In this case one two-rowed skew Latin square minor always exists, owing to the fact that $p = 4r + 1$ can be expressed as the sum of two squares. Here

$$\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -(2^2 + 3^2) = -13.$$

The quadratic residues for $p = 4r + 1$ always occur in pairs $\pm y$ and involve only half of the possible set of reduced integers: they may or may not lead to a non zero minor determinant. For $p = 13$, the six residues are $\pm 1, \pm 3, \pm 4$: for $p = 17$ the eight residues are $\pm 1, \pm 2, \pm 4, \pm 8$. We have

$$\begin{vmatrix} 1 & 4 & 3 \\ 4 & 3 & -1 \\ 3 & -1 & -4 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 8 & -1 \\ 4 & 8 & -1 & -2 \\ 8 & -1 & -2 & -4 \end{vmatrix} = 17^3.$$

THE UNIVERSITY,
ST ANDREWS.

Inequalities for Positive Series

By C. E. WALSH.

$$\text{Let } f(x) \equiv (1-x)^b + b\,\alpha^{b-1}\,x$$
$$\phi(x) \equiv x^c - c\,\beta^{c-1}\,x$$

where $b \geqq 1$, $c \geqq 1$, $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, and x is assumed to lie in the range $(0, 1)$. By differentiation, or otherwise, it is easily shewn that $f(x)$ and $\phi(x)$ have minima when $x = 1 - \alpha$ and when $x = \beta$, respectively. Hence

$$(1-x)^b + b\,\alpha^{b-1}\,x \geqq b\,\alpha^{b-1} + (1-b)\,\alpha^b$$
$$x^c - c\,\beta^{c-1}\,x \geqq (1-c)\,\beta^c.$$

Multiplying the first of these by $c\,\beta^{c-1}$, the second by $b\,\alpha^{b-1}$, and adding, we get

$$c\,\beta^{c-1}\,(1-x)^b + b\,\alpha^{b-1}\,x^c \geqq b\,\alpha^{b-1}\,\beta^{c-1}$$
$$+\, b\,(c-1)\,\alpha^{b-1}\,\beta^{c-1}\,(1-\beta) + c\,(1-b)\,\alpha^b\,\beta^{c-1}.$$

Dividing across by β^{c-1}, this is the same as

$$c\,(1-x) \geqq b\,\alpha^{b-1}\,(1-\beta^{1-c}\,x^c) + b\,(c-1)\,\alpha^{b-1}\,(1-\beta) + c\,(1-b)\,\alpha^b \quad (1).$$

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The use of (1) enables us to prove easily some inequalities of the sort obtained by Copson and Elliot[1] for positive series. An example may perhaps be given.

$$\text{Let } A_n = \sum_{r=1}^{n} \lambda_r a_r,\ \Lambda_n = \sum_{r=1}^{n} \lambda_r,\ P_n = \sum_{r=1}^{n} p_r \lambda_r$$

where, for all n, $a_n > 0$, $\lambda_n > 0$, $p_n > 0$. We assume further that

$$p_n \Lambda_n / P_n \geqq p_{n+1} \Lambda_{n+1} / P_{n+1}$$

always. Now make the following substitutions in (1).

$$\text{Let } x = A_{n-1}/A_n,\ \beta = P_{n-1}/P_n,\ \alpha = \mu \lambda_n / \Lambda_n$$

where μ is positive, and such that $\alpha = \mu \lambda_n/\Lambda_n < 1$, but is otherwise undetermined so far. As a result of these substitutions (1) becomes

$$c\left(\frac{\lambda_n a_n}{A_n}\right)^b \geqq b\left(\frac{\mu \lambda_n}{\Lambda_n}\right)^{b-1}\left(1 - \frac{P_{n-1}^{1-c}}{P_n^{1-c}} \frac{A_{n-1}^c}{A_n^c}\right) + b\,\frac{(c-1)\,\mu^{b-1} \lambda_n^{b-1} p_n \lambda_n}{\Lambda_n^{b-1} P_n} + \frac{c\,(1-b)\,\mu^b \lambda_n^b}{\Lambda_n^b}$$

Multiplying across by $\lambda_n^{1-b} A_n^c \Lambda_n^{b-1} P_n^{1-c}$ gives

$$c\,\lambda_n P_n^{1-c} \Lambda_n^{b-1} a_n^b A_n^{c-b} \geqq b\,\mu^{b-1} (P_n^{1-c} A_n^c - P_{n-1}^{1-c} A_{n-1}^c) + \lambda_n A_n^c \{b\,(c-1)\,\mu^{b-1} p_n P_n^{-c} + c\,(1-b)\,\mu^b P_n^{1-c} \Lambda_n^{-1}\}. \qquad (2)$$

Now the last factor on the right, namely

$$b\,(c-1)\,\mu^{b-1} p_n P_n^{-c} + c\,(1-b)\,\mu^b P_n^{1-c} \Lambda_n^{-1}$$

regarded as a function of μ has a maximum value when

$$\mu = \frac{(c-1)}{c} \frac{p_n \Lambda_n}{P_n} \qquad (3)$$

as is easily shewn by differentiation. Substituting this value for μ, the maximum value is found to be

$$\frac{(c-1)^b}{c^{b-1}} \frac{p_n^b \Lambda_n^{b-1}}{P_n^{b+c-1}}.$$

With μ having the value given by (3)

$$\alpha = \mu \frac{\lambda_n}{\Lambda_n} = \frac{(c-1)}{c} \frac{p_n \lambda_n}{P_n}$$

satisfies $0 < \alpha < 1$. Thus it is permissible to take

$$\mu = \frac{(c-1)}{c} \frac{p_n \Lambda_n}{P_n} = \mu_n$$

[1] Copson, *Journal London Math. Society* **2** (1927), 9-12; **3** (1928), 49-51. Elliott, *Ibid.*, **1** (1926), 93-96; **4** (1929), 21-23.

in (2). Having done so, the latter is now

$$c\,\lambda_n P_n^{1-c} \Lambda_n^{b-1} a_n^b A_n^{c-b} \geqq b\,\mu_n^{b-1} (P_n^{1-c} A_n^c - P_{n-1}^{1-c} A_{n-1}^c) + \frac{(c-1)^b}{c^{b-1}} \frac{\lambda_n p_n^b \Lambda_n^{b-1} A_n^c}{P_n^{b+c-1}}.$$

Hence, summing for $n = 1, 2, \ldots, N$

$$c \sum_{n=1}^{N} \lambda_n P_n^{1-c} \Lambda_n^{b-1} a_n^b A_n^{c-b} \geqq b \sum_{n=1}^{N-1} (\mu_n^{b-1} - \mu_{n+1}^{b-1}) P_n^{1-c} A_n^c + b\,\mu_N^{b-1} P_N^{1-c} A_N^c + \frac{(c-1)^b}{c^{b-1}} \sum_{n=1}^{N} \frac{\lambda_n p_n^b \Lambda_n^{b-1} A_n^c}{P_n^{b+c-1}} \qquad (4)$$

Since, by hypothesis, $p_n \Lambda_n / P_n$ never increases, it follows that $\mu_n \geqq \mu_{n+1}$ always. Accordingly, for all values of N

$$b \sum_{n=1}^{N-1} (\mu_n^{b-1} - \mu_{n+1}^{b-1}) P_n^{1-c} A_n^c \geqq 0.$$

Consequently, from (4)

$$c \sum_{1}^{N} \lambda_n P_n^{1-c} \Lambda_n^{b-1} a_n^b A_n^{c-b} > \frac{(c-1)^b}{c^{b-1}} \sum_{1}^{N} \frac{\lambda_n p_n^b \Lambda_n^{b-1} A_n^c}{P_n^{b+c-1}}.$$

Finally, letting $N \to \infty$, and dividing across by c

$$\sum_{n=1}^{\infty} \lambda_n P_n^{1-c} \Lambda_n^{b-1} a_n^b A_n^{c-b} \geqq \left(\frac{c-1}{c}\right)^b \sum_{1}^{\infty} \frac{\lambda_n p_n^b \Lambda_n^{b-1} A_n^c}{P_n^{b+c-1}} \qquad (5)$$

provided the series on the left-hand side converges, and $b \geqq 1$, $c \geqq 1$.

Now write $p_n = 1$ for all n, so that $P_n \equiv \Lambda_n$, and let $b = c$. Then (5) reduces to one of the theorems proved by Copson (in the first paper referred to) namely

If $a_n > 0$, $\lambda_n > 0$ for all n, and $b > 1$, then

$$\sum_{n=1}^{\infty} \lambda_n a_n^b \geqq \left(\frac{b-1}{b}\right)^b \sum_{1}^{\infty} \lambda_n (A_n/\Lambda_n)^b.$$

34 Chelmsford Road,
Ranelagh, Dublin.