Proceedings of the Edinburgh Mathematical Society (2010) **53**, 373–383 ©
DOI:10.1017/S0013091508000539 Printed in the United Kingdom

COMPLEX SUBMANIFOLDS, CONNECTIONS AND ASYMPTOTICS

TATYANA FOTH

Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada (tfoth@uwo.ca)

(Received 12 May 2008)

Abstract Let $L \to X$ be a positive line bundle on a compact complex manifold X. For compact submanifolds Y, S of X and a holomorphic submersion $Y \to S$ with compact fibre, we study curvature of a natural connection on certain line bundles on S.

Keywords: compact Kähler manifolds; holomorphic line bundles; curvature; asymptotics

2010 *Mathematics subject classification:* Primary 32Q15; 53C05; 53D50

1. Introduction

1.1. Preliminaries

Let X be a connected compact Kähler manifold of complex dimension $n \geqslant 2$, with integral Kähler form ω. Let $L \to X$ be a holomorphic Hermitian line bundle, with connection ∇, such that $\operatorname{curv}(\nabla) = -2\pi \mathrm{i}\omega$ (thus $c_1(L) = [\omega]$). The line bundle L is ample.

Suppose Y, S are connected compact complex submanifolds of X, $S \subset Y$, $1 \leqslant \dim S < \dim Y \leqslant n$, and $\pi : Y \to S$ is a surjective holomorphic map such that $\pi(z) = z$ for any $z \in S$, $Y = \bigcup_{z\in S} \pi^{-1}(z)$ and $\{\pi^{-1}(z) \mid z \in S\}$ is a complex analytic family of connected compact complex submanifolds of X. Define $Y_z = \pi^{-1}(z)$ for $z \in S$. Note that Y_z is diffeomorphic to $Y_{z'}$ for any $z, z' \in S$ (see, for example, [**9**]). For $z \in S$, denote by $\iota_z : Y_z \to X$ the inclusion map, and denote by m the (complex) dimension of Y_z. Also denote by $\sigma : S \to X$ the inclusion map.

Let k be a positive integer. Define $E_z^{(k)} = H^0(Y_z, \iota_z^*(L^{\otimes k}))$ for $z \in S$. For sufficiently large k the complex vector spaces $E_z^{(k)}$, $z \in S$, form a holomorphic vector bundle on S, which we shall denote by $E^{(k)}$; this follows by applying the Kodaira Vanishing Theorem and the Grothendieck–Riemann–Roch Theorem (also see, for example, [**11**], where the setting is more general). Define $r = r(k) = \dim E_z^{(k)}$ and $N = N(k) = \dim H^0(X, L^{\otimes k})$.

There exists k_0 such that, for all $k \geqslant k_0$ and for all $z \in S$, $\iota_z^*(L^{\otimes k})$ is very ample and the pull-back (restriction to Y_z) map $H^0(X, L^{\otimes k}) \to E_z^{(k)}$, $s \mapsto \iota_z^* s$ is surjective. The surjectivity statement means that every holomorphic section over Y_z can be extended to a holomorphic section over X [**4**, p. 74]. See also the following short explanation: denote by $\mathcal{J}_z^{(k)}$ the sheaf of holomorphic sections of $L^{\otimes k}$ vanishing on Y_z (i.e. $\mathcal{J}_z^{(k)}$ is the product of the ideal sheaf of Y_z and $\mathcal{O}(L^{\otimes k})$). Note that, for sufficiently large k, $H^1(X, \mathcal{J}_z^{(k)}) = 0$ (by the Serre Vanishing Theorem). Then the exact sequence of cohomology associated to the exact sequence of sheaves

$$0 \to \mathcal{J}_z^{(k)} \to \mathcal{O}(L^{\otimes k}) \to \mathcal{O}(\iota_z^*(L^{\otimes k})) \to 0$$

shows that the map $H^0(X, \mathcal{O}(L^{\otimes k})) \to H^0(Y_z, \mathcal{O}(\iota_z^*(L^{\otimes k})))$ is surjective (since the next map $H^0(Y_z, \mathcal{O}(\iota_z^*(L^{\otimes k}))) \to H^1(X, \mathcal{J}_z^{(k)})$ is the zero map).

Henceforth, we shall assume that $k \geqslant k_0$.

1.2. Statement of results and discussion

We shall introduce a natural metric and connection on $E^{(k)}$. It will provide a connection on the holomorphic line bundle $\det(E^{(k)})$, and we shall study its curvature asymptotically, as $k \to +\infty$.

This connection is the metric connection obtained from the Hermitian metric on $E^{(k)}$, defined as follows.

Denote the Hermitian inner product on the fibre of L over z by $b_z(\cdot\,,\cdot)$. This provides the Hermitian inner product on the fibre of $L^{\otimes k}$ over z, which we shall denote by $b_z^{(k)}(\cdot\,,\cdot)$. Then the Hermitian inner product on $L^2(X, L^{\otimes k})$ is

$$b^{(k)}(s, s') = \int_X b_z^{(k)}(s(z), s'(z)) \frac{\omega^n}{n!},$$

$s, s' \in L^2(X, L^{\otimes k})$, and this defines a Hermitian inner product on $H^0(X, L^{\otimes k}) \subset L^2(X, L^{\otimes k})$ which we shall also denote by $b^{(k)}(\cdot\,,\cdot)$.

One can define an inner product on $E_z^{(k)}$ as follows. Observe that for $z \in S$ the finite-dimensional complex vector space $H^0(X, L^{\otimes k})$, endowed with Hermitian inner product $b^{(k)}(\cdot\,,\cdot)$, splits into orthogonal sum of subspaces $\ker(\iota_z^*)$ and its orthogonal complement $V_z^{(k)} \cong E_z^{(k)}$. Thus, we obtain another inner product on $E_z^{(k)}$, obtained from $b^{(k)}(\cdot\,,\cdot)$. We shall denote this inner product by $(\cdot\,,\cdot)^{(k)}$, and we shall denote the corresponding connection on $E^{(k)}$ by $D^{(k)}$. Let us explain this in a little more detail. Suppose that $s, s' \in E_z^{(k)}$. Define *the extension* $\tilde{s}$ *of* s to X as $\tilde{s} \in H^0(X, L^{\otimes k})$ such that $\iota_z^* \tilde{s} = s$ and $\tilde{s}$ has the minimal norm (i.e. it has the minimal value of $b^{(k)}(\tilde{s}, \tilde{s})$).* Also, let $\tilde{s}'$ be the extension of s' to X. The definition above says that $(s, s')^{(k)} := b^{(k)}(\tilde{s}, \tilde{s}')$.

Note that $D^{(k)}$ is defined in the same way as the connection studied in [**6, 8**] (see also [**7**]), and this dates back to [**1**] (i.e. it is 'the same' connection, at least philosophically); see Lemma 3.1.

Another natural way to define an inner product on $E_\xi^{(k)}$, $\xi \in S$, is as follows:

$$\langle s, s' \rangle^{(k)} = \int_{Y_\xi} b_z^{(k)}(s(z), s'(z)) \frac{(\iota_\xi^* \omega)^m}{m!},$$

$s, s' \in E_\xi^{(k)}$. We shall denote the corresponding Hermitian connection on $E^{(k)}$ by $\nabla^{(k)}$. The connection on $\det(E^{(k)})$ obtained from $\nabla^{(k)}$ is the standard L^2 connection.

* I am grateful to Brian Hall for this remark.

https://doi.org/10.1017/S0013091508000539 Published online by Cambridge University Press

In the theorem below we prove that the curvature of the connection on $\det(E^{(k)})$ obtained from $D^{(k)}$ is asymptotic to $-2\pi \mathrm{i}\sigma^*\omega$, and as a corollary we obtain that it is also asymptotic to the curvature of the L^2 connection. We also give an explicit expression for the curvature of $D^{(k)}$ (Proposition 1.4).

Theorem 1.1. *Let $f_1,\dots,f_r$ be a local holomorphic frame for $E^{(k)}$ near $\xi \in S$, and let $h^{(k)} = h^{(k)}(z)$ be the $r\times r$ matrix with $h^{(k)}_{ij} = (f_i, f_j)^{(k)}$. Then, as $k \to +\infty$, at $z = \xi$*

$$\left\| -\frac{\mathrm{i}}{2\pi k r(k)}\partial\bar{\partial}\log\det h^{(k)} - \sigma^*\omega \right\|_{\xi} = O\left(\frac{1}{k}\right).$$

Lemma 1.2. *Let $f_1,\dots,f_r$ be a local holomorphic frame for $E^{(k)}$ near $\xi \in S$, and let $\gamma^{(k)} = \gamma^{(k)}(z)$ be the $r\times r$ matrix with $\gamma^{(k)}_{ij} = \langle f_i, f_j\rangle^{(k)}$, Then, as $k \to +\infty$, at $z = \xi$*

$$\left\| -\frac{\mathrm{i}}{2\pi k r(k)}\partial\bar{\partial}\log\det \gamma^{(k)} - \sigma^*\omega \right\|_{\xi} = O\left(\frac{1}{k}\right).$$

Corollary 1.3. *In the notation of Theorem 1.1 and Lemma 1.2, as $k \to +\infty$, at $z = \xi$*

$$\left\| \frac{1}{k r(k)}(\partial\bar{\partial}\log\det h^{(k)} - \partial\bar{\partial}\log\det \gamma^{(k)}) \right\|_{\xi} = O\left(\frac{1}{k}\right).$$

The precise meaning of the three asymptotic statements above is as follows: for a sequence $\{\Omega^{(k)}\}$ of 2-forms on S and $\xi \in S$,

$$\|\Omega^{(k)}\|_{\xi} = O\left(\frac{1}{k}\right), \quad k \to +\infty,$$

means that, for any tangent vectors $u, v \in T_{\xi}S$,

$$|\Omega^{(k)}_{\xi}(u,v)| = O\left(\frac{1}{k}\right) \quad \text{as } k \to +\infty$$

(clearly, uniformly on the unit sphere bundle in TS).

Also, to clarify: the curvature 2-form of the connection on $\det E^{(k)}$ obtained from $\nabla^{(k)}$ is $-\partial\bar{\partial}\log\det\gamma^{(k)}$ and $D^{(k)}$ gives the connection on $\det E^{(k)}$ with curvature $-\partial\bar{\partial}\log\det h^{(k)}$. Note that the asymptotics in Theorem 1.1 are consistent with equality (1) in [**11**] (a direct consequence of the Grothendieck–Riemann–Roch Theorem), which in our case gives

$$c_1(\det E^{(k)}) = \int_{Y_z} \frac{[\omega]^{m+1}}{(m+1)!} k^{m+1} + O(k^m)$$

as $k \to \infty$. This can be refined to a statement at the level of 2-forms (for the curvature of the L^2 connection, see Lemma 1.2 and its proof).

For an N-dimensional complex vector space W we shall denote by $G(r, W)$ the Grassmannian of r-dimensional linear subspaces of W. Choose and fix a basis $u_1,\dots,u_N$ in W. Recall that, given a basis $f_1,\dots,f_r$ in $U \in G(r,W)$, the Stiefel matrix A for U is the $N \times r$ matrix whose jth column consists of components of f_j with respect to the basis $u_1,\dots,u_N$. The matrix A is defined up to right multiplication by a non-singular $r\times r$ matrix.

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Proposition 1.4. *Suppose that $\xi \in S$, $z = z(t_1)$, $z = z(t_2)$ are paths in S such that*

$$z(t_1)|_{t_1=0} = z(t_2)|_{t_2=0} = \xi$$

and v_1, v_2 are the tangent vectors to S at ξ represented by these two paths. Suppose that an orthonormal basis in $H^0(X, L^{\otimes k}) =: W^{(k)}$ is chosen. For $j = 1, 2$ let $A^{(k)}(z(t_j))$ be Stiefel matrices for $V^{(k)}_{z(t_j)} \in G(r, W^{(k)})$ with respect to this basis, depending smoothly on t_j, and such that

$$A^{(k)}(z(t_1))|_{t_1=0} = A^{(k)}(z(t_2))|_{t_2=0} =: A^{(k)}_0,$$

and the basis in $V^{(k)}_\xi$ is chosen to be orthonormal. Then the curvature of $D^{(k)}$ is the following element of $\mathrm{End}(E^{(k)}_\xi)$:

$$\mathrm{Curv}_{\xi;v_1,v_2}(D^{(k)}) = \overline{\eta^{(k)}_1}^{\mathrm{T}}\Big(I_N - A^{(k)}_0\overline{A^{(k)}_0}^{\mathrm{T}}\Big)\eta^{(k)}_2 - \overline{\eta^{(k)}_2}^{\mathrm{T}}\Big(I_N - A^{(k)}_0\overline{A^{(k)}_0}^{\mathrm{T}}\Big)\eta^{(k)}_1, \tag{1.1}$$

where

$$\eta^{(k)}_j = \frac{\mathrm{d}A^{(k)}(z(t_j))}{\mathrm{d}t_j}\bigg|_{t_j=0}, \quad j = 1, 2,$$

and I_N is the $N \times N$ identity matrix.

Remark 1.5. If we choose the bases so that $f_j = u_j$ at ξ for $1 \leqslant j \leqslant r$, then the matrix (1.1) is $\bar{B}^{\mathrm{T}}_1 B_2 - \bar{B}^{\mathrm{T}}_2 B_1$, where B_l is the $(N - r) \times r$ matrix whose ijth entry is the $(r + i, j)$th entry of η_l, $l = 1, 2$.

2. Proofs of Theorem 1.1, Lemma 1.2 and Corollary 1.3

The following statement will be needed.

Lemma 2.1. *Let $\mathcal{L} \to M$ be a very ample Hermitian line bundle on a connected compact complex manifold M. Let $d = \dim H^0(M, \mathcal{L})$, and let $K(\cdot\,,\cdot)$ be the Bergman kernel for $\mathcal{L}$ (i.e. the kernel of the orthogonal projection $L^2(M, \mathcal{L}) \to H^0(M, \mathcal{L})$). Then there are points $p_1, \dots, p_d \in M$ such that $K(z, p_1), \dots, K(z, p_d)$ form a basis in $H^0(M, \mathcal{L})$.*

Proof. First, we recall that, for any $p \in M$, $K(z, p)$ is naturally identified with a holomorphic section of $\mathcal{L}$ and $s(p) = \langle s(z), K(z, p)\rangle$, where $p, z \in M$, $s \in H^0(M, \mathcal{L})$ and $\langle\cdot\,,\cdot\rangle$ denotes the inner product on $H^0(M, \mathcal{L})$.

Denote by $\iota : M \to \mathbb{P}^{d-1}$ the projective embedding given by $\mathcal{L}$. Choose points $p_1, \dots, p_d \in M$ such that $\iota(p_1), \dots, \iota(p_d)$ are not on the same hyperplane in $\mathbb{P}^{d-1}$.

Let us argue by contradiction. Assume that $K(z, p_1), \dots, K(z, p_d)$ do not form a basis in $H^0(M, \mathcal{L})$. Then there is $s \in H^0(M, \mathcal{L})$ which is not the zero section and such that for $1 \leqslant j \leqslant d$, $s(p_j) = \langle s(z), K(z, p_j)\rangle = 0$. The zero locus of s is the intersection of $\iota(M)$ with a hyperplane in $\mathbb{P}^{d-1}$, but $\iota(p_1), \dots, \iota(p_d)$ are not on the same hyperplane, and hence s must be the zero section, which is a contradiction. Thus, $K(z, p_1), \dots, K(z, p_d)$ form a basis in $H^0(M, \mathcal{L})$. □

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Proof of Lemma 1.2. There are various ways to prove this lemma. One way to derive the result is as follows: integrate (7) in [**11**, Theorem 2.1] over the fibre, note [**11**, (8)], also note that the curvature on $\det(E^{(k)})$ is the trace of the curvature on $E^{(k)}$ and the trace of the curvature operator is equal to the integral of its kernel restricted to the diagonal.

Or, instead, one could prove it using the Bismut–Gillet–Soulé Curvature Theorem [**2**, Theorem 1.9] and Bismut and Vasserot's result on the asymptotic expansion of the analytic torsion [**10**, Theorem 5.5.8].

Here we shall explain in detail another proof, which is based on asymptotic results for the Bergman kernel due to Zelditch and Borthwick and Uribe, to demonstrate the general idea of the proof of Theorem 1.1.

We recall that

$$\left\| \frac{1}{k}\phi_k^*(\omega_{\mathrm{FS}}^{(k)}) - \sigma^*\omega \right\| = O\left(\frac{1}{k}\right) \tag{2.1}$$

as $k \to +\infty$, where $\phi_k : S \to \mathbb{P}^q$ is the projective embedding given by $\sigma^*(L^{\otimes k})$, $q = q(k) = \dim H^0(S, \sigma^*(L^{\otimes k})) - 1$, and $\omega_{\mathrm{FS}}^{(k)}$ is the Fubini–Study form on $\mathbb{P}^q$ (see, [**12**], [**13**, Corollary 3], [**5**]).

For $z \in S$ denote by $K_z^{(k)}(\cdot\,,\cdot)$ the kernel of the orthogonal projection

$$L^2(Y_z, \iota_z^*(L^{\otimes k})) \to H^0(Y_z, \iota_z^*(L^{\otimes k}))$$

(the Bergman kernel for the line bundle $\iota_z^*(L^{\otimes k}) \to Y_z$).

For $E^{(k)}$ near ξ, let $s_1, \dots, s_r$ be a local unitary frame (with respect to $\langle\cdot\,,\cdot\rangle^{(k)}$), and also choose a local frame of the form

$$e_1 = K_z^{(k)}(\cdot\,, p_1(z)), \dots, e_r = K_z^{(k)}(\cdot\,, p_r(z)),$$

where $z \in S$, $p_1(z), \dots, p_r(z) \in Y_z$. Also assume that the points are chosen so that the $r \times r$ matrix $F = F(z,k)$ with ijth entry equal to $f_i(p_j)$ varies holomorphically near ξ. Note that we drop '(k)' from the notation for e_j, s_j, p_j and f_j, for simplicity.

Denote by $\gamma_{\mathrm{Berg}}^{(k)}$ the $r \times r$ matrix whose ijth entry is $\langle e_i, e_j\rangle^{(k)}$. Recall that

$$e_i = \sum_{j=1}^{r} s_j(z)\overline{s_j(p_i)}.$$

It is not difficult to see that $\gamma_{\mathrm{Berg}}^{(k)} = \bar{F}^{\mathrm{T}}(\gamma^{(k)})^{-1}F$; therefore,

$$\partial\bar{\partial}\log\det\gamma_{\mathrm{Berg}}^{(k)} = -\partial\bar{\partial}\log\det\gamma^{(k)}.$$

In the notation of [**3**] $e_j = \Pi_k(\cdot\,, p_j)$. Note that $\langle e_i(z), e_j(z)\rangle^{(k)} = K_z^{(k)}(p_j(z), p_i(z))$, $1 \leqslant i, j \leqslant r$. By [**3**, Lemma 4.5], as $k \to +\infty$,

$$K_z^{(k)}(p_j, p_i) = \nu_k(p_j)\mathrm{e}^{-kd(p_j,p_i)^2/2} + O(k^{m-1/2}), \tag{2.2}$$

where $d(p_j, p_i)$ is the distance between p_j and p_i, and

$$\nu_k(p_j) := K_z^{(k)}(p_j, p_j) \sim k^m + \text{lower-order terms}$$

(see [**13**] and [**3**, Theorem 4.1]).

We shall need one more assumption on the choice of points $p_1(z),\dots,p_r(z)$: $d(p_1,p_j) > \varepsilon$, $j=2,\dots,r$, for some $\varepsilon>0$, and for all k (this is possible by (2.1), and because $p_2,\dots,p_r$ can be chosen inside a ball of radius δ for any small $\delta>0$). Then, by (2.2), as $k\to+\infty$,

$$\det\gamma^{(k)}_{\mathrm{Berg}} \sim (k^m + O(k^{m-1/2}))^r\left(1+O\left(\frac{1}{\sqrt{k}}\right)\right)$$

and

$$\partial\bar\partial\log\det\gamma^{(k)} = -r\partial\bar\partial\log(1+O(k^{-1/2}))+O(1).$$

Let $u_0,\dots,u_q$ be an orthonormal basis in $H^0(S,\sigma^*(L^{\otimes k}))$. We have, as $k\to\infty$,

$$\sum_{j=0}^{q}|u_j(z)|^2 \sim k^q + O(k^{q-1})$$

by applying [**13**, Theorem 1] to the hyperplane bundle on $\mathbb{P}^q$. In addition,

$$\omega^{(k)}_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\partial\bar\partial\log\left(\sum_{j=0}^{q}|u_j(z)|^2\right).$$

Thus,

$$\phi_k^*\omega^{(k)}_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi}\partial\bar\partial\log\left(1+O\left(\frac{1}{k}\right)\right),$$

so

$$-\frac{\mathrm{i}}{2\pi r}\partial\bar\partial\log\det\gamma^{(k)} - \phi_k^*\omega^{(k)}_{\mathrm{FS}} = O(1)$$

and, using (2.1), we obtain the desired asymptotic statement. □

Proof of Theorem 1.1. The idea is essentially the same as in the proof of Lemma 1.2, but now we shall need a basis in $E^{(k)}_z$ that consists of sections obtained from the Bergman kernel for $L^{\otimes k}$ (rather than for $\iota_z^*(L^{\otimes k})$).

Let $f_1,\dots,f_r$ be, as above, a local holomorphic frame for $E^{(k)}$ and let $\tilde f_1,\dots,\tilde f_r$ be the corresponding basis in $V^{(k)}_z$ for $z\in S$ near ξ. Choose $\tilde f_{r+1},\dots,\tilde f_N$ in $\ker\iota_z^*$ so that $\tilde f_1,\dots,\tilde f_N$ is a basis in $H^0(X,L^{\otimes k})$, varying holomorphically with z. Let $s_1,\dots,s_N$ be an orthonormal basis in $H^0(X,L^{\otimes k})$, such that $\iota_z^*s_1,\dots,\iota_z^*s_r$ is a unitary frame for $E^{(k)}$ (with respect to $(\cdot,\cdot)^{(k)}$) near ξ, and such that s_j is the extension of $\iota_z^*s_j$ to X, $j=1,\dots,r$.

Denote by $K^{(k)}(\cdot,\cdot)$ the Bergman kernel for $L^{\otimes k}\to X$. Choose a basis

$$e_1 = K^{(k)}(\cdot,p_1),\dots,e_N = K^{(k)}(\cdot,p_N)\quad\text{in } H^0(X,L^{\otimes k})$$

for $z\in S$ near ξ, where the points $p_1=p_1(z),\dots,p_N=p_N(z)$ in X are chosen so that they satisfy the following assumptions: $p_1,\dots,p_r\in Y_z$, $\iota_z^*e_1,\dots,\iota_z^*e_r$ is a basis in $E^{(k)}_z$, the $\varepsilon-\delta$ condition for $p_1,\dots,p_r$, as above, is satisfied; also, $\tilde f_1(p_j)=\dots=\tilde f_r(p_j)=0$ for all $j=r+1,\dots,N$. Denote by $\tilde h^{(k)}$ the $N\times N$ matrix whose ijth entry is $b^{(k)}(\tilde f_i,\tilde f_j)$, denote by $\tilde{h}^{(k)}_{\mathrm{Berg}}$ the $N \times N$ matrix whose ijth entry is $b^{(k)}(e_i, e_j)$, and denote by F the $N \times N$ matrix whose ijth entry is $\tilde{f}_i(p_j)$. We have

$$\tilde{h}^{(k)} = F(\tilde{h}^{(k)}_{\mathrm{Berg}})^{-1}\bar{F}^{\mathrm{T}},$$
$$h^{(k)} = F_r(h^{(k)}_{\mathrm{Berg}})^{-1}\bar{F}_r^{\mathrm{T}},$$

where F_r is the $r \times N$ matrix which consists of the first r rows of F, and $h^{(k)}_{\mathrm{Berg}}$ is the $r \times r$ matrix whose ijth entry is

$$(\iota_z^* e_i, \iota_z^* e_j)^{(k)} = b^{(k)}(e_i, e_j)$$

(the $1 \leqslant i, j \leqslant r$ part of $\tilde{h}^{(k)}_{\mathrm{Berg}}$). Then, as above, as $k \to \infty$ we obtain

$$\det h^{(k)}_{\mathrm{Berg}} = (k^n + O(k^{n-1/2}))^r \biggl(1 + O\biggl(\frac{1}{\sqrt{k}}\biggr)\biggr),$$
$$\partial\bar{\partial} \log \det h^{(k)} = -\partial\bar{\partial} \log \det h^{(k)}_{\mathrm{Berg}} = -r\partial\bar{\partial} \log(1 + O(k^{-1/2})) + O(1),$$

and now the statement follows by the same argument as above, at the end of the proof of Lemma 1.2. □

Proof of Corollary 1.3. This follows immediately from Theorem 1.1 and Lemma 1.2. □

3. $D^{(k)}$, projectors and the proof of Proposition 1.4

The following lemma gives an equivalent way to define the connection $D^{(k)}$ (in terms of projectors).

Lemma 3.1. *Suppose that $\xi \in S$, $z(t)$ is a path in S such that $z(0) = \xi$ and v is the tangent vector to S at ξ represented by $z(t)$. Let $\Pi^{(k)}_t$ be the one-parameter family of orthogonal projectors $\Pi^{(k)}_t : H^0(X, L^{\otimes k}) \to V^{(k)}_{z(t)}$. Let s be a holomorphic section of $E^{(k)}$. Then*

$$D^{(k)}_v s = \Pi^{(k)}_0 \frac{\mathrm{d}}{\mathrm{d}t} s(z(t))\bigg|_{t=0}. \tag{3.1}$$

Corollary 3.2. *Suppose that $\xi \in S$, $z = z(t_1)$, $z = z(t_2)$ are two paths in S such that*

$$z(t_1)|_{t_1=0} = z(t_2)|_{t_2=0} = \xi$$

and let v_1, v_2 be the tangent vectors to S at ξ represented by $z(t_1)$, $z(t_2)$. Let $\Pi^{(k)}_{t_1}$ and $\Pi^{(k)}_{t_2}$ be the one-parameter families of orthogonal projectors

$$\Pi^{(k)}_{t_j} : H^0(X, L^{\otimes k}) \to V^{(k)}_{z(t_j)}, \quad j = 1, 2.$$

Then the curvature of $D^{(k)}$ is

$$\mathrm{Curv}_{\xi; v_1, v_2}(D^{(k)}) = \Pi^{(k)}_0 \biggl[\frac{\partial \Pi^{(k)}_{t_1}}{\partial t_1}\bigg|_{t_1=0}, \frac{\partial \Pi^{(k)}_{t_2}}{\partial t_2}\bigg|_{t_2=0}\biggr] \Pi^{(k)}_0|_{V^{(k)}_\xi} \in \mathrm{End}(V^{(k)}_\xi).$$

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Suppose that the chosen orthonormal basis in $W^{(k)}$ is $u_1,\dots,u_N$, and that the Stiefel matrices are written with the use of a local holomorphic frame $f_1,\dots,f_r$ for $E^{(k)}$. As before, $h^{(k)}=(h^{(k)}_{ij})$ is the $r\times r$ matrix with $h^{(k)}_{ij}=(f_i,f_j)^{(k)}$, and $\tilde f_i$ denotes the element of $V^{(k)}_z$ corresponding to f_i.

Define $A^{(k)}=A^{(k)}(z)=(a_{ji}(z))$, the $N\times r$ matrix whose ith column consists of components of $\tilde f_i$ with respect to the basis $u_1,\dots,u_N$, i.e. $\tilde f_i=\sum_{j=1}^N a_{ji}u_j$.

Again, for convenience, we drop the '(k)' from the notation for f_j, $\tilde f_j$, u_j, a_{ji}. The matrix of the orthogonal projection $W^{(k)}\to V^{(k)}_z$ in the basis $u_1,\dots,u_N$ is

$$A^{(k)}(z)((h^{(k)}(z))^{-1})^{\mathrm T}\overline{A^{(k)}(z)}^{\mathrm T}.$$

Additionally, $h^{(k)}(z)=(A^{(k)}(z))^{\mathrm T}\overline{A^{(k)}(z)}$, so at $z=\xi$ we have $(A^{(k)}_0)^{\mathrm T}\overline{A^{(k)}_0}=I_r$, where I_r is the $r\times r$ identity matrix.

Proof of Lemma 3.1. It is sufficient to prove (3.1) for $s=f_i$, $1\leqslant i\leqslant r$. We have

$$D^{(k)}f_i=\sum_{j=1}^r \Theta^{(k)}_{ij}f_j,$$

where $(\Theta^{(k)}_{ij})=\Theta^{(k)}=\partial h^{(k)}\cdot(h^{(k)})^{-1}$ is the connection matrix of $D^{(k)}$.

We need to show that

$$\sum_{j=1}^r(v\rfloor\Theta^{(k)}_{ij})f_j=\Pi^{(k)}_0\frac{\mathrm d}{\mathrm dt}s(z(t))\bigg|_{t=0}.$$

Let us work in $V^{(k)}_z\subset W^{(k)}$, so we replace s by $\tilde s$ and f_j by $\tilde f_j$.

The left-hand side of the desired equality is

$$\sum_{j=1}^r(v\rfloor\Theta^{(k)}_{ij})\sum_{p=1}^N a_{pj}u_p,$$

the right-hand side is

$$\sum_{j=1}^N\frac{\mathrm da_{ji}}{\mathrm dt}\bigg|_{t=0}\Pi^{(k)}_0u_j=\sum_{j=1}^N\frac{\mathrm da_{ji}}{\mathrm dt}\bigg|_{t=0}\sum_{p=1}^N\Big(A^{(k)}_0\overline{A^{(k)}_0}^{\mathrm T}\Big)_{pj}u_p$$

and they are equal because

$$\sum_{j=1}^r(v\rfloor\Theta^{(k)}_{ij})a_{pj}=\sum_{j=1}^r\sum_{l=1}^N a_{pj}\bar a_{lj}\frac{\mathrm da_{li}}{\mathrm dt}\bigg|_{t=0}=\sum_{j=1}^N\frac{\mathrm da_{ji}}{\mathrm dt}\bigg|_{t=0}\Big(A^{(k)}_0\overline{A^{(k)}_0}^{\mathrm T}\Big)_{pj}.$$

□

Proof of Corollary 3.2. See [**6**, **7**], where the setting is different (the fixed Hilbert space and its subspaces are defined in a different way) but the calculation is the same, since the connection is defined by (3.1), in terms of orthogonal projectors from a fixed Hilbert space to its subspaces. □

Proof of Proposition 1.4. Use Corollary 3.2. The matrix of

$$\Pi_0^{(k)}\left[\frac{\partial \Pi_{t_1}^{(k)}}{\partial t_1}, \frac{\partial \Pi_{t_2}^{(k)}}{\partial t_2}\right]\bigg|_{t_1=t_2=0} \Pi_0^{(k)}$$

regarded as an element of $\mathrm{End}(W^{(k)})$ in the basis $u_1, \ldots, u_N$ is

$$\begin{aligned}
A_0^{(k)}\overline{A_0^{(k)}}^{\mathrm{T}}&\Big(\Big(\eta_1\overline{A_0^{(k)}}^{\mathrm{T}} - A_0^{(k)}\Big(\overline{\eta_1}^{\mathrm{T}}A_0^{(k)} + \overline{A_0^{(k)}}^{\mathrm{T}}\eta_1\Big)\overline{A_0^{(k)}}^{\mathrm{T}} + A_0^{(k)}\overline{\eta_1}^{\mathrm{T}}\Big)\\
&\times\Big(\eta_2\overline{A_0^{(k)}}^{\mathrm{T}} - A_0^{(k)}\Big(\overline{\eta_2}^{\mathrm{T}}A_0^{(k)} + \overline{A_0^{(k)}}^{\mathrm{T}}\eta_2\Big)\overline{A_0^{(k)}}^{\mathrm{T}} + A_0^{(k)}\overline{\eta_2}^{\mathrm{T}}\Big)\\
&-\Big(\eta_2\overline{A_0^{(k)}}^{\mathrm{T}} - A_0^{(k)}\Big(\overline{\eta_2}^{\mathrm{T}}A_0^{(k)} + \overline{A_0^{(k)}}^{\mathrm{T}}\eta_2\Big)\overline{A_0^{(k)}}^{\mathrm{T}} + A_0^{(k)}\overline{\eta_2}^{\mathrm{T}}\Big)\\
&\times\Big(\eta_1\overline{A_0^{(k)}}^{\mathrm{T}} - A_0^{(k)}\Big(\overline{\eta_1}^{\mathrm{T}}A_0^{(k)} + \overline{A_0^{(k)}}^{\mathrm{T}}\eta_1\Big)\overline{A_0^{(k)}}^{\mathrm{T}} + A_0^{(k)}\overline{\eta_1}^{\mathrm{T}}\Big)\Big)A_0^{(k)}\overline{A_0^{(k)}}^{\mathrm{T}}\\
&\qquad = A_0^{(k)}\Big(\bar{\eta}_1^{\mathrm{T}}\Big(I_N - A_0^{(k)}\overline{A_0^{(k)}}^{\mathrm{T}}\Big)\eta_2^{(k)} - \bar{\eta}_2^{\mathrm{T}}\Big(I_N - A_0^{(k)}\overline{A_0^{(k)}}^{\mathrm{T}}\Big)\eta_1^{(k)}\Big)\overline{A_0^{(k)}}^{\mathrm{T}}.
\end{aligned}$$

Hence, the matrix of the corresponding endomorphism of $V_\xi^{(k)}$ in the basis $\tilde{f}_1, \ldots, \tilde{f}_r$ is given by (1.1). $\square$

4. Example: calculation for a Segré variety

Let $Y = X = \{[u_0 : u_1 : u_2 : u_3] \in \mathbb{P}^3 \mid u_0u_3 - u_1u_2 = 0\}$. This quadric hypersurface in $\mathbb{P}^3$ is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ under the Segré embedding

$$\begin{gathered}
\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3,\\
([z_0 : z_1], [w_0 : w_1]) \mapsto [z_0w_0 : z_0w_1 : z_1w_0 : z_1w_1].
\end{gathered}$$

Let L be the hyperplane bundle on X, let

$$\omega = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log(|z_0w_0|^2 + |z_0w_1|^2 + |z_1w_0|^2 + |z_1w_1|^2)$$

(the pull-back of the Fubini–Study form on $\mathbb{P}^3$ to X) and let $S = \phi(\{([z_0 : z_1], [1 : 0]) \in \mathbb{P}^1 \times \mathbb{P}^1\})$. Recall that $\sigma : S \to X$ denotes the inclusion map. We shall carry out calculations for $k = k_0 = 1$. The vector bundle $E^{(1)}$ is of rank $r(1) = 2$. For $z = [z_0 : 0 : z_1 : 0] \in S$, $Y_z = \{[z_0w_0 : z_0w_1 : z_1w_0 : z_1w_1] \in \mathbb{P}^3 \mid [w_0 : w_1] \in \mathbb{P}^1\}$, w_0, w_1 form a basis in $H^0(Y_z, \iota_z^*L)$ and this is a local holomorphic frame for $E^{(1)}$.

Note that

$$\langle s, s'\rangle^{(1)} = \int_{Y_z} \frac{s\overline{s'}}{|z_0w_0|^2 + |z_0w_1|^2 + |z_1w_0|^2 + |z_1w_1|^2}\iota_z^*\omega$$

for $s, s' \in H^0(Y_z, \iota_z^*L)$. By straightforward computation we obtain $\langle w_0, w_1\rangle^{(1)} = 0$ and

$$\langle w_0, w_0\rangle^{(1)} = \langle w_1, w_1\rangle^{(1)} = \frac{1}{2(|z_0|^2 + |z_1|^2)}.$$

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Thus,

$$\det \gamma^{(1)} = \frac{1}{4(|z_0|^2 + |z_1|^2)^2}$$

and

$$-\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log\det\gamma^{(1)} = \frac{\mathrm{i}}{\pi}\partial\bar{\partial}\log(|z_0|^2 + |z_1|^2) = kr(k)\sigma^*\omega.$$

Now let us discuss the other connection. Set $\zeta_1 = 2z_0w_0$, $\zeta_2 = 2z_0w_1$, $\zeta_3 = 2z_1w_0$, $\zeta_4 = 2z_0w_0$. We note that

$$b^{(1)}(s, s') = \int_X \frac{s\overline{s'}}{|z_0w_0|^2 + |z_0w_1|^2 + |z_1w_0|^2 + |z_1w_1|^2}\frac{\omega^2}{2}$$

for $s, s' \in H^0(X, L)$. A tedious but straightforward computation shows that $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ is an orthonormal basis in $H^0(X, L)$. Clearly, $\ker \iota_z^*$ is the span of $z_1\zeta_1 - z_0\zeta_3$ and $z_1\zeta_2 - z_0\zeta_4$, and recall that $V_z^{(1)}$ is its orthogonal complement. Let us now carry out explicit computations, without loss of generality, near $z = [z_0 : z_1]$ with $z_0 \neq 0$. Choose a local holomorphic frame in $E^{(1)}$ as follows: $f_1 = 2z_0w_0$, $f_2 = 2z_0w_1$. To compute $(f_i, f_j)^{(1)}$, $1 \leqslant i, j \leqslant 2$, we need to find extensions $\tilde{f}_1$ and $\tilde{f}_2$. We have $\tilde{f}_1 = a\zeta_1 + c\zeta_3$, where a and c are determined by the equality

$$a + \frac{z_1}{z_0}c = 1$$

and by the requirement that $a\bar{a} + c\bar{c}$ has the minimum value. We get

$$a = \frac{z_0\bar{z}_0}{z_0\bar{z}_0 + z_1\bar{z}_1}, \qquad c = \frac{z_0\bar{z}_1}{z_0\bar{z}_0 + z_1\bar{z}_1}.$$

Similarly, $\tilde{f}_2 = a\zeta_2 + c\zeta_4$ with a and c as above. Then $(f_1, f_2)^{(1)} = 0$,

$$(f_1, f_1)^{(1)} = (f_2, f_2)^{(1)} = a\bar{a} + c\bar{c} = \frac{z_0\bar{z}_0}{|z_0|^2 + |z_1|^2}$$

and

$$-\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log\det h^{(1)} = -\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log\frac{z_0^2\bar{z}_0^2}{(|z_0|^2 + |z_1|^2)^2} = kr(k)\sigma^*\omega.$$

Acknowledgements. I thank A. Dhillon and I. Dolgachev for related discussions and P. Foth, B. Hall and A. Uribe for brief helpful comments. I am grateful to the referee for interesting and useful comments and suggestions. This research was supported in part by the NSERC, Canada.

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