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THE CLASSICAL LIMIT OF DYNAMICS FOR SPACES QUANTIZED BY AN ACTION OF $\mathbb{R}^d$

MARC A. RIEFFEL

ABSTRACT. We have previously shown how to construct a deformation quantization of any locally compact space on which a vector group acts. Within this framework we show here that, for a natural class of Hamiltonians, the quantum evolutions will have the classical evolution as their classical limit.

Introduction. Let M be a locally compact space, and let α be an action of $V = \mathbb{R}^d$ on M. Let $A = C_\infty(M)$, the C*-algebra of complex-valued continuous functions on M which vanish at infinity, and let α denote also the corresponding action of V on A. Let J be a skew-symmetric operator on V. Then J determines a "Poisson bracket" on A, and in [Rf] we have shown how to construct a strict deformation quantization of A into a one-parameter family, $A_\hbar$, of non-commutative C*-algebras, in "the direction of this Poisson bracket". The purpose of the present paper is to show that within this framework, the quantum evolution of the system which is determined by any Hamiltonian from a natural class, converges as $\hbar \longrightarrow 0$ to the classical evolution for that Hamiltonian, as one would expect.

Our aim here is not at all to obtain the strongest possible results—a more elaborate and lengthy analysis could deal with a far wider class of Hamiltonians than the ones we consider here. Rather our aim is to show how naturally this matter fits within the framework of [Rf]. The main argument, given in Section 2, is relatively simple and brief (though heavily dependent on the results in [Rf]).

The rest of this paper, contained in Sections 3 and 4, is concerned simply with showing that the Hamiltonians which we consider do have classical flows (evolutions) which exist for all time, and that these classical flows have the smoothness properties needed for our analysis in Section 2. This is a matter of some independent interest, as indicated in [B], and our situation permits us to remove the restriction to locally free actions which is required in some of the relevant places in [B]. (I thank Alan Weinstein for a suggestion which simplified my proof of the existence of the classical flow.) This gives a partial answer to Question 2.4.29 in [B]. In the Appendix we sketch how to obtain the existence of our classical flows as a consequence of a powerful Theorem of D. Robinson [Rs1]. (I thank George Elliott and Ola Bratteli for comments which led me to look at this paper of Robinson.) It is natural to carry this out in the more general context of an arbitrary Lie group acting on a space. This gives a more substantial partial answer to Question 2.4.29

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of [B]. But I have not seen how to use this approach to conveniently give the smoothness properties which we need for the proof of our main theorem (see Question 1 of [Rs1]), and so I have found it best to include the much more elementary approach given in Section 3, since it develops most of the tools needed for Section 4.

There is already an enormous literature concerned with the classical limit of quantum evolutions, mostly on $\mathbb{R}^{2n}$, and we will not try to review it here. Many references can be found by chasing back the references given in [E, Rr, W].

The construction of strict deformation quantizations developed in [Rf] works equally well for non-commutative C*-algebras, and so one can ask whether the results of the present paper extend to that case. The difficulty is that usually the Poisson bracket applied to a Hamiltonian does not give a derivation of the non-commutative algebra, and so one cannot expect it to generate a group of automorphisms analogous to the classical flow. In other words, I don't know how to even pose the question we consider here, for the more general situation. (In the very special case where all is sufficiently related to the center of the algebra one will obtain a derivation, and presumably the results of the present paper can be extended to that case; but at present it is not clear to me that this is of any particular interest and so I have not pursued it here.)

1. **The classical flows.** The purpose of this section is to describe the classical vector fields, and corresponding classical flows, which we will consider, and to state those properties of these classical flows which we will need later. We will defer the proofs of these properties until after our discussion of the classical limits of the quantum flows in the next section (where we will also relate our classical vector fields to Hamiltonians).

Let M, V, A and α be as in the introduction. It is the action α which gives M its "differential" structure. (It would certainly be of interest to consider actions of more general Lie groups than V, but I don't know how to construct deformation quantizations in that generality.) Let $C_b(M)$ denote the algebra of bounded continuous functions on M, the multiplier algebra of A. The evident action α of V on $C_b(M)$ is not in general strongly continuous. Let B (or $B(M)$, or $B(M,\alpha)$) denote the subspace of elements of $C_b(M)$ on which α is norm-continuous, so that B is the largest C*-subalgebra of $C_b(M)$ on which α is strongly continuous. Note that B is a unital C*-algebra containing A as an essential ideal. (We could now view B as the algebra of continuous functions on its maximal ideal space, which is compact and on which α gives an action, but this is not technically advantageous at this point.)

Let A^∞ and B^∞ denote the dense $*$-subalgebras of A and B consisting of smooth (*i.e.* infinitely differentiable) vectors [B] for α. As in Chapter 9 of [Rf] we will distinguish between V and its Lie algebra, denoting the latter by L. Thus for each $X \in L$ we have a corresponding derivation, α_X, on A^∞ and B^∞, given by the infinitesimal generator of the one-parameter group of operators corresponding to X. We will heuristically think of α_X as a smooth tangent vector field on M, and think of the tangent space at each point m of M as corresponding to L, by means of the α_X's followed by a point evaluation at m. Then we will think of a continuous real vector-field on M as being just a continuous L-valued function on M.

Fix for the rest of this paper an arbitrary positive-definite inner product on L. Heuristically this makes M into a Riemannian "manifold". Let $C_b(M,L)$ denote the Banach space of continuous bounded L-valued functions on M, equipped with the supremum norm using the inner product on L. We have the evident action α of V on $C_b(M,L)$. Much as above, denote by $B(M,L)$ the largest subspace of $C_b(M,L)$ on which α is strongly continuous, and by $B^\infty(M,L)$ the subspace of smooth vectors for α. We think of $B^\infty(M,L)$ as the space of smooth bounded vector fields on M.

Let $\Phi \in B^\infty(M,L)$. For any $f \in B^\infty$ and any $m \in M$ the function $x \longmapsto f\big(\alpha_x^{-1}(m)\big)$ is smooth on V, and in particular it will have a finite total derivative, $(Df)_m$, at $x = 0$. This is a linear functional on L. Thus we can define a map, δ_Φ, from B^∞ to itself by

$$(\delta_\Phi f)(m) = (Df)_m\big(\Phi(m)\big). \tag{1.1}$$

Then δ_Φ is a $*$-derivation of B^∞, in accordance with our heuristic view that Φ is a smooth vector field. To see this, note that the above is just a coordinate-free way of saying the following. Let $\{E_j\}$ be a basis for L, and let $\{\Phi_j\}$ denote the corresponding components of Φ. Note that $\Phi_j \in B^\infty$ for each j. Let $\partial_j = \alpha_{E_j}$, a $*$-derivation of B^∞. Then

$$\delta_\Phi f = \Sigma\, \Phi_j(\partial_j f) \tag{1.2}$$

for $f \in B^\infty$. It is now clear that δ_Φ is a $*$-derivation of B^∞, and that it carries A^∞ into itself. (If A were non-commutative, we could not expect δ_Φ to be a derivation unless each Φ_j were in the center of B.)

The main fact which we need is that each $\Phi \in B^\infty(M,L)$ determines a flow on M which exists for all time, and which carries B into itself. We formulate this as:

THEOREM 1.3. *Let M, α and B be as above, and let $\Phi \in B^\infty(M,L)$. Then δ_Φ is a pregenerator, that is, there is a (unique) strongly continuous one-parameter group, β, of automorphisms of B whose generator is the closure of δ_Φ. Furthermore, β carries A into itself, and so β comes from a flow on M (which we will also denote by β).*

We will defer the proof of this theorem to Section 3.

We also need control over the higher derivatives associated with the flow β. For each $f \in B^\infty$ we have the higher total derivatives, $D^k f$, of f. Thus each $D^k f$ is a function on M into the (symmetric) k-linear functionals from L to the complex numbers, defined by

$$(D^k f)_m(X_1, \ldots, X_k) = (\alpha_{X_1} \cdots \alpha_{X_k} f)(m)$$

for each $m \in M$. Each $D^k f$ is smooth and bounded, because of the definition of B^∞. Thus, by using the inner product on L to define the norm of k-linear functionals on L, we can define semi-norms $\|\ \|_{(k)}$ on B^∞ by

$$\|f\|_{(k)} = \|D^k f\|_\infty = \sup\{\|(D^k f)_m\| : m \in M\}.$$

We will need:

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THEOREM 1.4. *With notation as above, let β be the action on B^∞ for $\Phi \in B^\infty(M, L)$ as in Theorem 1.3. Then the action β is strongly continuous for each of the semi-norms $\| \ \|_{(k)}$ on B^∞. Furthermore, for any $f \in B^\infty$ the function $t \longmapsto \beta_t f$ is smooth for these semi-norms, and its first derivative is $(\delta_\Phi f) \circ \beta_t$.*

We remark that β will not usually be uniformly bounded for the above semi-norms.

We defer the proof of this theorem to Section 4.

2. **The classical limit.** As above, we let α be an action of V on a locally compact space M. Thus we have the algebras A and B, and their smooth versions A^∞ and B^∞. We let J be a skew-symmetric operator on L, so that J determines a Poisson bracket, $\{\ ,\ \}$, on A^∞ and B^∞. It is defined, in terms of a basis $\{E_j\}$ for L, by

$$\{f, g\} = \sum J_{jk} \alpha_{E_j}(f) \alpha_{E_k}(g).$$

For each "Planck's constant" $\hbar$ we let $A_\hbar$ and $B_\hbar$ denote the corresponding deformed C*-algebras, as constructed in [Rf]. Thus $A_\hbar$ has A^∞ as dense subspace, with product given there by

$$f \times_\hbar g = \iint \alpha_{\hbar J u}(f) \alpha_v(g) e(u \cdot v)\, du\, dv$$

(an oscillatory integral, with $e(t) = e^{2\pi i t}$), and with corresponding C*-norm. The involution is still complex-conjugation. We define $B_\hbar$ similarly. Then $A_\hbar$ will be an essential ideal in $B_\hbar$ by Proposition 5.9 of [Rf]. Furthermore, α gives an action of V on $A_\hbar$ and $B_\hbar$ by Proposition 5.11 of [Rf], and the corresponding subspaces of smooth vectors will be exactly A^∞ and B^∞ as vector spaces, by Theorem 7.1 of [Rf].

The Hamiltonians which we will consider consist of the real-valued functions in B^∞. So fix such a real-valued $H \in B^\infty$. The mapping $f \longmapsto \{H, f\}$ is a derivation of A^∞ and B^∞. If for a basis $\{E_j\}$ for L we set

$$\Phi(m) = \sum J_{jk} \Big(\alpha_{E_j}(H) \Big)(m) E_k,$$

we see that Φ is a function from M to L of the kind considered in the first section. In particular its coefficient functions are in B^∞ (and real) so that $\Phi \in B^\infty(M, L)$. Furthermore, it is clear that $\{H, f\} = \delta_\Phi(f)$ for each $f \in B^\infty$. Thus Φ is the "Hamiltonian vector field" for H.

According to Theorem 1.3, δ_Φ determines a flow, β, on M, with corresponding strongly continuous one-parameter action on A and B. This is the "Hamiltonian flow" for H.

For each $\hbar$ we let $[\ ,\ ]_\hbar$ denote the ordinary commutator for the corresponding product in B^∞, so that

$$[f, g]_\hbar = f \times_\hbar g - g \times_\hbar f.$$

Set $H^\hbar = (-\pi/\hbar)H$, viewed as a self-adjoint element of $B_\hbar$. (The $-\pi$ comes from our conventions in [Rf] for the definition of $\times_\hbar$, given above.) Then the map $f \longmapsto [iH^\hbar, f]_\hbar$ is a $*$-derivation of $A_\hbar$ and $B_\hbar$ which is bounded (but with norms going to $+\infty$ as $\hbar \longrightarrow 0$). Thus $H^\hbar$ determines a one-parameter group, $\beta^\hbar$, of $*$-automorphisms of $A_\hbar$ and $B_\hbar$, the

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corresponding quantum flow. This flow consists of inner automorphisms of $B_\hbar$. For let $u_t^\hbar = \exp_\hbar(itH^\hbar)$ for each $t \in \mathbb{R}$, where $\exp_\hbar$ denotes the exponential defined by the usual power series, but using the product $\times_\hbar$ in $B_\hbar$. Then $u_t^\hbar$ is a unitary element of $B_\hbar$. By the usual calculations we will have

$$\beta_t^\hbar(f) = u_t^\hbar \times_\hbar f \times_\hbar u_{-t}^\hbar$$

for all $f \in B_\hbar$. Again it is clear that $\beta^\hbar$ carries the ideal $A_\hbar$ into itself. But notice that $\beta^\hbar$ is not only strongly continuous, but actually norm (*i.e.* uniformly) continuous (since $H^\hbar$ is bounded).

The main theorem of this paper is:

THEOREM 2.1. *With notation as above, for any $f \in B^\infty$ we have*

$$\|\beta_t^\hbar f - \beta_t f\|_\hbar \longrightarrow 0 \quad \text{as} \quad \hbar \longrightarrow 0$$

for each $t \in \mathbb{R}$, with the convergence being uniform in t over any finite interval.

It is in this sense that, within our framework, the quantum flow has the classical flow as its classical limit.

We remark that in the proof we will see how to obtain specific estimates for the convergence.

PROOF. Let I denote the interval $[-1,1]$. We only need consider $\hbar$'s in I. We will denote by $B^{(k)}$ the space B^∞ equipped with the norm $\| \ \|_k$ which is the sum of $\| \ \|_\infty$ with the semi-norms $\| \ \|_{(j)}$ (defined near the end of Section 1) for $j \leq k$. This norm is equivalent to the norm used in [Rf], defined on p. 1 of [Rf]. We choose k large enough that we can apply the little argument near the beginning of the Proof of Theorem 9.3 of [Rf] which shows that there is a constant, c, independent of $f \in B^\infty$, such that

$$\|f\|_\hbar \leq c\|f\|_k \tag{2.2}$$

for all $\hbar \in I$. Fix $f \in B^\infty$. From Theorem 1.4 we know that $t \longmapsto \beta_t f$ can be viewed as a smooth function on V with values in $B^{(k)}$, whose first derivative is $\{H, \beta_t f\}$. From 2.2 it follows that $t \longmapsto \beta_t^f$ is smooth as a function with values in $B_\hbar$, for each $\hbar \in I$, with the same first derivative.

Fix $\hbar \in I$. Then the smooth function $t \longmapsto u_t^\hbar$ with values in $B_\hbar$ clearly has as derivative $iH^\hbar \times_\hbar u_t^\hbar$. We now adapt to our situation a device which is commonly used to compare semigroups of operators. As an example quite close to our present situation, see the Proof of Equation 16 of [E]. (Undoubtedly the full expansion of Equation 16 could be obtained in our framework too.)

Fix t, and define ϕ for this t by

$$\phi(s) = u_s^\hbar \times_\hbar (\beta_{t-s} f) \times_\hbar u_{-s}^\hbar.$$

From the comments above, ϕ is a differentiable function with values in $B_\hbar$, whose derivative is given by

$$\phi'(s) = u_s^\hbar \times_\hbar \Big((\pi / i\hbar)[H, \beta_{t-s} f]_\hbar - \{H, \beta_{t-s} f\} \Big) \times_\hbar u_{-s}^\hbar.$$

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Notice that $\phi(0) = \beta_t f$ while $\phi(t) = \beta_t^{\hbar} f$. Thus

$$\begin{aligned}\|\beta_t^{\hbar} f - \beta_t f\|_{\hbar} &= \Big\| \int_0^t \phi'(s)\, ds \Big\|_{\hbar} \\ &\leq |t| \sup\{\|(\pi/i\hbar)[H, \beta_{t-s} f]_{\hbar} - \{H, \beta_{t-s} f\}\|_{\hbar} : |s| \leq |t|\}.\end{aligned}$$

From 2.2 above it is clear that it now suffices to control the size of

$$\|(\pi/i\hbar)[H,g]_{\hbar} - \{H,g\}\|_k$$

where $g = \beta_{t-s} f$.

We now need to use the same arguments as in the Proof of Theorem 9.3 of [Rf], but keeping track of β_{t-s} so as to get an estimate which is uniform in s. For any multi-index μ let ∂^μ denote the corresponding (higher) partial derivative for the basis $\{E_j\}$ chosen earlier. The norm $\|\ \|_k$ is equivalent to a finite linear combination of the semi-norms $f \longmapsto \|\partial^\mu f\|_\infty$ for various μ's. So it suffices to obtain suitable estimates for these semi-norms. But, just as in the Proof of Theorem 9.3 of [Rf], repeated application of Leibniz' rule shows that

$$\Big\|\partial^\mu\big((\pi/i\hbar)[H,g]_{\hbar} - \{H,g\}\big)\Big\|_\infty$$

is dominated by a finite linear combination of terms of form

$$\|(\pi/i\hbar)[\partial^\nu H, \partial^\lambda g]_{\hbar} - \{\partial^\nu H, \partial^\lambda g\}\|_\infty,$$

where the coefficients of the linear combination do not depend on H, g or $\hbar$. But H is fixed throughout, and so for notational simplicity we can set $F = \partial^\nu H$ for any given ν. Then we see that it suffices to obtain for any given multi-index λ, a suitable estimate for the size of

$$(\pi/i\hbar)[F, \partial^\lambda g]_{\hbar} - \{F, \partial^\lambda g\},$$

where we remember that $g = \beta_{t-s} f$.

To bring all this even closer to the Proof of Theorem 9.3 of [Rf], we use the commutativity of B to write

$$[F, \partial^\lambda g]_{\hbar} = \big(F \times_{\hbar} (\partial^\lambda g) - F(\partial^\lambda g)\big) - \big((\partial^\lambda g) \times_{\hbar} F - (\partial^\lambda g)F\big).$$

Then we see that it suffices to obtain a suitable estimate for the size of

$$(2.3) \qquad (2\pi/i\hbar)\big(F \times_{\hbar} (\partial^\lambda g) - F(\partial^\lambda g)\big) - \{F, \partial^\lambda g\}$$

and a similar term. But by the last displayed equation in the Proof of Theorem 9.3 of [Rf] we find that (2.3) is equal to $\hbar 2\pi i R(\hbar)$ where (after omitting an erroneous subscript J)

$$R(\hbar) = (2\pi i)^{-2} \sum J_{pj} J_{qk} \int_0^1 \int_0^1 (a_{kj} \times_{\hbar rs} b_{qp})\, ds\, r\, dr,$$

and where $a_{kj} = \alpha_{E_k} \alpha_{E_j}(F)$ and $b_{qp} = \alpha_{E_q} \alpha_{E_p}(\partial^\lambda q)$. From Proposition 2.2 of [Rf] it then follows that each summand of $R(\hbar)$ is dominated for the norm $\|\ \|_k$ by $c\|F\|_m \|\partial^\lambda g\|_m$ for

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an integer m and constant c independent of F and g. Note further that $\|\partial^\lambda g\|_m \leq d\|g\|_n$ for a suitable integer n and constant d. But $\|g\|_n = \|\beta_{t-s} f\|_n$, which is uniformly bounded for $t - s$ ranging in any finite interval, because of the continuity given by Theorem 1.4. It follows that $\|R(\hbar)\|_n$ is uniformly bounded for $\hbar \in I$, for our fixed f, and for t in any fixed finite interval. Because our error term involved $\hbar R(\hbar)$, we thus obtain the desired convergence as $\hbar \longrightarrow 0$. ∎

We remark that one can follow the above analysis more carefully to obtain a specific bound for $\|R(\hbar)\|_n$.

We also remark that with somewhat more care we could use the commutativity of B and the symmetry of $[H, \beta_{t-s} f]$ and $\{H, \beta_{t-s} f\}$ to obtain an error term of form $\hbar^2 R(\hbar)$ rather than $\hbar R(\hbar)$, as is usually obtained in discussions of related situations in the literature, such as expansion 16 of [E]. This possibility was not discussed in [Rf] since it is not available when A is not commutative.

The following comments were stimulated by conversations with A. Vershik. Consider the ordinary 2-torus T^2, and let L_0 denote $C^\infty(T^2)$ as Lie algebras with the standard Poisson bracket. The process of associating to elements of L_0 their Hamiltonian vector fields is a Lie algebra homomorphism of L_0 onto the Lie algebra of those smooth vector fields which generate area-preserving diffeomorphisms of T^2. (This homomorphism is an isomorphism once one factors by the subspace of constant functions, the center of L_0.) As seen in Example 10.2 of [Rf], the deformation quantization of the symplectic space T^2 for the action of $\mathbb{R}^2$ gives the quantum 2-tori (the rotation algebras) A_θ (where $\theta = \hbar$). Let L_θ denote A_θ viewed just as a Lie algebra with its commutator bracket, forgetting the associative algebra structure. It is remarked in example 3e of [V] (and in references given there and in [S]) that L_θ tends to L_0 as θ goes to 0 (with similar statements for other crossed product algebras). We can view Theorem 9.3 of [Rf], applied to T^2, as then making this intuition rigorous. In the same way, Theorem 2.1 of the present paper, applied to T^2, goes in the direction of saying rigorously that the group of inner automorphisms of A_θ coming from unitaries in the connected component of the unitary group of A_θ, tends to the group of area-preserving diffeomorphisms of T^2 as θ goes to 0. (For information on the structure of this unitary group of A_θ see [Rf1].)

3. **The Proof of Theorem 1.3.**

We remark that with suitable care the steps below can be carried out with V replaced by a general connected Lie group. For simplicity of exposition we treat only V here since this is all we need, but see the appendix for the general case.

LEMMA 3.1. *Let N be a locally compact space, let $M = V \times N$, and let α be the action of V on M coming from the translation action of V on itself. Let $\Phi \in B^\infty(M, L)$, and view δ_Φ as a derivation of A^∞ (not B^∞). Then δ_Φ is the pregenerator of a one-parameter action β on A, with corresponding flow β on M. Furthermore, β carries A^∞ into itself, and the flow β on M carries each leaf $V \times \{n\}$ into itself.*

PROOF. Because here the action α of V on M is free, this lemma is essentially a special case of Theorem 2.4.26 of [B] (which is closely related to results in [BD]). Its proof

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is basically just a matter of applying the usual existence theorem for flows generated by Lipschitz vector fields on $\mathbb{R}^d$ to obtain a global flow on each leaf $V \times \{n\}$. Then one applies the theorem concerning the continuous dependence of such flows on their initial conditions to show that, as n varies, the corresponding flows fit together continuously to give a flow on M. From this approach we see that β carries each leaf of M into itself. Theorem 2.4.26 of [B] also gives that A^∞ is a core for the generator of the action β, and that for each $f \in A^\infty$ we have

$$(d/dt)|_{t=s}\beta_t(f) = \beta_s\big(\delta_\Phi(f)\big). \tag{3.2}$$

Since in our case δ_Φ carries A^∞ into itself, a simple induction argument shows that β carries A^∞ into itself. ∎

LEMMA 3.3. *Let N, M, α, Φ and β be as in the previous lemma. Let β also denote the corresponding one-parameter action on $C_b(M)$. Then β carries $B = B(M, \alpha)$ into itself, is strongly continuous on B, and carries B^∞ into itself. Let δ_Φ be the derivation of B^∞ defined earlier. Then δ_Φ is a pregenerator for β acting on B.*

PROOF. Note that we cannot directly invoke Theorem 2.4.26 of [B] here because, in general, the action α on the maximal ideal space of B will not be locally free. In fact, the most difficult part of the proof is to show that each β_t actually carries B into itself.

Because of the special form of M, we can initially work on each leaf $V \times \{n\}$ separately. For simplicity of notation we temporarily consider our n to be fixed, and omit it from the notation, and thus work on V itself. But we must be careful to obtain estimates which are uniform in $n \in N$.

By restriction we view Φ as an element of $B^\infty(V, L)$. Thus it is smooth on V in the usual sense. Now $B = B(V)$ will consist exactly of the uniformly continuous functions on V. Thus to show that B is carried into itself by β it clearly suffices to obtain an estimate of the form

$$\|\beta_t(x) - \beta_t(y)\| \leq K_t\|x - y\|$$

for all $x, y \in V$, where K_t is a constant independent of x and y. Fix $x, y \in V$ with $x \neq y$, and let $w = y - x$. Let g be the V-valued function on $\mathbb{R}^2$ defined by

$$g(t, r) = \beta_t(x + rw).$$

From the usual facts about solutions of differential equations, g is smooth since Φ is. Note that for fixed t the path $g(t, r)$ goes from $\beta_t(x)$ to $\beta_t(y)$ as r goes from 0 to 1. We consider the length, $L(t)$, of this path. We use ideas from the first variational equation for ordinary differential equations (*e.g.* p. 190 of [A]).

Let $h = \partial g/\partial r$, so that

$$L(t) = \int_0^1 \|h(t, r)\|\, dr.$$

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Now, by the fact that partial derivatives commute, we have

$$\begin{aligned}\partial h/\partial t &= (\partial/\partial r)(\partial g/\partial t)\\ &= (\partial/\partial r)(\Phi\circ g)\\ &= \big((D\Phi)\circ g\big)\circ(\partial g/\partial r)\\ &= \big((D\Phi)\circ g\big)\circ h,\end{aligned}$$

where $D\Phi$ is the usual total derivative of Φ. Note that since β_t is a diffeomorphism and $w \neq 0$, h never takes value 0, and so the function $\|h(t,r)\|$ is smooth, as is then L. A little calculation then shows that

$$|(dL/dt)(t)| \leq \int_0^1 \|(\partial h/\partial t)(t,r)\|\,dr \leq \|D\Phi\|_\infty L(t).$$

Consequently

$$L(t) \leq L(0)\,e^{t\|D\Phi\|_\infty} = \|x-y\|\,e^{t\|D\Phi\|_\infty}.$$

Since $L(t)$ is the length of some curve from $\beta_t(x)$ to $\beta_t(y)$, it follows that

$$\|\beta_t(x)-\beta_t(y)\| \leq \|x-y\|\,e^{t\|D\Phi\|_\infty}.$$

This is an estimate of the desired type, and so as indicated above, β carries B into itself. Notice that we have used the hypotheses that $\Phi \in B^\infty(V,L)$ to ensure that $\|D\Phi\|_\infty$ is finite.

We return now to the general case in which $M = V \times N$. As long as we now interpret $\|D\Phi\|_\infty$ as a supremum over all of M, which is still finite since $\Phi \in B^\infty(V,L)$, we see that the above inequality is uniform over all the leaves. It follows easily that β carries B into itself in this case also.

We must now show that the action β on B is strongly continuous. By multiplying elements of B^∞ by elements of A^∞ which have value 1 on neighborhoods of various points, we see that every element of B^∞ agrees locally with an element of A^∞. It follows that for any $f \in B^\infty$ and any $m \in M$ we have

$$(d/dt)\Big(f\big(\beta_t(m)\big)\Big) = (\delta_\Phi f)\big(\beta_t(m)\big),$$

since this can be viewed as a local statement. In particular, the derivative on the left exists. Consequently

$$f\big(\beta_t(m)\big) = f(m) + \int_0^t (\delta_\Phi f)\big(\beta_s(m)\big)\,ds, \tag{3.4}$$

so that

$$\|\beta_t f - f\|_\infty \leq |t|\,\|\delta_\Phi f\|_\infty.$$

Thus β is strongly continuous on B^∞. Since β is isometric and B^∞ is dense in B, it follows that β is strongly continuous on B.

We must now show that B^∞ is contained in the domain of the infinitesimal generator of β, and that on B^∞ this generator agrees with δ_Φ. We argue much as in the Proof of

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Lemma 2.4.3 of [B]. Let $f \in B^\infty$, so that $\delta_\Phi f \in B^\infty$. Because we now know that β is strongly continuous on B, the integral $\int_0^t \beta_s(\delta_\Phi f)\,ds$ is well-defined for the supremum norm on B. Now evaluation at any point $m \in M$ is continuous for this norm, and so can be brought inside the integral. From (3.4) it then follows that

$$\beta_t f - f = \int_0^t \beta_s(\delta_\Phi f)\,ds.$$

From this it follows immediately that

$$(d/dt)|_{t=0}(\beta_t f) = \delta_\Phi f$$

for the norm on B, so that f is in the domain of the generator of β. Furthermore, we see that on B^∞ this generator agrees with δ_Φ.

It follows readily that equation (3.2) holds for any $f \in B^\infty$. From this equation and the fact that δ_Φ carries B^∞ into itself, it follows by a simple induction argument that β carries B^∞ into itself. We can now apply Corollary 3.1.7 of [BR] to conclude that B^∞ is a core for the generator of β, *i.e.* that this generator is the closure of δ_Φ. ∎

CONCLUSION OF THE PROOF OF THEOREM 1.3. Let M, α, A and B be as in the statement of Theorem 1.3. Let $P = V \times M$, and let τ denote the action of V on P coming from translation on V. Let η be the map from P to M defined by $\eta(x,m) = \alpha_x(m)$. Since η is surjective, it gives an isometric isomorphism, still denoted by η, of $B(M)$ onto a subalgebra of $B(P)$. When convenient we will simply identify $B(M)$ with this subalgebra. Note that η is equivariant for α and τ. Thus $B(M)$ is a τ-invariant subalgebra of $B(P)$. Define (as suggested to me by Alan Weinstein) an action, γ, of V on P by $\gamma_y(x,m) = \big(x-y, \alpha_y(m)\big)$. Note that $\eta \circ \gamma_y = \eta$ for any $y \in V$, and that the γ-orbits of points in P are exactly the η-preimages of points in M. Let $\Phi \in B^\infty(M,L)$ be given, and set $\hat{\Phi} = \Phi \circ \eta$. It is easily seen that $\hat{\Phi} \in B^\infty(P,L)$, and clearly $\hat{\Phi} \circ \gamma_y = \hat{\Phi}$ for all $y \in V$. Let $\hat{\beta}$ denote the flow for $\hat{\Phi}$ on P, whose existence is assured by Lemma 3.1, and which carries $B^\infty(P)$ into itself by Lemma 3.3. Fix $m \in M$ and $y \in V$. Then γ_y gives a bijection of $V \times \{m\}$ onto $V \times \{\alpha_y(m)\}$, and under this bijection the restrictions of $\hat{\Phi}$ agree. By the uniqueness theorem for ordinary differential equations, the corresponding flows must agree. But by construction these flows are just given by $\hat{\beta}$. Thus $\hat{\beta}$ commutes with each γ_y. It follows that for each t the homeomorphism $\hat{\beta}_t$ carries each γ-orbit into exactly another γ-orbit, and so determines a "flow", β, on M. It is easily seen that η is an open map (γ is a free and proper action). From this and the continuity of $\hat{\beta}$ it follows that β is continuous, so that it really is a flow.

Note that by construction η is equivariant for β and $\hat{\beta}$. Since η carries $B(M)$ isometrically into $B(P)$ and $\hat{\beta}$ carries $B(P)$ into itself and is strongly continuous on $B(P)$, it follows that β carries $B(M)$ into itself and is strongly continuous there. (Note that η does not carry $A(M)$ into $A(P)$.) For the same reasons, β will carry $B^\infty(M)$ into itself. A straight-forward calculation using the equivariance of η for τ and α shows that for $f \in B^\infty(M)$ we have

$$\delta_{\hat{\Phi}}(f \circ \eta) = (\delta_\Phi f) \circ \eta.$$

Now for fixed $m \in M$ we have $(\beta_t f)(m) = \big(\hat{\beta}_t(f \circ \eta)\big)(0, m)$, and so

$$(d/dt)|_{t=0}(\beta_t f)(m) = (d/dt)|_{t=0}\big(\hat{\beta}_t(f \circ \eta)\big)(0, m) = (\delta_\Phi f)(m).$$

We can now argue as in the last parts of the Proof of Lemma 3.3 to conclude that $B^\infty(M)$ is in the domain of the generator of β, that on $B^\infty(M)$ this generator agrees with δ_Φ, and that $B^\infty(M)$ is a core for this generator. ■

4. **The proof of Theorem 1.4.** Exactly as in the conclusion of the Proof of Theorem 1.3, let $P = V \times M$ with action τ of V, so that $B(M)$ is identified via ν with a C*-subalgebra of $B(P)$, and β on $B(M)$ is just the restriction of $\hat{\beta}$ on $B(P)$. Since the action α of V on $B(M)$ is just the restriction of the action τ on $B(P)$, the semi-norms defined earlier in terms of α will just be the restrictions to $B(M)$ of the corresponding semi-norms for τ on $B(P)$. Thus we see that it suffices to prove Theorem 1.4 for the setting of Lemma 3.3. This means that it suffices to prove the theorem on each leaf $V \times \{m\}$, as long as we obtain uniform estimates in m. Thus we consider first the case $M = V$ with α the action τ of translation, and we consider Φ and β as being on V. Then β can be viewed as a function from $\mathbb{R} \times V$ to V which is smooth. We will let D^k denote k-th derivative for variables in V. We identify L and V in the usual way. Then for each fixed $(t, x) \in \mathbb{R} \times V$ the expression $(D^k\beta)(t, x)$ is a symmetric k-linear map from V to V. We use the inner product on V to define the norm of this map. The proof of the following lemma is of a type familiar in the theory of ODE's.

LEMMA 4.1. *For any $k > 0$ and any finite interval I about 0 there is a constant K such that*

$$\|(D^k\beta)(t, x)\| \leq K$$

for all $x \in V$ and $t \in I$.

PROOF. We argue by induction on k. Let ∂ denote derivatives with respect to t. Thus $\partial\beta = \Phi \circ \beta$, where here and in the following we work pointwise. Now D commutes with ∂, so

$$\partial(D\beta) = D(\partial\beta) = D(\Phi \circ \beta) = \big((D\Phi) \circ \beta\big) \circ D\beta$$

by the chain rule. Thus

$$(D\beta)(t, x) = (D\beta)(0, x) + \int_0^t \Big(D\Phi\big(\beta_s(x)\big)\Big) \circ \big((D\beta)(s, x)\big)\, ds,$$

and so, since $\beta_0(y) = y$ for all y,

$$\|D\beta(t, x)\| \leq 1 + \|D\Phi\|_\infty \int_0^t \|(D\beta)(s, x)\|\, ds.$$

By Gronwall's inequality (Lemma 4.1.7 of [A]) we obtain

$$\|D\beta(t, x)\| \leq \exp(t\|D\Phi\|_\infty),$$

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with the right-hand side independent of x. Thus the Proof of Lemma 4.1 is complete for $k = 1$.

In the proof for higher k we argue by induction. Thus assume that there is a constant K_k such that for each $j \le k-1$ we have $\|(D^j\beta)(t,x)\| \le K_k$ for $t \in I$ and $x \in V$. Much as above, we have

$$\partial(D^k\beta) = D^k(\partial\beta) = D^k(\Phi \circ \beta).$$

We now invoke the chain rule for higher derivatives, as given for example on p. 92 of [A]. Suppressing some of the notation given there, we obtain

$$\partial(D^k\beta) = \big((D\Phi) \circ \beta\big) \circ (D^k\beta) + G_k,$$

where

$$G_k = \sum_{m=2}^{k} \sum \big((D^m\Phi) \circ \beta\big) \circ \big(\{(D^{\ell_m}\beta)\}\big),$$

with each $\ell_m \le k-1$. By the induction hypothesis, $\|(D^{\ell_m}\beta)(t,x)\| \le K_k$ for each $t \in I$ and $x \in V$. By our hypothesis on Φ each $\|(D^m\Phi)(x)\|$ is uniformly bounded over V. It follows that there is a constant, L_k, such that

$$\|G_k(t,x)\| \le L_k$$

for all $t \in I$ and $x \in V$. Now

$$(D^k\beta)(t,x) = (D^k\beta)(0,x) + \int_0^t \big(D^k(\Phi \circ \beta)\big)(s,x)\, ds.$$

But for $k \ge 2$ we have $(D^k\beta)(0,x) = 0$. Thus

$$\begin{aligned}\|(D^k\beta)(t,x)\| &\le \int_0^t \Big(\big\|(D\Phi)\big(\beta_s(x)\big)\big\|\, \|(D^k\beta)(s,x)\| + \|G_k(s,x)\|\Big)\, ds \\ &\le \|D\Phi\|_\infty \int_0^t \|(D^k\beta)(s,x)\|\, ds + cL_k,\end{aligned}$$

where c is the length of the interval I. Thus again by Gronwall's inequality we find that

$$\|(D^k\beta)(t,x)\| \le cL_k \exp(t\|D\Phi\|_\infty). \qquad \blacksquare$$

PROOF OF THEOREM 1.4. We deal first with the case $M = V$. So for $f \in B^\infty$ we view $D^k f$ as a function from V into the normed space of k-linear maps from V into the complex numbers. Let I be a finite interval about 0 in $\mathbb{R}$. As in the above lemma, we let ∂ denote derivatives in t. Then, working pointwise, we have for any $f \in B^\infty$

$$\partial\big(D^k(f \circ \beta)\big) = D^k\big(\partial(f \circ \beta)\big) = D^k\big((\delta_\Phi f) \circ \beta\big).$$

Set $g = \delta_\Phi f$, so that $g \in B^\infty$. Then if we apply the chain rule to $D^k(g \circ \beta)$, much as in the Proof of Lemma 4.1, and if we apply the conclusion of Lemma 4.1, we find that there is a constant, K, independent of g and $x \in V$, such that

$$\|\big(D^k(g \circ \beta)\big)(t,x)\| \le K\Big(\sum_{j \le k} \|D^j g\|_\infty\Big)$$

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for all $t \in I$. Application of the chain rule to $D^j g = D^j(\delta_\Phi f)$ shows that there is a constant L, depending only on Φ and k, such that

$$\sum_{j\le k} \|D^j g\|_\infty \le L \sum_{j\le k+1} \|D^j f\|_\infty.$$

Since

$$\big(D^k(f\circ\beta)\big)(t,x) - (D^k f)(x) = \int_0^t D^k(g\circ\beta)(s,x)\,ds,$$

it follows that

$$\|\big(D^k(f\circ\beta_t - f)\big)(x)\| \le tKL \sum_{j\le k+1} \|D^j f\|_\infty$$

for all $t \in I$ and $x \in V$. But the right-hand side is independent of x, so we obtain the desired strong continuity in this case.

Now consider the case where $M = V \times N$ as in Lemma 3.3. On each leaf $V \times \{n\}$ we will have the above inequality, and the constant KL depends only on Φ and its derivatives on that leaf. But by examining a bit more carefully the origin of KL and by using the fact that Φ and its derivatives are assumed to be uniformly bounded over all of M, we see that we can find finite KL which works uniformly over all of M. The comments at the beginning of this section complete the proof of strong continuity. The proof of the remaining facts in the statement of Theorem 1.4 is then essentially the same as the proof of the similar facts in Lemma 3.3. ∎

Appendix.

Lipschitz Flow. We sketch here how to use a powerful theorem of Derek Robinson [Rs1] to prove Theorem 1.3. We carry this out in the more general setting of an action of an arbitrary (connected) Lie group G, with Lie algebra $\mathfrak{g}$. Let α be an action of G on a locally compact space M, with corresponding action α on $A = C_\infty(M)$, and on B, the largest algebra of bounded continuous functions on M on which α is strongly continuous. In the same way, we define $B(M,\mathfrak{g})$, a generalization of our earlier $B(M,L)$. We have the corresponding spaces of smooth vectors A^∞, B^∞, and $B^\infty(M,\mathfrak{g})$. Much as earlier, we view elements of $B^\infty(M,\mathfrak{g})$ as "smooth bounded vector fields" on M.

Let $\Phi \in B^\infty(M,\mathfrak{g})$. Then, much as done earlier, Φ determines a derivation, δ_Φ, on A^∞ and B^∞.

THEOREM A1. *The derivation δ_Φ is the pregenerator of a one-parameter group, β, of automorphisms of B, which carries A into itself, and so determines a flow, β, on M.*

PROOF. Robinson's theorem tells us that to show that δ_Φ is a pregenerator it suffices to verify two conditions. The first is the usual condition that δ_Φ be conservative. The second and crucial condition is that δ_Φ be Lipschitz for the action α. We recall here briefly what this means.

As we did earlier for L, choose an arbitrary inner product on $\mathfrak{g}$. This can be translated around G to define a left-invariant Riemannian metric on G. We denote the corresponding length function [Rs2] by $|x|$. Let γ be an action of G on a Banach space U. Let U^∞

denote the smooth vectors for γ. For $u \in U^\infty$ let Du denote the linear map from $\mathfrak{g}$ to U defined by $(Du)(X) = \gamma_X(u)$. We use the inner product on $\mathfrak{g}$ to define $\|Du\|$, and we set $\|u\|_1 = \|u\| + \|Du\|$.

DEFINITION A2 [RS1]. With notation as above, an operator $T\colon U^\infty \longrightarrow U$ is said to be Lipschitz if there are constants $\delta > 0$ and K such that

$$\| \operatorname{ad}_{\gamma_x}(T)u\| \leq K|x|\|u\|_1$$

for all $u \in U^\infty$ and all $x \in G$ with $|x| < \delta$. (Here $\operatorname{ad}_{\gamma_x}(T) = \gamma_x \circ T - T \circ \gamma_x$.)

We will need the following fact, whose proof is a straightforward argument using the lengths of curves in G.

PROPOSITION A3. *With notation as above, each operator γ_X for $X \in \mathfrak{g}$ is a Lipschitz operator for γ.*

We now check that δ_Φ and α satisfy Robinson's conditions. That δ_Φ is conservative is seen in the usual way by considering, for any $f \in B^\infty$, the functional consisting of evaluation at a point of the maximal ideal space of B at which f takes its maximal absolute value. We now sketch the verification that δ_Φ is a Lipschitz operator for α. It is clear from the definition that sums of Lipschitz operators are again Lipschitz operators. Thus it suffices to show that any operator of the form $h\alpha_X$ for $h \in B^\infty$ and $X \in \mathfrak{g}$ is Lipschitz. But such an operator is the composition of the operators corresponding to (multiplication by) h and α_X. It is now convenient for us to make:

DEFINITION A4. With notation as earlier, we say that $u \in U$ is a Lipschitz vector for the action γ if there are constants $\delta > 0$ and K such that

$$\|\gamma_x(u) - u\| \leq K|x|$$

for $|x| < \delta$.

Then a straightforward argument again using the length of curves in G yields the first part of the following proposition. The second part then follows easily from the first.

PROPOSITION A5. *With notation as earlier, any $h \in B^\infty$ is a Lipschitz vector for α. The operator, M_h, of multiplication on B^∞ by h is a Lipschitz vector for* Ad_α *and the operator norm.*

We will say that an operator T on U^∞ is of order 1 if there is an inequality of the form

$$\|Tu\| \leq K\|u\|_1$$

for $u \in U^\infty$. We remark that the operators α_X for $X \in \mathfrak{g}$ are clearly of order 1. By a straightforward argument we then obtain:

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PROPOSITION A6. *Let γ be an action of G on a Banach space U. Let T be a Lipschitz operator on U^∞ for γ, of order 1. Let S be a bounded operator on U which is an operator-norm Lipschitz vector for* Ad_γ, *and carries U^∞ into itself. Then ST is a Lipschitz operator on U^∞ for γ.*

From Propositions A3, A5 and A6 it follows that δ_Φ is a Lipschitz operator for α. We phrased the above discussion for B, but all holds equally well for A. We can thus apply Robinson's theorem to obtain a one-parameter action on B which carries A into itself, and so gives a flow on M. ∎

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Department of Mathematics
University of California
Berkeley, CA
USA 94720-3840
e-mail: rieffel@math.berkeley.edu

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