligned}$$

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Again, let Θ_{21} and Θ_{22} denote the last two theta function expressions, respectively. The first becomes

$$\begin{aligned}\Theta_{21} &= \vartheta_3\Big(-\pi r\theta - \pi\theta + \pi(-d - b\theta) + i\pi\big(-d - b\theta - (r+1)\theta\big), 8u\Big)\\ &= \vartheta_3(-\pi r\theta - \pi\theta - \pi d - \pi b\theta - i\pi d - i\pi b\theta - i\pi r\theta - i\pi\theta, 8u)\\ &= \vartheta_3\big(-\pi d(i+1) - \pi b\theta(i+1) - (r+1)\pi\theta(i+1), 8u\big)\\ &= \vartheta_3\big(2v - 2w - 4\pi(r+1)u, 8u\big)\\ &= \mathrm{e}^{-2\pi iu(1+3r)}\mathrm{e}^{i(2v-2w)(1+r)} \cdot \vartheta_{2+r}(2v - 2w, 8u)\end{aligned}$$

where the last equality holds by (A8) when $r = 0$ or by (A10) when $r = 1$. Similarly,

$$\begin{aligned}\Theta_{22} &= \vartheta_3\big(r\pi\theta - \pi\theta(r+1) + \pi(-d + b\theta) + i\pi(-d + b\theta - \theta), 8u\big)\\ &= \vartheta_3\big(-\pi d - i\pi d + \pi b\theta + i\pi b\theta + r\pi\theta - \pi\theta(r+1) - i\pi\theta, 8u\big)\\ &= \vartheta_3(2v + 2w - 4\pi u, 8u)\\ &= \mathrm{e}^{2\pi iu}\mathrm{e}^{i(2v+2w)}\mathrm{e}^{-4\pi iu} \cdot \vartheta_2(2v + 2w, 8u)\\ &= \mathrm{e}^{-2\pi iu}\mathrm{e}^{i(2v+2w)} \cdot \vartheta_2(2v + 2w, 8u).\end{aligned}$$

Hence B_2 becomes

$$\begin{aligned}B_2 = \mathrm{e}(bd)\frac{\theta^{3/2}}{\sqrt{2}}\lambda^{(r+1)/2}\mathrm{e}(K^2/2\theta)\mathrm{e}\Big(-\big(K + (r+1)L\big)/2\Big)\exp\big(-\pi(L^2 + K^2)/2\theta\big)\\ \cdot \exp\big(\pi L(r+1) + \pi K\big)\exp\Big(-\pi\theta\big((r+1)^2 + 1\big)/2\Big)\\ \cdot \exp\big(-2\pi iu(1 + 3r) + i(2v - 2w)(1 + r)\big)\\ \cdot \exp\big(-2\pi iu + i(2v + 2w)\big) \cdot \vartheta_{2+r}(2v - 2w, 8u)\vartheta_2(2v + 2w, 8u)\end{aligned}$$

which we shall write as

$$B_2 = \frac{\theta^{3/2}}{\sqrt{2}}N\vartheta_{2+r}(2v - 2w, 8u)\vartheta_2(2v + 2w, 8u)$$

where

$$\begin{aligned}N := \mathrm{e}(bd)\lambda^{(r+1)/2}\mathrm{e}(K^2/2\theta)\mathrm{e}\Big(-\big(K + (r+1)L\big)/2\Big)\exp\big(-\pi(L^2 + K^2)/2\theta\big)\\ \cdot \exp\big(\pi L(r+1) + \pi K\big)\exp\Big(-\pi\theta\big((r+1)^2 + 1\big)/2\Big)\\ \cdot \exp\big(-2\pi iu(1 + 3r) + i(2v - 2w)(1 + r)\big)\\ \cdot \exp\big(-2\pi iu + i(2v + 2w)\big).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\psi_{1r}(\langle f, fW\rangle_{A}) &= B_1 + B_2\\ &= \frac{\theta^{3/2}}{\sqrt{2}} M\vartheta_{3-r}(2v-2w, 8u)\vartheta_3(2v+2w, 8u)\\ &\quad + \frac{\theta^{3/2}}{\sqrt{2}} N\vartheta_{2+r}(2v-2w, 8u)\vartheta_2(2v+2w, 8u)\end{aligned} \tag{5.3}$$

It is in fact not hard (but a little tedious) to verify that

$$M = N = \exp\Big(-\frac{\pi}{2} ir\theta + \frac{\pi}{\theta}(b^2\theta^2 + d^2)(i-1)\Big).$$

The Left Side of Equation (5.2)

Next, let us work out the left side of (5.2) by first computing the term $\overline{\psi}_{30}(\langle f, f\rangle_{A_\beta})$. In terms of the bracket notation of Section 2, we have

$$\begin{aligned}\overline{\psi}_{30} &= \overline{[-1,0,0;0,0]} + \overline{[-1,0,0;-1,1]},\\ \overline{\psi}_{31} &= \overline{[-1,1,1;1,0]} + \overline{[-1,1,1;0,1]}.\end{aligned}$$

Computation of $\overline{\psi}_{30}(\langle f, f\rangle_{A_\beta})$

We have

$$\overline{\psi}_{30}(\langle f, f\rangle_{A_\beta}) = \overline{[-1,0,0;0,0]}\,(\langle f, f\rangle_{A_\beta}) + \overline{[-1,0,0;-1,1]}\,(\langle f, f\rangle_{A_\beta}).$$

Let us call the latter two terms A_{01} and A_{02}, respectively. By Lemma 6.2 we have

$$\begin{aligned}A_{01} &= \overline{[-1,0,0;0,0]}\,(\langle f, f\rangle_{A_\beta})\\ &= \sqrt{\frac{\theta}{2}}\cdot\vartheta_3\Big(-2\pi b, \frac{2}{\theta}(i-1)\Big)\vartheta_3\Big(2\pi\frac{d}{\theta}, \frac{2}{\theta}(i-1)\Big).\end{aligned}$$

Here, one applies (A7) to this product of thetas. But first put

$$t = \frac{2}{\theta}(i-1) \quad \text{so that} \quad t^{-1} = -\frac{\theta}{4}(i+1) = -u.$$

Then it is easy to verify the following simple relations (which will be used freely below):

$$\begin{aligned}2\pi\frac{b}{t} = -w, &\quad 2\pi\frac{d}{\theta t} = v,\\ \frac{\pi}{\theta t} = -\frac{\pi}{4}(i+1), &\quad \frac{\pi}{\theta t}(i-1) = \frac{\pi}{2}.\end{aligned}$$

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In terms of these, one gets

$$\begin{aligned}\vartheta_3\Big(-2\pi b, \frac{2}{\theta}(i-1)\Big)\vartheta_3\Big(2\pi\frac{d}{\theta}, \frac{2}{\theta}(i-1)\Big)\\ &= -\frac{i\theta}{4}(i+1)\exp\big(i\pi(b^2\theta^2+d^2)(i+1)/\theta\big)\\ &\quad\cdot\vartheta_3\Big(\pi b\frac{\theta}{2}(i+1), \frac{\theta}{4}(i+1)\Big)\cdot\vartheta_3\Big(-\pi\frac{d}{2}(i+1), \frac{\theta}{4}(i+1)\Big)\\ &= \frac{\theta}{4}(1-i)\exp\big(i\pi(b^2\theta^2+d^2)(i+1)/\theta\big)\cdot\vartheta_3(w,u)\cdot\vartheta_3(v,u)\end{aligned}$$

so that

$$A_{01} = \frac{\theta^{3/2}}{4\sqrt{2}}(1-i)\exp\big(i\pi(b^2\theta^2+d^2)(i+1)/\theta\big)\cdot\vartheta_3(w,u)\cdot\vartheta_3(v,u).$$

In the same way we find A_{02}

$$\begin{aligned}A_{02} &= \overline{[-1,0,0;-1,1]}\,(\langle f,f\rangle_{A_\beta})\\ &= \sqrt{\frac{\theta}{2}}\mu^{-1/2}\mathrm{e}(-b)\mathrm{e}(-d/\theta)\exp(-\pi/\theta)\\ &\quad\cdot\vartheta_3\Big(-\frac{\pi}{\theta}-2\pi b+i\frac{\pi}{\theta}, \frac{2}{\theta}(i-1)\Big)\vartheta_3\Big(\frac{\pi}{\theta}+2\pi\frac{d}{\theta}-i\frac{\pi}{\theta}, \frac{2}{\theta}(i-1)\Big)\\ &= \sqrt{\frac{\theta}{2}}\mu^{-1/2}\mathrm{e}(-b)\mathrm{e}(-d/\theta)\exp(-\pi/\theta)\cdot\vartheta_3\Big(-2\pi b+\frac{\pi}{2}t, t\Big)\vartheta_3\Big(2\pi\frac{d}{\theta}-\frac{\pi}{2}t, t\Big)\\ &= \sqrt{\frac{\theta}{2}}\mu^{-1/2}\mathrm{e}(-b)\mathrm{e}(-d/\theta)\exp(-\pi/\theta)\\ &\quad\cdot\frac{i}{t}\exp\big([(-2\pi b+\pi t/2)^2+(2\pi d/\theta-\pi t/2)^2]/(\pi i t)\big)\\ &\quad\cdot\vartheta_3\Big(-2\pi\frac{b}{t}+\frac{\pi}{2}, -t^{-1}\Big)\vartheta_3\Big(2\pi\frac{d}{\theta t}-\frac{\pi}{2}, -t^{-1}\Big)\\ &= \sqrt{\frac{\theta}{2}}\mu^{-1/2}\mathrm{e}(-b)\mathrm{e}(-d/\theta)\exp(-\pi/\theta)\\ &\quad\cdot\frac{i}{t}\exp\big([(-2\pi b+\pi t/2)^2+(2\pi d/\theta-\pi t/2)^2]/(\pi i t)\big)\\ &\quad\cdot\vartheta_3\Big(w+\frac{\pi}{2}, u\Big)\vartheta_3\Big(v-\frac{\pi}{2}, u\Big)\\ &= \frac{i}{t}\sqrt{\frac{\theta}{2}}\mu^{-1/2}\mathrm{e}(-b)\mathrm{e}(-d/\theta)\exp(-\pi/\theta)\\ &\quad\exp\Big(-4\pi i\frac{b^2}{t}+2\pi i b+\frac{\pi t}{4i}-4\pi i\frac{d^2}{\theta^2 t}+2\pi i\frac{d}{\theta}-i\frac{\pi t}{4}\Big)\cdot\vartheta_4(w,u)\cdot\vartheta_4(v,u)\end{aligned}$$

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where use was made of (A7) in the fourth equality and (A12) in the last equality to go from ϑ_3 to ϑ_4. It is not hard to check that several of the exponential constants appearing in this last expression cancel out and so reducing to

$$A_{02} = \frac{\theta^{3/2}}{4\sqrt{2}}(1-i)\exp\big(i\pi(b^2\theta^2+d^2)(i+1)/\theta\big)\cdot\vartheta_4(w,u)\vartheta_4(v,u).$$

Putting together A_{01} and A_{02} yields

$$\begin{aligned}(5.4)\quad \overline{\psi}_{30}(\langle f,f\rangle_{A_\beta}) &= A_{01}+A_{02}\\ &= \frac{\theta^{3/2}}{4\sqrt{2}}(1-i)\mathrm{e}^{i\pi(b^2\theta^2+d^2)(i+1)/\theta}\big(\vartheta_3(w,u)\vartheta_3(v,u)+\vartheta_4(w,u)\vartheta_4(v,u)\big)\end{aligned}$$

Computation of $\overline{\psi}_{31}(\langle f,f\rangle_{A_\beta})$

Here, we have

$$\overline{\psi}_{31}(\langle f,f\rangle_{A_\beta}) = \overline{[-1,1,1;1,0]}\,(\langle f,f\rangle_{A_\beta}) + \overline{[-1,1,1;0,1]}\,(\langle f,f\rangle_{A_\beta})$$

and call these two bracket terms A_{11} and A_{12}, respectively. For the first (by Lemma 6.2)

$$\begin{aligned}A_{11} &= \overline{[-1,1,1;1,0]}\,(\langle f,f\rangle_{A_\beta})\\ &= \sqrt{\frac{\theta}{2}}\mathrm{e}(b)\exp(-\pi/2\theta)\cdot\vartheta_3\Big(\frac{\pi}{\theta}-2\pi b-i\frac{\pi}{\theta},\frac{2}{\theta}(i-1)\Big)\vartheta_3\Big(2\pi\frac{d}{\theta},\frac{2}{\theta}(i-1)\Big)\\ &= \sqrt{\frac{\theta}{2}}\mathrm{e}(b)\exp(-\pi/2\theta)\cdot\vartheta_3\Big(2\pi b+\frac{\pi}{2}t,t\Big)\vartheta_3\Big(2\pi\frac{d}{\theta},t\Big)\\ &= \sqrt{\frac{\theta}{2}}\mathrm{e}(b)\exp(-\pi/2\theta)\cdot\frac{i}{t}\cdot\exp\bigg(\Big[\Big(2\pi b+\frac{\pi}{2}t\Big)^2+\Big(2\pi\frac{d}{\theta}\Big)^2\Big]\Big/(\pi it)\bigg)\\ &\qquad\cdot\vartheta_3\Big(2\pi\frac{b}{t}+\frac{\pi}{2},-\frac{1}{t}\Big)\vartheta_3\Big(2\pi\frac{d}{\theta t},-\frac{1}{t}\Big)\\ &= \sqrt{\frac{\theta}{2}}\mathrm{e}(b)\exp(-\pi/2\theta)\cdot\frac{i}{t}\cdot\exp\bigg(\Big[\Big(2\pi b+\frac{\pi}{2}t\Big)^2+\Big(2\pi\frac{d}{\theta}\Big)^2\Big]\Big/(\pi it)\bigg)\\ &\qquad\cdot\vartheta_4(w,u)\vartheta_3(v,u).\end{aligned}$$

After expanding and simplifying the exponentials this becomes

$$A_{11} = \frac{\theta^{3/2}}{4\sqrt{2}}(1-i)\exp\Big(\frac{\pi}{2\theta}i+\frac{\pi}{\theta}(b^2\theta^2+d^2)(i-1)\Big)\cdot\vartheta_4(w,u)\vartheta_3(v,u).$$

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Similarly, for A_{12} one has

$$
\begin{aligned}
A_{12} &= \overline{[-1,1,1;0,1]}\,(\langle f,f\rangle_{A_\beta}) \\
&= \sqrt{\frac{\theta}{2}}\mathrm{e}(-d/\theta)\exp(-\pi/2\theta)\cdot\vartheta_3\Big(-2\pi b,\frac{2}{\theta}(i-1)\Big)\vartheta_3\Big(\frac{\pi}{\theta}+2\pi\frac{d}{\theta}-i\frac{\pi}{\theta},\frac{2}{\theta}(i-1)\Big) \\
&= \sqrt{\frac{\theta}{2}}\mathrm{e}(-d/\theta)\exp(-\pi/2\theta)\cdot\vartheta_3(-2\pi b,t)\vartheta_3\Big(2\pi\frac{d}{\theta}-\frac{\pi}{2}t,t\Big) \\
&= \sqrt{\frac{\theta}{2}}\mathrm{e}(-d/\theta)\exp(-\pi/2\theta)\cdot\frac{i}{t}\cdot\exp\bigg(\Big[4\pi^2b^2+\Big(2\pi\frac{d}{\theta}-\frac{\pi}{2}t\Big)^2\Big]\Big/(\pi i t)\bigg) \\
&\qquad\cdot\vartheta_3\Big(-2\pi\frac{b}{t},-t^{-1}\Big)\vartheta_3\Big(2\pi\frac{d}{\theta t}-\frac{\pi}{2},-t^{-1}\Big) \\
&= \sqrt{\frac{\theta}{2}}\mathrm{e}(-d/\theta)\exp(-\pi/2\theta)\cdot\frac{i}{t}\cdot\exp\bigg(\Big[4\pi^2b^2+\Big(2\pi\frac{d}{\theta}-\frac{\pi}{2}t\Big)^2\Big]\Big/(\pi i t)\bigg) \\
&\qquad\cdot\vartheta_3(w,u)\vartheta_4(v,u)
\end{aligned}
$$

which when simplified as before becomes

$$A_{12}=\frac{\theta^{3/2}}{4\sqrt{2}}(1-i)\exp\Big(\frac{\pi}{2\theta}i+\frac{\pi}{\theta}(b^2\theta^2+d^2)(i-1)\Big)\vartheta_3(w,u)\vartheta_4(v,u).$$

Putting together A_{11} and A_{12} one has

$$\overline{\psi}_{31}(\langle f,f\rangle_{A_\beta})=\frac{\theta^{3/2}}{4\sqrt{2}}(1-i)P\big(\vartheta_4(w,u)\vartheta_3(v,u)+\vartheta_3(w,u)\vartheta_4(v,u)\big) \tag{5.5}$$

where

$$P=\exp\Big(\frac{\pi}{2\theta}i+\frac{\pi}{\theta}(b^2\theta^2+d^2)(i-1)\Big).$$

Therefore, equation (5.2) becomes, after inserting the expressions in (5.3), (5.4), and (5.5),

$$
\begin{aligned}
&\frac{\theta^{3/2}}{\sqrt{2}}M\cdot\vartheta_{3-r}(2v-2w,8u)\vartheta_3(2v+2w,8u)+\frac{\theta^{3/2}}{\sqrt{2}}M\cdot\vartheta_{2+r}(2v-2w,8u)\vartheta_2(2v+2w,8u) \\
&\quad= C\frac{\theta^{3/2}}{4\sqrt{2}}(1-i)\exp\big(i\pi(b^2\theta^2+d^2)(i+1)/\theta\big)\big(\vartheta_3(w,u)\vartheta_3(v,u)+\vartheta_4(w,u)\vartheta_4(v,u)\big) \\
&\qquad+D\frac{\theta^{3/2}}{4\sqrt{2}}(1-i)P\big(\vartheta_4(w,u)\vartheta_3(v,u)+\vartheta_3(w,u)\vartheta_4(v,u)\big)
\end{aligned}
$$

or

$$
\begin{aligned}
&M\big(\vartheta_{3-r}(2v-2w,8u)\vartheta_3(2v+2w,8u)+\vartheta_{2+r}(2v-2w,8u)\vartheta_2(2v+2w,8u)\big) \\
&\quad=\frac{C}{4}(1-i)\exp\big(i\pi(b^2\theta^2+d^2)(i+1)/\theta\big)\big(\vartheta_3(w,u)\vartheta_3(v,u)+\vartheta_4(w,u)\vartheta_4(v,u)\big) \\
&\qquad+\frac{D}{4}(1-i)P\big(\vartheta_4(w,u)\vartheta_3(v,u)+\vartheta_3(w,u)\vartheta_4(v,u)\big)
\end{aligned}
$$

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where

$$M = \exp\Big(-\frac{\pi}{2} ir\theta + \frac{\pi}{\theta}(b^2\theta^2 + d^2)(i-1)\Big).$$

Noting the cancellations, one has

$$\begin{aligned} &e^{-\pi ir\theta/2}\cdot\big(\vartheta_{3-r}(2v-2w,8u)\vartheta_3(2v+2w,8u) + \vartheta_{2+r}(2v-2w,8u)\vartheta_2(2v+2w,8u)\big) \\ &\quad = \frac{C}{4}(1-i)\big(\vartheta_3(w,u)\vartheta_3(v,u) + \vartheta_4(w,u)\vartheta_4(v,u)\big) \\ &\qquad + \frac{D}{4}(1-i)e^{\pi i/2\theta}\big(\vartheta_4(w,u)\vartheta_3(v,u) + \vartheta_3(w,u)\vartheta_4(v,u)\big) \end{aligned}$$

or

(5.6)

$$\begin{aligned} &\vartheta_{3-r}(2v-2w,8u)\vartheta_3(2v+2w,8u) + \vartheta_{2+r}(2v-2w,8u)\vartheta_2(2v+2w,8u) \\ &\quad = A\big(\vartheta_3(w,u)\vartheta_3(v,u) + \vartheta_4(w,u)\vartheta_4(v,u)\big) + B\big(\vartheta_4(w,u)\vartheta_3(v,u) + \vartheta_3(w,u)\vartheta_4(v,u)\big) \end{aligned}$$

where

$$A = \frac{C}{4}(1-i)e^{\pi ir\theta/2}, \quad \text{and} \quad B = \frac{D}{4}(1-i)e^{\pi i/2\theta}e^{\pi ir\theta/2}$$

are constants depending only on r and θ, but not on v nor w. In the following lemma we take $w = v$ in (5.6) in order to find the constants A and B.

Lemma 5.1 *If θ is fixed (and hence u) and if A and B are constants depending only on r and θ such that*

$$\begin{aligned} &\vartheta_{3-r}(0,8u)\vartheta_3(4v,8u) + \vartheta_{2+r}(0,8u)\vartheta_2(4v,8u) \\ &\qquad = A\big(\vartheta_3(v,u)^2 + \vartheta_4(v,u)^2\big) + 2B\vartheta_3(v,u)\vartheta_4(v,u) \end{aligned} \tag{5.7}$$

for all complex v, then $A = \frac{1}{4}$ and $B = \frac{(-1)^r}{4}$. In fact, this equation holds, a priori, for these values of A and B.

Proof It is not hard to see that (5.7) holds *a priori* with $A = \frac{1}{4}$ and $B = \frac{(-1)^r}{4}$. To see that these values are necessary choose v such that $\vartheta_3(v,u) = 0$. Then (5.7) becomes

$$\vartheta_{3-r}(0,8u)\vartheta_3(4v,8u) + \vartheta_{2+r}(0,8u)\vartheta_2(4v,8u) = A\vartheta_4(v,u)^2.$$

But we also know (from the *a priori* equation (5.7)) that

$$\vartheta_{3-r}(0,8u)\vartheta_3(4v,8u) + \vartheta_{2+r}(0,8u)\vartheta_2(4v,8u) = \frac{1}{4}\vartheta_4(v,u)^2$$

so by comparison, and since $\vartheta_4(v,u)^2 \neq 0$ (by our choice of v), it follows that $A = \frac{1}{4}$. In a similar fashion one obtains $B = \frac{(-1)^r}{4}$. ∎

From this lemma it follows that

$$C(1-i)e^{\pi i r\theta/2} = 1, \quad D(1-i)e^{\pi i/2\theta}e^{\pi i r\theta/2} = (-1)^r$$

hence

$$C = \frac{1}{2}(1+i)e^{-\pi i r\theta/2}, \quad D = \frac{1}{2}(-1)^r(1+i)e^{-\pi i/2\theta}e^{-\pi i r\theta/2}.$$

In particular,

$$\psi'_{1r}(1) = C = \frac{1}{2}(1+i)e^{-\pi i r\theta/2} = \frac{1}{2}(1+i)\lambda^{-r/4}$$

from which one obtains the Chern character values

$$(T_{1r})_*([\mathcal{F}]) = \frac{1}{4}\psi'_{1r}(1) = \frac{1}{8}(1+i)\lambda^{-r/4}.$$

Also note that after the constants A and B are substituted back into (5.6) one actually gets some of Jacobi's identities relating theta functions. For example taking $r = 0$ in (5.6) it becomes [†]

$$\frac{1}{4}\{\vartheta_3(s,u) + \vartheta_4(s,u)\}\{\vartheta_3(t,u) + \vartheta_4(t,u)\}$$
$$= \vartheta_3(2t-2s, 8u)\cdot\vartheta_3(2t+2s, 8u) + \vartheta_2(2t-2s, 8u)\cdot\vartheta_2(2t+2s, 8u).$$

We can now substitute the values for C and D found above into (5.2) to obtain the explicit relations connecting the unbounded traces across Morita equivalence (which was promised in the Introduction). All these results can now be summarized follows.

Theorem 5.2 *The Chern characters of the Fourier module $\mathcal{F}$ which arise from the σ-traces have the values*

$$T_{10}[\mathcal{F}] = \frac{1}{8}(1+i), \quad T_{11}[\mathcal{F}] = \frac{1}{8}(1+i)\lambda^{-1/4}.$$

Furthermore, we have, for all $f, g \in S(\mathbb{R})$, the following equations relating the σ-traces on A and the σ^3-traces on A_β:

$$\psi_{10}(\langle g, fW\rangle_A) = \frac{1}{2}(i+1)\overline{\psi}_{30}(\langle f,g\rangle_{A_\beta}) + \frac{1}{2}(i+1)e^{-\pi i/2\theta}\overline{\psi}_{31}(\langle f,g\rangle_{A_\beta})$$

$$\psi_{11}(\langle g, fW\rangle_A) = \frac{1}{2}(1+i)e^{-\pi i\theta/2}\overline{\psi}_{30}(\langle f,g\rangle_{A_\beta}) - \frac{1}{2}(1+i)e^{-\pi i/2\theta - \pi i\theta/2}\overline{\psi}_{31}(\langle f,g\rangle_{A_\beta})$$

$$\psi_{30}(\langle g, fW^*\rangle_A) = \frac{1}{2}(1-i)\overline{\psi}_{10}(\langle f,g\rangle_{A_\beta}) + \frac{1}{2}(1-i)e^{\pi i/2\theta}\overline{\psi}_{11}(\langle f,g\rangle_{A_\beta})$$

$$\psi_{31}(\langle g, fW^*\rangle_A) = \frac{1}{2}(1-i)e^{\pi i\theta/2}\overline{\psi}_{10}(\langle f,g\rangle_{A_\beta}) - \frac{1}{2}(1-i)e^{\pi i/2\theta + \pi i\theta/2}\overline{\psi}_{11}(\langle f,g\rangle_{A_\beta})$$

for all $f, g \in \mathcal{S}(\mathbb{R})$. These are non-commutative generalizations of some of Jacobi's theta function equations.

[†]The author wishes to thank Jonathan Borwein for pointing out to him that this equation actually follows from Example 1 on page 464 of [18] combined with Exercise 2 on page 488.

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6 Proofs of Lemmas

The proof of the following lemma is straightforward, but we have included it for the sake of clarity. The function ϑ_3 appearing in Lemma 6.2 is the third theta function recalled in the Appendix (Section 7)—the notation there is being used here. In both lemmas we have used the function $f_{b,d}$ defined at the beginning of Section 4.

6.1 Lemma *We have*

$$\begin{aligned}\langle f_{b,d}, f_{b',d'}\rangle_{A}(m,n) = \frac{\theta^{3/2}}{\sqrt{2}}\lambda^{mn/2}\mathrm{e}(-bm\theta)\mathrm{e}\Big(\frac{1}{2}m(b-b')\theta\Big)\mathrm{e}\Big(-\frac{1}{2}(b-b')(d+d')\Big)\\ \cdot\,\mathrm{e}\Big(-\frac{1}{2}n(d+d')\Big)\exp\Big(-\frac{\pi}{2\theta}(m\theta-d+d')^2\Big)\\ \cdot\exp\Big(-\frac{\pi\theta}{2}(b-b'+n)^2\Big)\end{aligned}$$

and

$$\langle f_{b,d}, f_{b,d}\rangle_{C_A}(m,n) = \sqrt{\frac{\theta}{2}}\mu^{mn/2}\mathrm{e}\Big(-bm+\frac{nd}{\theta}\Big)\exp\Big(-\frac{\pi}{2\theta}m^2\Big)\exp\Big(-\frac{\pi}{2\theta}n^2\Big).$$

Proof We will use the Fourier transform of the function

$$g(t) = \exp\Big(-\frac{2\pi}{\theta}t^2\Big)$$

which is easily checked to be

$$\hat{g}(s) = \sqrt{\frac{\theta}{2}}\exp\Big(-\frac{\pi\theta}{2}s^2\Big).$$

To establish the first equality, one has

$$\begin{aligned}&\langle f_{b,d}, f_{b',d'}\rangle_{A}(m,n)\\ &\quad= \theta\int \overline{f_{b,d}(t+m\theta)}\, f_{b',d'}(t)\mathrm{e}(-nt)\,dt\\ &\quad= \theta\int \mathrm{e}\big(-b(t+m\theta)\big)\exp\Big(-\frac{\pi}{\theta}(t+m\theta-d)^2\Big)\\ &\qquad\cdot\mathrm{e}(b't)\exp\Big(-\frac{\pi}{\theta}(t-d')^2\Big)\cdot\mathrm{e}(-nt)\,dt\\ &\quad= \theta\mathrm{e}(-bm\theta)\int\exp\Big(-\frac{\pi}{\theta}[(t+m\theta-d)^2+(t-d')^2]\Big)\cdot\mathrm{e}\big((-b+b'-n)t\big)\,dt\end{aligned}$$

and after completing the square becomes

$$\begin{aligned}&= \theta\mathrm{e}(-bm\theta)\cdot\exp\Big(-\frac{\pi}{2\theta}(m\theta-d+d')^2\Big)\\ &\qquad\cdot\int\exp\Big(-\frac{2\pi}{\theta}\Big(t+\frac{m\theta-d-d'}{2}\Big)^2\Big)\cdot\mathrm{e}\big((-b+b'-n)t\big)\,dt\end{aligned}$$

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which, after translation and using g above, becomes

$$
\begin{aligned}
&= \theta \mathrm{e}(-bm\theta) \cdot \exp\Big(-\frac{\pi}{2\theta}(m\theta - d + d')^2\Big) \mathrm{e}\Big(\frac{1}{2}(b - b' + n)(m\theta - d - d')\Big) \\
&\quad \cdot \int \exp\Big(-\frac{2\pi}{\theta} t^2\Big) \mathrm{e}\big((-b + b' - n)t\big)\, dt \\
&= \theta \mathrm{e}(-bm\theta) \cdot \exp\Big(-\frac{\pi}{2\theta}(m\theta - d + d')^2\Big) \mathrm{e}\Big(\frac{1}{2}(b - b' + n)(m\theta - d - d')\Big) \\
&\quad \cdot \hat{g}(b - b' + n) \\
&= \theta \mathrm{e}(-bm\theta) \cdot \exp\Big(-\frac{\pi}{2\theta}(m\theta - d + d')^2\Big) \mathrm{e}\Big(\frac{1}{2}(b - b' + n)(m\theta - d - d')\Big) \\
&\quad \cdot \sqrt{\frac{\theta}{2}} \exp\Big(-\frac{\pi\theta}{2}(b - b' + n)^2\Big),
\end{aligned}
$$

giving the first equality. For the second equality,

$$
\begin{aligned}
\langle f_{b,d}, f_{b,d} \rangle_{C_A}(m, n) &= \int f_{b,d}(t - m)\, \overline{f_{b,d}(t)}\, \mathrm{e}(nt/\theta)\, dt \\
&= \int \mathrm{e}\big(b(t - m)\big) \exp\Big(-\frac{\pi}{\theta}(t - m - d)^2\Big) \\
&\qquad \cdot \mathrm{e}(-bt) \exp\Big(-\frac{\pi}{\theta}(t - d)^2\Big) \mathrm{e}(nt/\theta)\, dt \\
&= \mathrm{e}(-bm) \int \exp\Big(-\frac{\pi}{\theta}(t - m - d)^2 - \frac{\pi}{\theta}(t - d)^2\Big) \cdot \mathrm{e}(nt/\theta)\, dt \\
&= \mathrm{e}(-bm)\mathrm{e}(nd/\theta) \int \exp\Big(-\frac{\pi}{\theta}[t^2 + (t - m)^2]\Big) \mathrm{e}(nt/\theta)\, dt \\
&= \mathrm{e}(-bm + nd/\theta) \exp\Big(-\frac{\pi}{2\theta} m^2\Big) \int \exp\Big(-\frac{2\pi}{\theta}\Big(t - \frac{m}{2}\Big)^2\Big) \mathrm{e}(nt/\theta)\, dt \\
&= \mu^{mn/2} \mathrm{e}(-bm + nd/\theta) \exp\Big(-\frac{\pi}{2\theta} m^2\Big) \cdot \int \exp\Big(-\frac{2\pi}{\theta} t^2\Big) \mathrm{e}(nt/\theta)\, dt \\
&= \mu^{mn/2} \mathrm{e}(-bm + nd/\theta) \exp\Big(-\frac{\pi}{2\theta} m^2\Big) \cdot \hat{g}(-n/\theta) \\
&= \mu^{mn/2} \mathrm{e}(-bm + nd/\theta) \exp\Big(-\frac{\pi}{2\theta} m^2\Big) \cdot \sqrt{\frac{\theta}{2}} \exp\Big(-\frac{\pi}{2\theta} n^2\Big),
\end{aligned}
$$

as required. ■

Lemma 6.2 *Letting*

$$
E = -\frac{1}{2}(b + b')\theta, \quad F = -\frac{1}{2}(d + d')
$$
$$
K = (b - b')\theta, \quad L = -d + d'
$$

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one has

$$[\nu, a, c; k, t](\langle f_{b,d}, f_{b',d'}\rangle_{A}) = \frac{\theta^{3/2}}{\sqrt{2}} e(FK/\theta)\lambda^{kt/2} e(-Ek - Ft)\exp\big(-\pi(L^2 + K^2)/2\theta\big)$$
$$\cdot \exp(\pi Lk + \pi Kt)\exp\big(-\pi\theta(k^2 + t^2)/2\big)$$
$$\cdot \vartheta_3(2\pi E + \pi\theta a - \pi\theta t - \pi i\theta k + \pi i L, 2\theta\nu + 2i\theta)$$
$$\cdot \vartheta_3(2\pi F + \pi\theta c - \pi\theta k - \pi i\theta t + \pi i K, 2\theta\nu + 2i\theta)$$

and

$$\overline{[\nu, a, c; k, t]}\,(\langle f_{b,d}, f_{b,d}\rangle_{A_\beta})$$
$$= \sqrt{\frac{\theta}{2}}\mu^{kt/2} e(bk - dt/\theta)\exp\big(-\pi(k^2 + t^2)/2\theta\big)$$
$$\cdot \vartheta_3\Big(\frac{\pi a}{\theta} - \frac{\pi t}{\theta} - 2\pi b - i\frac{\pi k}{\theta}, \frac{2\nu}{\theta} + i\frac{2}{\theta}\Big)\vartheta_3\Big(\frac{\pi c}{\theta} - \frac{\pi k}{\theta} + 2\pi\frac{d}{\theta} - i\frac{\pi t}{\theta}, \frac{2\nu}{\theta} + i\frac{2}{\theta}\Big)$$

where ϑ_3 is the third theta function described in the Appendix.

Proof Consider the first of these. One has

$$[\nu, a, c; k, t](\langle f_{b,d}, f_{b',d'}\rangle_{A}) = \sum_{p,q} \lambda^{\nu(p^2+q^2)-2pq+ap+cq} J(2p - k, 2q - t)$$

where $J = \langle f_{b,d}, f_{b',d'}\rangle_{A}$. Lemma 6.1 gives

$$J(m, n) = \frac{\theta^{3/2}}{\sqrt{2}}\lambda^{mn/2} e(-bm\theta) e\Big(\frac{1}{2}m(b - b')\theta\Big) e\Big(-\frac{1}{2}(b - b')(d + d')\Big)$$
$$\cdot e\Big(-\frac{1}{2}n(d + d')\Big)\exp\Big(-\frac{\pi}{2\theta}(m\theta - d + d')^2\Big)\exp\Big(-\frac{\pi\theta}{2}(b - b' + n)^2\Big)$$

and, with E, F, K, and L as given in the statement of the lemma, this becomes

$$J(m, n) = \frac{\theta^{3/2}}{\sqrt{2}}\lambda^{mn/2} e(FK/\theta) e(Em) e(Fn) \cdot \exp\Big(-\frac{\pi}{2\theta}(m\theta + L)^2\Big)\exp\Big(-\frac{\pi}{2\theta}(n\theta + K)^2\Big).$$

So

$$J(2p - k, 2q - t) = \frac{\theta^{3/2}}{\sqrt{2}}\lambda^{2pq-pt-qk+kt/2} e(FK/\theta) e\big(E(2p - k)\big) e\big(F(2q - t)\big)$$
$$\cdot \exp\Big(-\frac{\pi}{2\theta}\big((2p - k)\theta + L\big)^2\Big)\exp\Big(-\frac{\pi}{2\theta}\big((2q - t)\theta + K\big)^2\Big)$$

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which when simplified, and after collecting the p and p^2 factors and q and q^2 factors separately, becomes

$$\begin{aligned}
&\lambda^{\nu(p^2+q^2)-2pq+ap+cq} J(2p-k,2q-t)\\
&\qquad = \frac{\theta^{3/2}}{\sqrt{2}} \mathrm{e}(FK/\theta)\lambda^{kt/2}\mathrm{e}(-Ek-Ft)\exp\bigl(-\pi(L^2+K^2)/2\theta\bigr)\\
&\qquad\quad \cdot \exp(\pi Lk+\pi Kt)\exp\bigl(-\pi\theta(k^2+t^2)/2\bigr)\\
&\qquad\quad \cdot \mathrm{e}^{\pi i(2\theta\nu+2i\theta)p^2} \exp\bigl(2ip(2\pi E-\pi\theta t-\pi i\theta k+\pi iL+\pi\theta a)\bigr)\\
&\qquad\quad \cdot \mathrm{e}^{\pi i(2\theta\nu+2i\theta)q^2} \exp\bigl(2iq(2\pi F-\pi\theta k-\pi i\theta t+\pi iK+\pi\theta c)\bigr).
\end{aligned}$$

Hence, using the definition of ϑ_3 given in the Appendix one has

$$\begin{aligned}
&[\nu,a,c;k,t](\langle f_{b,d}, f_{b',d'}\rangle_A)\\
&\quad = \frac{\theta^{3/2}}{\sqrt{2}} \mathrm{e}(FK/\theta)\lambda^{kt/2}\mathrm{e}(-Ek-Ft)\exp\bigl(-\pi(L^2+K^2)/2\theta\bigr)\\
&\qquad \cdot \exp(\pi Lk+\pi Kt)\exp\bigl(-\pi\theta(k^2+t^2)/2\bigr)\\
&\qquad \cdot \sum_p \mathrm{e}^{\pi i(2\theta\nu+2i\theta)p^2} \exp\bigl(2ip(2\pi E-\pi\theta t-\pi i\theta k+\pi iL+\pi\theta a)\bigr)\\
&\qquad \cdot \sum_q \mathrm{e}^{\pi i(2\theta\nu+2i\theta)q^2} \exp\bigl(2iq(2\pi F-\pi\theta k-\pi i\theta t+\pi iK+\pi\theta c)\bigr)\\
&\quad = \frac{\theta^{3/2}}{\sqrt{2}} \mathrm{e}(FK/\theta)\lambda^{kt/2}\mathrm{e}(-Ek-Ft)\exp\bigl(-\pi(L^2+K^2)/2\theta\bigr)\\
&\qquad \cdot \exp(\pi Lk+\pi Kt)\exp\bigl(-\pi\theta(k^2+t^2)/2\bigr)\\
&\qquad \cdot \vartheta_3(2\pi E-\pi\theta t-\pi i\theta k+\pi iL+\pi\theta a, 2\theta\nu+2i\theta)\\
&\qquad \cdot \vartheta_3(2\pi F-\pi\theta k-\pi i\theta t+\pi iK+\pi\theta c, 2\theta\nu+2i\theta)
\end{aligned}$$

which establishes the first equation of the Lemma. To establish the second relation, from Lemma 6.1 one has

$$\begin{aligned}
&\overline{[\nu,a,c;k,t]}(\langle f_{b,d}, f_{b,d}\rangle_{C_A})\\
&\qquad = \sum_{p,q} \mu^{\nu(p^2+q^2)-2pq+ap+cq}\langle f_{b,d}, f_{b,d}\rangle_{C_A}(2p-k,2q-t)\\
&\qquad = \sqrt{\frac{\theta}{2}} \sum_{p,q} \mu^{\nu(p^2+q^2)-2pq+ap+cq}\mu^{(2p-k)(2q-t)/2}\mathrm{e}\Bigl(-b(2p-k)+\frac{d}{\theta}(2q-t)\Bigr)\\
&\qquad\quad \cdot \exp\Bigl(-\frac{\pi}{2\theta}(2p-k)^2\Bigr)\exp\Bigl(-\frac{\pi}{2\theta}(2q-t)^2\Bigr)
\end{aligned}$$

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and expanding all the exponentials, including those of $\mu = \mathrm{e}(1/\theta)$, we can collect those factors containing p and p^2 and separating them from those containing q and q^2 as follows

$$= \sqrt{\frac{\theta}{2}}\mu^{kt/2}\mathrm{e}\Big(bk - \frac{dt}{\theta}\Big)\exp\Big(-\frac{\pi}{2\theta}(k^2+t^2)\Big)$$
$$\cdot \sum_p \exp\Big(2\pi i\frac{\nu}{\theta}p^2 - \frac{2\pi}{\theta}p^2\Big)\exp\Big(2\pi i\frac{a}{\theta}p - 2\pi i\frac{t}{\theta}p - 4\pi i b p + 2\pi\frac{k}{\theta}p\Big)$$
$$\cdot \sum_q \exp\Big(2\pi i\frac{\nu}{\theta}q^2 - \frac{2\pi}{\theta}q^2\Big)\exp\Big(2\pi i\frac{c}{\theta}q - 2\pi i\frac{k}{\theta}q + 4\pi i\frac{d}{\theta}q + 2\pi\frac{t}{\theta}q\Big)$$
$$= \sqrt{\frac{\theta}{2}}\mu^{kt/2}\mathrm{e}\Big(bk - \frac{dt}{\theta}\Big)\exp\Big(-\frac{\pi}{2\theta}(k^2+t^2)\Big)$$
$$\cdot \sum_p \exp\big(\pi i p^2(2\nu+2i)/\theta)\big)\exp\Big(i2p\Big\{\frac{\pi a}{\theta} - \frac{\pi t}{\theta} - 2\pi b - i\pi\frac{k}{\theta}\Big\}\Big)$$
$$\cdot \sum_q \exp\big(\pi i q^2(2\nu+2i)/\theta)\big)\exp\Big(i2q\Big\{\frac{\pi c}{\theta} - \frac{\pi k}{\theta} + 2\pi\frac{d}{\theta} - i\pi\frac{t}{\theta}\Big\}\Big)$$
$$= \sqrt{\frac{\theta}{2}}\mu^{kt/2}\mathrm{e}\Big(bk - \frac{dt}{\theta}\Big)\exp\Big(-\frac{\pi}{2\theta}(k^2+t^2)\Big)$$
$$\cdot \vartheta_3\Big(\frac{\pi a}{\theta} - \frac{\pi t}{\theta} - 2\pi b - i\pi\frac{k}{\theta}, \frac{2\nu+2i}{\theta}\Big)\cdot\vartheta_3\Big(\frac{\pi c}{\theta} - \frac{\pi k}{\theta} + 2\pi\frac{d}{\theta} - i\pi\frac{t}{\theta}, \frac{2\nu+2i}{\theta}\Big)$$

as required. ∎

7 Appendix: Theta Functions

For $z, t \in \mathbb{C}$ where $\mathrm{Im}(t) > 0$, the theta functions are given by (see for example [18, Chapter XXI])

(A1) $$\vartheta_1(z,t) = -i\sum_n (-1)^n \mathrm{e}^{\pi i t(n+\frac{1}{2})^2}\mathrm{e}^{i(2n+1)z}$$

(A2) $$\vartheta_2(z,t) = \sum_n \mathrm{e}^{\pi i t(n+\frac{1}{2})^2}\mathrm{e}^{i(2n+1)z}$$

(A3) $$\vartheta_3(z,t) = \sum_n \mathrm{e}^{\pi i t n^2}\mathrm{e}^{i2nz}$$

(A4) $$\vartheta_4(z,t) = \sum_n (-1)^n \mathrm{e}^{\pi i t n^2}\mathrm{e}^{i2nz},$$

where all summations are over the integers. It is easy to see that ϑ_2, ϑ_3, and ϑ_4 are even functions in the variable z. It is known that all the zeros of ϑ_3 are given by

(A5) $$\Big(\frac{\pi}{2} + m\pi + \Big(\frac{\pi}{2} + n\pi\Big)t, t\Big),$$

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where $m, n \in \mathbb{Z}$ and $\operatorname{Im}(t) > 0$ are arbitrary. We also have the following transformation formula of Jacobi for ϑ_3 [18, Section 21.51]

$$\vartheta_3(z,t) = (-it)^{-1/2} \mathrm{e}^{z^2/(\pi i t)} \vartheta_3\Big(\frac{z}{t}, -\frac{1}{t}\Big), \tag{A6}$$

which holds for arbitrary complex z and for $\operatorname{Im}(t) > 0$. (Here, one takes the principal square root.) The exact same transformation formula holds for ϑ_2 and ϑ_4. We shall call this the "inversion" formula for theta functions. It will be found useful to write down a similar formula for products of two theta functions:

$$\begin{aligned} \vartheta_3(x,t)\vartheta_3(y,t) &= (-it)^{-1/2} \mathrm{e}^{x^2/(\pi i t)} \vartheta_3\Big(\frac{x}{t}, -\frac{1}{t}\Big) (-it)^{-1/2} \mathrm{e}^{y^2/(\pi i t)} \vartheta_3\Big(\frac{y}{t}, -\frac{1}{t}\Big) \\ &= \frac{i}{t} \mathrm{e}^{(x^2+y^2)/(\pi i t)} \vartheta_3\Big(\frac{x}{t}, -\frac{1}{t}\Big) \vartheta_3\Big(\frac{y}{t}, -\frac{1}{t}\Big) \end{aligned} \tag{A7}$$

which we shall call the "double inversion" formula. We also have need for the (simple) relations

$$\vartheta_3(z,t) = \mathrm{e}^{\pi i t/4} \mathrm{e}^{iz} \vartheta_2\Big(z + \frac{\pi}{2} t, t\Big), \tag{A8}$$

$$\vartheta_3(z,t) = \mathrm{e}^{\pi i t/4} \mathrm{e}^{-iz} \vartheta_2\Big(z - \frac{\pi}{2} t, t\Big). \tag{A9}$$

$$\vartheta_3(z,t) = \mathrm{e}^{\pi i t} \mathrm{e}^{2iz} \vartheta_3(z + \pi t, t) \tag{A10}$$

$$\vartheta_3(z \pm \pi, t) = \vartheta_3(z,t) \tag{A11}$$

$$\vartheta_3\Big(z \pm \frac{\pi}{2}, t\Big) = \vartheta_4(z,t) \tag{A12}$$

$$\vartheta_2(z + \pi, t) = -\vartheta_2(z,t). \tag{A13}$$

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References

[1] F. P. Boca, *On the flip fixed point algebra in certain noncommutative tori*. Indiana J. Math. **45**(1996), 253–272.

[2] ———, *Projections in rotation algebras and theta functions*. Comm. Math. Phys. (1999), to appear.

[3] O. Bratteli, G. A. Elliott, D. E. Evans and A. Kishimoto, *Non-commutative spheres I*. Internat. J. Math. (2) **2**(1990), 139–166.

[4] ———, *Non-commutative spheres II: rational rotations*. J. Operator Theory **27**(1992), 53–85.

[5] O. Bratteli and A. Kishimoto, *Non-commutative spheres III. Irrational Rotations*. Comm. Math. Phys. **147**(1992), 605–624.

[6] A. Connes, *C^* algebre et geometrie differentielle*. C. R. Acad. Sci. Paris Ser. A–B **290**(1980), 599–604.

[7] ———, *Noncommutative Geometry*. Academic Press, 1994.

[8] G. Elliott and D. Evans, *The structure of the irrational rotation C^*-algebra*. Ann. of Math. **138**(1993), 477–501.

[9] C. Farsi and N. Watling, *Quartic algebras*. Canad. J. Math. (6) **44**(1992), 1167–1191.

[10] J. A. Jeong and H. Osaka, *Extremally rich C^*-crossed products and cancellation property*. J. Austral. Math. Soc. Ser. A (3) **64**(1998), 285–301.

[11] M. Rieffel, *C^*-algebras associated with irrational rotations*. Pacific J. Math. (2) **93**(1981), 415–429.

[12] ———, *The cancellation theorem for projective modules over irrational rotation algebras*. Proc. London Math. Soc. (3) **47**(1983), 285–302.

[13] W. Rudin, *Functional Analysis*. McGraw-Hill, 2nd edition, 1991.

[14] S. Walters, *Projective modules over the non-commutative sphere*. J. London Math. Soc. (2) **51**(1995), 589–602.

[15] ———, *Inductive limit automorphisms of the irrational rotation algebra*. Comm. Math. Phys. **171**(1995), 365–381.

[16] ———, *K-theory of non commutative spheres arising from the Fourier automorphism*. (1999), 39 pages, preprint.

[17] ———, *On the irrational quartic algebra*. C. R. Math. Rep. Acad. Sci. Canada (1999), 5 pages, to appear.

[18] E. T. Whittaker and G. N. Watson, *A course in modern analysis*. Cambridge Univ. Press, 4th edition, 1950.

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